THREE – DIMENSIONAL DYNAMIC STRESS ANALYSIS OF SANDWICH PANELS

DISSERTATION

Victor Y. Perel, B.S., M.S.

AFIT/DS/ENY/00-02

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

The views expressed in this dissertation are those of the author and do not reflect the official policy or position of the Department of Defense or the U.S. Government.

AFIT/DS/ENY/00-02

Three-Dimensional Dynamic Stress Analysis of Sandwich Panels

Dissertation

Presented to the Faculty
Graduate School of Engineering and Management
Air Force Institute of Technology
Air University
Air Education and Training Command
In Partial Fulfillment of the Requirements for the
Degree of Doctor of Philosophy

Victor Y. Perel, B.S., M.S.

June, 2000

AFIT/DS/ENY/00- 2

Three-Dimensional Dynamic Stress Analysis of Sandwich Panels

Victor Y. Perel

Approved:

Anthony N. Palazotto ______ 30 June 2000

ANTHONY N. PALAZOTTO

Committee Chairman

Robert A. Brockman ______ 30 Jun 2000

ROBERT A. BROCKMAN

(University of Dayton)

Committee Member

William P. Baker ______ 27 Jun 2000

WILLIAM P. BAKER

Committee Member

Mark E. Oxley ______ 27 Jun 2000

MARK E. OXLEY

Dean's Representative

Accepted:

Robert A. Calico ______ 11 July 2000

ROBERT A. CALICO

Dean, School of Engineering

Acknowledgments and Dedication

This research was sponsored by Dr. Steve Walker of the Air Force Office of Scientific Research. His sponsorship is very much appreciated.

I express my sincere gratitude to my faculty advisor Dr. Anthony N. Palazotto. His profound expertise in structural mechanics and finite element methods guided me through these vast areas of knowledge. Besides, without his help in providing me the financial assistance, the completion of my graduate studies would be impossible.

I thank also my committee members Dr. Robert A. Brockman, Dr. William P. Baker and Dr. Mark E. Oxley for their interest in this work, reading and evaluation of the dissertation and valuable suggestions on its improvement.

I also acknowledge Dr. Joseph P. Gallagher of University of Dayton for his financial support at the initial stages of my graduate studies.

The information contained in the dissertation of Major Eric Herup was very useful for me in the process of work on my dissertation. Major Herup was very friendly to me and readily provided any help that I was asking him for.

I want to express my gratitude and appreciation to my wife [redacted], my parents [redacted] my brother [redacted] and my son [redacted] for their infinite love and encouragement for this work.

During the period of work on this dissertation, my close relative [redacted] died in a tragic accident at the age of twenty. He was loved by everybody who knew him. He was also a very talented young man and a brilliant student. This dissertation is dedicated to the memory of [redacted]

Contents

List of Figures

List of Tables

AFIT/DS/ENY/00-6

Abstract

A layerwise geometrically nonlinear theory for a thick sandwich plate was developed by introducing assumptions on a variation of transverse strains in the thickness direction of the faces and the core of the plate. An effect of transverse extensibility or compressibility of the core and the face sheets is taken into account, and the terms associated with transverse shear strain of the face sheets and the core are included into the expression for the strain energy. Displacements, obtained by integration of the strain-displacement relations, depend nonlinearly on a coordinate in the thickness direction, and are continuous at the boundaries between the face sheets and the core. The nonlinear von-Karman strain-displacement relations are used in order to provide a representation of the moderately large rotations. The in-plane stresses are computed from the constitutive relations in each ply of the face sheets, using each ply's material properties, and the transverse stresses are computed by substituting the in-plane stresses into equations of motion and by integrating the equations of motion. Such a method of computation of the transverse stress components allows one to obtain accurate results, because this method leads to satisfaction of conditions of continuity of the transverse stresses at the boundaries between the face sheets and the core, at the boundaries between the plies of the face sheets, and allows one to satisfy stress boundary conditions at both the upper and lower external surfaces.

A finite element formulation was developed for a sandwich cargo platform under its impact against the ground, modelled as an elastic Winkler foundation. This formulation was done for a plate in cylindrical bending, and a finite element program was written on the basis of this formulation, with the capability of taking account of damage progression in time. The damage prediciton is performed with the use of the Hashin's and Tsai-Wu criteria by reducing, at each step of time integration, the appropriate material characteristics of those plies within a finite element in which failure occurs. The stresses and displacements, computed by this program, are shown to be in good agreement with the known exact solutions of various static and dynamic problems. Example problems of stress and failure analysis of sandwich cargo-delivery patforms during their impact against the elastic foundations are considered. In these example problems, the stresses as functions of time are computed at certain locations in the platforms with account of damage progression, i.e. with account of degradation of material characteristics of the failing plies. The locations of the failures, the modes of failures and the times of their occurence are defined by the program.

The theory of the sandwich plates, presented in the dissertation, does not require many degrees of freedom in the finite element formulation and has a wide range of applicability. It can be used for analysis of both thick and thin sandwich plates, with thick and thin face sheets, with transversely rigid and transversely flexible faces and cores.

Chapter 1

Introduction

Thick sandwich composite panels have many Air Force applications. One such application is the design of cargo delivery platforms that undergo extensive failure though ground impact. A study of this phenomenon requires an analysis of the sandwich plates with the development of an exact state of stress. A finite element analysis with the use of solid elements can provide information about all stress components, but such an approach is often unacceptable for real structures, because it requires many degrees of freedom. A computational cost can be reduced by using two-dimensional plate formulations. The formulations of thick sandwich plates in the past, using two-dimensional approaches, lack the ability to predict the necessary stress components that can lead to realistic states of stress for use in failure analysis. The work developed in this dissertation overcomes the shortcomings of the two-dimensionality by incorporating a method which contains the associated conditions of through-the-thickness strains and failure response. In addition, this work takes into account the appropriate equations of motion pertaining to the plate under the impact with an elastic foundation.

The sandwich plates have a well pronounced zigzag variation of the in-plane displacements in the thickness direction, due to their high ratios of thickness to in-plane dimensions and large difference of elastic moduli of the face sheets and the core. Such characteristics of the sandwich plates make it necessary to use a layerwise approach in their analysis, the idea of which is to introduce the separate simplifying assumptions regarding through-the-thickness variation of either displacements, or strains or stresses within each face sheet and the core. Besides, in order to achieve a high accuracy of stress computation, in a model of the sandwich plate with a thick core and thick face sheets, it must be

assumed that the in-plane displacements vary nonlinearly in the thickness direction of both the core and the face sheets, and, in the expression for the strain energy, the transverse direct and shear strains need to be taken into account not only in the core, but also in the face sheets.

The two-dimensional layerwise finite element formulations of this type, for analysis of thick sandwich plates with transversely compressible or extensible cores and face sheets and with nonlinear variation of the in-plane displacements in the thickness direction of both the core and the face sheets, have not been presented extensively in literature so far. Development of such a finite element formulation and its application to progressive failure analysis of sandwich cargo-delivery platforms under their ground impact, is the objective of this dissertation.

To determine a load-carrying capacity and service life of a composite structure, it is necessary to predict the initiation and evolution of the damage. When the stresses, as functions of time, in the composite structure are known as a result of solving the plate-bending problem, then the onset of failure can be predicted by applying an appropriate failure criterion. It has been observed that after the initial failure in a single layer of a composite structure, loading can still be carried. Therefore, the subsequent failure prediction is required to determine the dynamic response of the platform in the presence of some damage. There are many proposed theories to predict the onset of failures and their progression. A set of failure criteria that can predict modes of failures in the composite laminates, and in which failures are due to the combination of in-plane and transverse stresses were suggested by Hashin (1980). In our study these criteria are used for the face sheets of the sandwich platform. For the core of the sandwich platform, we use the Tsai-Wu criterion.

It is possible to predict the first occurrence of failure (first-ply failure) in a composite laminate without much difficulty (Reifsnider and Masters (1982), Highsmith and Reisfinder (1982), Talreja (1985), Hashin (1985), Reddy and Pandley (1987), Reddy, Y.S.N. and Reddy, J. N. (1992), Daniel and Ishai (1994), Barbero (1999)). But it is more difficult to predict the subsequent failures after the initial damage has occurred, since the detailed stress analysis of a composite laminate with thousands of small cracks becomes practically impossible. In the progressive failure analysis, this problem is dealt with in an indirect way. It is assumed that the damaged material can be replaced with an equivalent material with degraded properties, and the stress analysis of a composite laminate with degraded properties is conducted without taking into account the singularities of the stress field near the crack tips.

One of the first attempts to model the failure behavior of composite laminates by progressive failure analysis was done by Petit and Waddoups (1969). They used the classical laminated plate

theory for stress analysis and an incremental loading procedure for failure analysis. As the incremental loading proceeded, the individual lamina elastic constants were updated. Ultimate failure of a laminate was assumed to occur when the in-plane laminate stiffness matrix [A] became singular, or when a diagonal term of [A] became negative.

Chang et al. (1984, 1987) performed progressive failure analysis of notched composite laminates in tension and compression by using the finite element model based on the classical plate theory. Stiffness reduction was carried out at the element level and a failure criterion originally proposed by Yamada and Sun was used.

Tan (1991) included the effect of thermal residual stresses and hygroscopic stresses in his progressive failure model. The classical laminated plate theory was used for stress analysis, and the Tsai-Wu criterion was used for failure prediction.

The progressive failure models, considered so far, were based on computation only of the in-plane stresses and could not take into account the delamination type of failure. Ochoa and Engblom (1987) used a higher order plate theory for stress analysis and computed the transverse shear and normal stresses from the equilibrium equations. Stiffness reduction was carried out at Gauss points, and Hashin's failure criterion was used for the failure prediction.

Lee (1982) performed a fully three-dimensional failure analysis of biaxially loaded laminates with a central hole. The finite element mesh consisted of 8-node brick elements, and a special kind of loading condition was used that made it possible to analyze only a quarter of the entire laminate. The stiffness reduction was carried out at the element level, taking into account three types of damage models: fiber breakage, transverse cracking and delamination.

Sun (1989) performed progressive failure analysis of angle-ply laminates by using an iterative three-dimensional finite element approach. The average stress in each element and the Hashin's failure criterion were used for failure prediction.

Tolson and Zabaras (1991) developed a seven degree of freedom finite element model for laminated composite plates. The model utilizes three displacements, two rotations of normals about the plate midplane, and two rotations of the normals to the datum surface. The in-plane stresses were calculated from the constitutive equations, and the transverse stresses - from the three-dimensional equilibrium equations. The maximum stress, Lee, Hashin, Hoffman and Tsai-Wu failure criteria were used. The procedure for determining the strength of a laminate involved an incremental load analysis. For a given load the stresses in each lamina were determined with respect to the material coordinates. When failure in a lamina occurred, the stiffness was modified and the load increased

until the final failure was reached.

Eason and Ochoa (1996) incorporated a shear deformable composite element with built-in progressive damage capability into a commercial finite element program ABAQUS, as a user element. The constitutive equations were used for calculation of the in-plane stresses, and the equilibrium equations were used to calculate the transverse shear and normal stresses. When a damage was detected at a quadrature point, damage was accounted for by reducing stiffness of the lamina at the quadrature point, in correspondence with the failure mode. The criterion with quadratic interaction between stresses and the maximum stress criterion were used for failure prediction.

In all the above references, the material failure was considered for structures under static deformations, and not much work has been done to study the influence of geometric nonlinearity and transverse normal stress on the failure behavior of composite laminates subjected to bending loads. In the present work the stress and failure analysis is conducted for a dynamic problem, and both the geometric nonlinearity and the transverse normal stress are taken into account.

In the dynamic finite element program, that is developed for the analysis of our problem, the damage progression is taken into account by reducing, at each step of time integration, the values of appropriate material constants of those plies within a finite element in which failure occurs. After that, the element and global stiffness matrices are recomputed, and the finite element analysis is restarted at the same time step, i.e. stresses are calculated at the same moment of time with a new stiffness matrix. If no failure occurs, analysis proceeds to the next time step. Otherwise, the appropriate material constants are reduced again. The degraded material characteristics of a failed ply within a finite element are assumed to be small fractions of the original material characteristics of the undamaged material, but not equal to zero, in order to avoid ill-conditioning of the finite element equations (large differences of relative magnitudes of terms in the stiffness matrix, that results in large computational errors). The average stress in each element and the Hashin failure criterion for the face sheets together with the Tsai-Wu criterion for the core are used for failure prediction. In this work, all integrations required in the calculation of the element stiffness matrices are performed in closed form, using programs for symbolic computation (MAPLE and "Scientific Workplace"). No numerical quadratures were used. This feature leads to savings in computations, that is important in this work, where the finite element method is used for nonlinear dynamic analysis, that requires updating of the stiffness matrix in each step of time integration.

To study the impact-generated damage, it is important to get accurate information not only for the in-plane stresses, but also for the transverse stresses, which are not negligible in thick sandwich

panels. The transverse stresses play a significant role in the various modes of failure. Therefore, we have a three-dimensional problem. Theoretically, one can model fiber composite structures with three-dimensional finite elements, representing a thickness of each ply with a thickness of at least one element. But practically this leads either to the elements with large aspect ratios, resulting in ill-conditioning of the finite element equations, or to an excessively large number of degrees of freedom in the model, if the large element aspect ratios are avoided by making in-plane dimensions of the three-dimensional elements not much larger than their thickness. Therefore, the composite structures are usually modeled by putting several plies into the thickness of one element. This can be achieved by dividing the laminate into a number of sublaminates, each of which contains several plies, and by introducing some simplifying assumptions regarding the through-the-thickness distribution of displacements, strains or stresses within each sublaminate. This leads to the layer-wise (or discrete-layer) plate theories, in which each sublaminate is analyzed as a single layer with the averaged through-the-thickness material properties. In the post-processing procedure the stresses are computed in each ply, using each ply's material properties (not the averaged though-the-thickness material properties). The layerwise theories of the laminated plates, beams or shells, based on different assumptions, were developed, for example, by Whitney (1969), Mau (1973), Chou and Carleone (1973), Swift, G. W. and Heller, R. A. (1974), Durocher and Solecki (1975), Seide (1980), Di Sciuva (1984, 1986, 1987), Mukarami (1986), Ren (1986), Hinrichsen and Palazotto (1986), Chaudhuri and Seide (1987), Reddy (1987). The layerwise theories can represent the zigzag variation of the in-plane displacements in the thickness direction. This zigzag variation is more pronounced for thick laminates, where the transverse shear moduli change abruptly through the thickness, and it can be seen in the exact three-dimensional elasticity solutions, obtained by Pagano (1969, 1970), Pagano and Hatfield (1972), Srinivas, Joga Rao and Rao (1970), Srinivas and Rao (1970), Noor (1973), Pikul (1977), Savoia and Reddy (1992). The sandwich plate, that is considered in this dissertation, has the characteristics that make the discrete-layer approach necessary, namely high thickness-to-length ratio, and large difference in values of elastic moduli of the face sheets and the core. In this layerwise model of the sandwich plate there are three sublaminates: the face sheets and the core.

According to the existing exact three-dimensional elasticity solutions for composite laminates, mentioned earlier, and the exact elasticity solutions for homogeneous isotropic beams and plates (Saada (1993), Vlasov (1957)) the strains, stresses and in-plane displacements in the plates vary nonlinearly in the thickness direction of the plate. In two-dimensional plate or shell theories these

nonlinear variations can be captured by maintaining the higher-order terms in the expansions of displacements in the thickness coordinate. Such theories were proposed by Sun and Whitney (1973), Lo et. al. (1977, 1978), Reddy (1984), Reddy and Liu (1985), Murthy and Vellaichamy (1987), Hinrichsen and Palazotto (1986, 1988), Tessler (1991), Greer and Palazotto (1996) and others. In all these references, except that of Greer and Palazotto (1996), the transverse displacement is assumed to be constant in the thickness direction, or, in other words, the direct transverse strain ε_{zz} is assumed to be equal to zero. In our model of the sandwich plate, the in-plane displacements vary quadratically in the thickness direction, and the transverse displacement varies linearly in the thickness direction within the thickness of a sublaminate. This is achieved by assuming that the transverse strains ε_{zz}, ε_{xz} and ε_{yz} are constant in the thickness direction, and by integrating the strain-displacement relations in order to obtain displacements in terms of the unknown functions and the transverse coordinate (the unknown functions depend on the in-plane coordinates x and y). In this procedure of integrating the strain-displacement relations the constants of integration are chosen such that conditions of continuity of the displacements at the interfaces between the sublaminates are satisfied.

In the plate theories, the transverse stresses, obtained from constitutive equations, turn out discontinuous at the interfaces between the plies of a sublaminate with different material properties (or between the plies of the whole laminate in single-layer theories), due to assumed continuity of strains at the interfaces between these plies. This is a violation of the third Newton's law. Therefore, the accuracy of the transverse stresses, computed from the constitutive equations, is not sufficient to use them in failure criteria.

That is why, many authors e.g. Lo et al. (1978), Lajczok (1986), Chaudhuri (1986), Chaudhuri and Seide (1987), Reddy (1984), Barbero and Reddy (1989), Barbero et al.(1990), Byun and Kapania (1991), obtained only the in-plane stresses from the constitutive relations, and expressed the transverse stresses in terms of the in-plane stresses by integrating three-dimensional equilibrium equations (or equations of motion in dynamic problems). In this case, the continuity of the transverse stresses can be enforced by defining the constants of integration from these conditions of continuity.

Many researchers studied the sandwich plates with thick, vertically incompressible cores and thin incompressible face sheets, using layerwise models. Most of the layerwise models of such structures are based on the piecewise linear through the thickness approximations of in-plane displacement, in

addition to constant (though the thickness) transverse displacements (Reissner (1948), Yu (1959), Plantema (1966), Allen (1969), Kanematsu, Hirano *et al* (1969), Monforton and Ibrahim (1975), Mukhopadhyay and Sierakowski (1990), Lee, Xavier *et al* (1993)).

The assumption of linear variation of the in-plane displacements in the thickness direction, i. e. the assumption, that the cross-sections of the core and the face sheets remain plane after deformation, holds only for the cross-sections that are far from supports or locations of concentrated and partially distributed loads. Therefore, the discrete-layer models with higher-order through-the-thickness displacement approximations for each layer (Chan and Foo (1977), Gutierrez and Webber (1980), Kutilowski and Myslecki (1991), Liu and Chen (1991), Herup (1996)) produce more accurate results. In all of the models of the sandwich plates discussed above, the transverse displacement does not vary in the thickness direction, i.e. the plates are assumed to be incompressible in the thickness direction.

The modern cores are usually made of plastic foams and non-metallic honeycombs, like Aramid and Nomex. These cores have properties similar to those used traditionally (for example, metallic honeycombs), but due to their transverse compressibility (i.e. ability of such cores to change height under applied loads) the direct transverse strain ε_{zz} becomes important. Therefore, the models of the sandwich plates with the cores made of plastic foams or non-metallic honeycombs must not exclude the change of height of the core. Frostig, Baruch *et al (1992, 1996)* developed a theory of a sandwich beam with thin face sheets in which account is taken of transverse compressibility of the core, and the longitudinal displacement in the core varies nonlinearly in the thickness direction. In this theory the longitudinal displacement in the face sheets varies linearly in the thickness direction, and the transverse displacement of the face sheets does not vary in the thickness direction (i. e. the transverse direct strain ε_{zz} in the face sheets is assumed to be equal to zero in the expression for the strain energy). The transverse shear strain ε_{xz} in the face sheets is also considered to be negligibly small in the expression for the strain energy, that is used for variational derivation of the differential equations for the unknown functions. The transverse shear stress in the face sheets can be computed by integration of the pointwise equilibrium equation $\sigma_{xx,x} + \sigma_{xz,z} = 0$.

Under certain circumstances, when the face sheets are thick, when the plate is loaded by a concentrated or partially distributed load, or when the plate is on an elastic foundation, taking account of the direct transverse strain ε_{zz} in the face sheets and the transverse shear strain ε_{xz} in the face sheets in the expression for the strain energy allows one to obtain a higher accuracy of the stress computation. Besides, in order to achieve a high accuracy of stress computation in the thick

face sheets, a model for such a plate must assume or lead to the nonlinear through-the-thickness variation of the in-plane displacements not only in the core (as in the works of Frostig, Baruch *et al*), but also in the face sheets.

Construction of a computational scheme that satisfies these requirements can be approached, for example, with the help of the layerwise laminated plate theory of Reddy (1996), which is a generalization of many other displacement-based layerwise theories of laminated plates. In this theory the displacement field in the k-th layer is written as

$$u^{(k)}(x,y,z,t) = \sum_{j=1}^{m} u_j^{(k)}(x,y,t)\,\phi_j^{(k)}(z),$$

$$v^{(k)}(x,y,z,t) = \sum_{j=1}^{m} v_j^{(k)}(x,y,t)\,\phi_j^{(k)}(z),$$

$$w^{(k)}(x,y,z,t) = \sum_{j=1}^{n} w_j^{(k)}(x,y,t)\,\psi_j^{(k)}(z),$$

where $u_j^{(k)}(x,y,t)$, $v_j^{(k)}(x,y,t)$, $w_j^{(k)}(x,y,t)$ are the unknown functions and $\phi_j^{(k)}(z)$ and $\psi_j^{(k)}(z)$ are chosen to be the Lagrange interpolation functions of the thickness coordinate, in order to provide the required continuity of displacements and discontinuity of the transverse strains across the interface between adjacent thickness subdivisions. This theory allows one to achieve a high accuracy of computation of all stress components in the composite laminates, but for this purpose it requires a large number of thickness subdivisions of the laminate. This leads to a large number of the unknown functions and degrees of freedom in a finite element model. In effect, the finite element model, based on this generalized layerwise laminated plate theory is equivalent to the three-dimensional finite element model. In order to reduce a number of the unknown functions in the layerwise model of a laminated plate, one can use a concept of a sublaminate, i.e. make the number of thickness subdivisions less than the number of material layers, and deal with the material properties, averaged through the thickness of a sublaminate. In a model of the sandwich plate it is natural to choose three sublaminates: the two face sheets and the core. With such a small number of the sublaminates, the nature of assumptions on the through-the-thickness variation of displacements can have a large effect on the accuracy of the computed stresses. Therefore, in a layerwise model of the sandwich plate with only three sublaminates, it is desirable to have a flexibility in the choice of the functions that represent through-the-thickness variation of displacements.

Pikul (1995) suggested an approach to construction of a layered shell theory, based on representation of the transverse components of the strain tensor in the k-th layer of the shell in the following

approximate form

$$\varepsilon_{xz}^{(k)}(x,y,z,t) = f_1^{(k)}(z)\phi_1^{(k)}(x,y),$$

$$\varepsilon_{yz}^{(k)}(x,y,z,t) = f_2^{(k)}(z)\phi_2^{(k)}(x,y),$$

$$\varepsilon_{zz}^{(k)}(x,y,z,t) = f_3^{(k)}(z)\phi_3^{(k)}(x,y),$$

where $\phi_i^{(k)}(x,y)$ are the unknown functions of the tangential coordinates and $f_1^{(k)}(z)$, $f_2^{(k)}(z)$, $f_3^{(k)}(z)$ are some known functions that represent variation of the transverse strains in the thickness direction. The differential equations for the unknown functions were derived from the boundary conditions on one of the external surfaces and from the conditions of minimization of the discrepancy between the assumed transverse strains and the transverse strains obtained from the strain-stress relations with transverse stresses being expressed in terms of the unknown functions with the use of the equilibrium equations. A finite element formulation based on this approach was not performed by this author.

In the dissertation, in order to construct a layerwise sandwich plate theory that takes account of the transverse strains in both the face sheets and the core but has fewer unknown functions (and therefore fewer degrees of freedom in the finite model) than the Reddy's layerwise theory, a computational scheme is constructed in which the simplifying assumptions, that lead to a plate-type theory, are made for the transverse strains, similarly to the Pikul's theory, but, unlike the Pikul's theory, these assumptions are introduced into the virtual work principle in order to construct a finite element formulation. The assumptions are made with respect to the variation of the transverse strains in the thickness direction of the faces and the core of the sandwich plate. The displacements are then obtained by integration of these assumed transverse strains, and the constants of integration are chosen to satisfy the conditions of continuity of the displacements across the borders between the face sheets and the core. In such a method, the required continuity of the displacements in the thickness direction is satisfied regardless of the assumed type of through-the-thickness distribution of the transverse strains. This leads to a larger number of choices of simplifying assumptions about the variation of strains (and, therefore, displacements) in the thickness direction, and, therefore allows a better adjustment of the computational scheme to the conditions under which the sandwich plate is analyzed by a layerwise method with only three sublaminates (being the face sheets and the core). This allows one to achieve any desired degree of nonlinearity of the through-the -thickness variation of the displacements without an increase of the number of the unknown functions, and, therefore, without an increase of the number of the degrees of freedom in finite element models. The transverse stresses are computed by integration of the pointwise equilibrium equations, that leads

to satisfaction of conditions of continuity of the transverse stresses across the boundaries between the face sheets and the core and satisfaction of stress boundary conditions on the upper and lower surfaces of the plate.

In the present work, a model was developed based on the simplest of such assumptions that do not ignore in the expression for the strain energy the transverse shear and normal strains in both the face sheets and the core. It is assumed that the transverse strains do not vary in the thickness direction within the face sheets and the core, but can be different functions of the in-plane coordinates in the face sheets and the core. In the post-process stage, these first approximations of the transverse strains can be improved by substituting the transverse stresses, obtained by integration of the pointwise equilibrium equations into the strain-stress relations. The improved values of the transverse strains depend on the z-coordinate (z-axis is in the thickness direction). In this model, the transverse displacement, obtained by integration of the assumed transverse normal strain, varies linearly in the thickness direction within a sublaminate, and the in-plane displacements, obtained by integration of the assumed transverse shear strains, vary quadratically within the thickness of a sublaminate.

The theory of the sandwich plate, presented in the dissertation, does not require many degrees of freedom in the finite element formulation and has a wide range of applicability. It can be used for analysis of both thick and thin sandwich plates, with thick and thin face sheets, with transversely rigid and transversely flexible faces and cores. Besides, in the finite element analysis of the thin sandwich plates, the shear locking phenomenon does not occur.

In our model we use the Green-Lagrange strain tensor and the energy-conjugate to it second Piola-Kirchhoff stress tensor. Due to relatively high velocities of the platform when it hits the ground, we need to provide the capability of the model to represent moderately large displacements and rotations (displacements of the order of thickness of the platform, and rotations of the order of 10-15 degrees). For the problems with such characteristics, in the strain-displacement relations for the Green's strain tensor, the non-linear terms $\left(\frac{\partial w}{\partial x}\right)^2, \left(\frac{\partial w}{\partial y}\right)^2, \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)$ are not negligible as compared to $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}$, and all other non-linear terms in the strain-displacement relations are negligible (von Karman (1910), Palazotto and Dennis (1992), Reddy (1996)). So, the strain-displacement relations, used in our model, are (von Karman strains):

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \quad \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right),$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad \varepsilon_{yy} = \frac{\partial u}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{1.1}$$

In our study the expressions for the transverse stresses in terms of the unknown functions, obtained by integration of the pointwise equations of motion[1]

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + F_x = \rho \ddot{u}, \qquad \sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + F_y = \rho \ddot{v},$$

$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \frac{\partial}{\partial x}\left(\sigma_{xx} w_{,x} + \sigma_{yx} w_{,y}\right) + \frac{\partial}{\partial y}\left(\sigma_{xy}\ w_{,x} + \sigma_{yy}\ w_{,y}\right) + F_z = \rho \ddot{w}, \tag{1.2}$$

contain derivatives of the unknown functions of the order higher than the degree of the interpolation polynomials, used in the finite element formulation. These higher order derivatives can not be computed as derivatives of the piecewise[2] interpolation polynomials, used in the finite element formulation, because such method would lead to vanishing of these higher order derivatives, that can be wrong for a particular problem. Therefore, some numerical procedure is necessary to construct these higher-order derivatives from the nodal values of the unknown functions, obtained as a result of the finite element solution. Byun and Kapania (1991) used a least -squares global [3] polynomial approximation of the nodal values of displacements[4], and calculated various higher order derivatives of the displacements as derivatives of these global approximation polynomials. The two types of polynomials were used in the global displacement approximation: Chebishev polynomials and a class of orthogonal polynomials, defined by the following recurrence formula:

$$P_0(x) = 1, \quad P_{-1}(x) = 0, \quad P_{r+1}(x) = (x - \alpha_{r+1}) P_r(x) - \beta_r P_{r-1}(x) \qquad (r = 0, 1, 2, ...), \tag{1.3}$$

where

$$\alpha_{r+1} = \frac{\sum_{i=1}^{m} a_i x_i \left[P_r(x_i)\right]^2}{\sum_{i=1}^{m} a_i \left[P_r(x_i)\right]^2}, \qquad \beta_r = \frac{\sum_{i=1}^{m} a_i \left[P_r(x_i)\right]^2}{\sum_{i=1}^{m} a_i \left[P_{r-1}(x_i)\right]^2} \tag{1.4}$$

[1] these are equations of motion, variationally consistent with the Von-Karman strain-dispacement equations.

[2] defined over a domain of a finite element

[3] defined over the whole area of the plate

[4] these nodal values of the unknown functions were taken directly from the finite element solution

and a_i are values of weighting functions at data points x_i. The higher order derivatives, computed by both methods were in good agreement with the values of derivatives obtained from exact solutions.

In this study, for computation of the higher order derivatives a finite-difference scheme was applied, using the nodal data from the finite element solution. The numerical experiments showed that, despite the simplicity of such a method, it can produce quite accurate values of the derivatives, if the finite element mesh is sufficiently fine.

In the dissertation, we will consider a dynamic response of the cargo platform dropped on the ground modelled as elastic foundation. The Winkler (1867) model of the elastic foundation is the simplest model for expressing relationship of pressure and deflection of the foundation surface. This relationship can be expressed as

$$p(x,y) = -kw(x,y), \tag{1.5}$$

where k is a modulus of surface reaction with units of force per cubic length, and $w(x,y)$ is a ground surface displacement. The characteristic feature of this soil mechanics relationship is that it leads to discontinuity of the surface displacement. It is obvious that a correction had to be found since the surface displacement is present beyond the loaded region. Pasternak (1954) developed a relationship in which some interaction between the spring elements occurred. The proposed response equation was:

$$p(x,y) = -kw(x,y) - G\Delta^2 w(x,y), \tag{1.6}$$

where k and G are two foundation parameters, and Δ^2 is the Laplace operator. Unfortunately, this relationship produces concentrated reactions along the free edges of the structure. Kerr (1964) proposed a correction to the Pasternak model by adding a spring layer on top of a shearing layer, that is considered more appropriate for elastic foundation analyses, but the expression is much more complicated, resulting in a sixth order partial differential equation (Kneifati, 1985). It was decided that as an initial attempt at representing the overall problem, we would only consider the simpler Winkler foundation representation. It is possible to consider the more accurate expressions.

The subsequent chapters contain some preliminary considerations regarding construction of plate theories, based on assumed transverse strains, development of the two-dimensional geometrically nonlinear computational model of the composite sandwich plate with account of transverse stresses, transverse flexibility and damage progression, development of a simplified model of the sandwich composite plate and the corresponding finite element formulation, description of the finite element

program based on this formulation and discussion of results for an example problem, obtained with help of this finite element program.

Chapter 2

Preliminary Considerations Regarding Construction of a Plate Theory, Based on Assumed Transverse Strains

In thick sandwich plates the transverse shear strains ε_{xz}, ε_{yz} and transverse direct strain ε_{zz} can be **not** negligibly small as compared to the in-plane strains ε_{xx}, ε_{yy}, ε_{xy} (it is implied that the z-axis is in the direction of the plate's thickness, and x- and y- coordinates are in the plane of the plate's middle surface, as shown, for example in Figure 2.1). This is especially true if the plate is on an elastic foundation or is loaded by a partially distributed load, as it is in the case of a cargo platform, dropped on the ground. The corresponding transverse stresses σ_{xz}, σ_{yz} and σ_{zz} can also be not negligibly small under the same conditions. Therefore, in analysis of such plates it can be important that in the expressions for the strain energy density the terms $\sigma_{xz}\varepsilon_{xz}$, $\sigma_{yz}\varepsilon_{yz}$ and $\sigma_{zz}\varepsilon_{zz}$ are taken into account.

The two-dimensional computational models of plates are usually deduced from the three-dimensional formulations by making some assumptions about through-the-thickness distribution of either displacements or strains or stresses in the plates. We construct a plate theory of the sandwich plate

by making assumptions on distribution of the transverse strains ε_{zz}, ε_{xz} and ε_{yz} in the thickness of the face sheets and the core, i.e. by assuming that these transverse strains are some known functions of the z-coordinate within the face sheets and the core. Such method of constructing a two-dimensional plate theory provides a convenient way to make displacements continuous across the boundaries between the face sheets and the core: the expressions for the displacements are obtained by integration of the strain-displacement relations and the constants of integration are chosen such that the conditions of continuity of displacements are satisfied.

But before proceeding to the actual problem of the dissertation, we will study and compare, in a simpler problem of cylindrical bending of a sandwich plate with homogeneous isotropic face sheets and the core, the accuracy and computational efficiency of theories based on two different kinds of assumptions on transverse strains:

1) the transverse strains are non-zero in both the face sheets and the core, do not depend on z-coordinate within the face sheets and the core, but each of these strains is a different function of the in-plane coordinate within each sublaminate (a face sheet or a core);

2) the assumed transverse strains in the face sheets, that enter into the expression for the strain energy density, are equal to zero, and the assumed transverse strains in the core do not vary in z-direction.

As it was mentioned in the first chapter, the transverse stresses will be computed by integration of the pointwise equilibrium equations for each sublaminate. In this integration the number of constants of integration is equal to the number of interfaces between the sublaminates plus one. Therefore, these constants of integration can be chosen to satisfy the conditions of continuity of the transverse stresses at the interfaces between the sublaminates and the boundary conditions on one of the external surfaces (upper or lower). In this chapter it will be shown that if the governing differential equations for the unknown functions have an exact solution, then the transverse stresses, obtained by integration of the pointwise equilibrium equations, satisfy exactly the boundary conditions on **both** the upper and lower external surfaces. Proving this fact requires less voluminous derivations if a homogeneous plate is considered, rather than the sandwich plate. Therefore, this chapter is started by considering a model of a homogeneous isotropic plate in cylindrical bending, based on assumptions, similar to those that will be applied to the sandwich plates: the transverse strains will be assumed to be non-zero and not dependent on the z-coordinate (not varying in the thickness direction). In this chapter it will be shown also that if the unknown functions of the model of the sandwich plate are computed by the finite element method (which is equivalent to approximate

solving the differential equations for the unknown functions), then the boundary conditions on one of the external surfaces (upper or lower) are satisfied approximately by the transverse stresses obtained from the pointwise equilibrium equations, in addition to exact satisfaction of the boundary conditions on the other external surface and conditions of continuity of the transverse stresses between the sublaminates.

2.1 Cylindrical Bending of a Homogeneous Isotropic Plate

In this section we will consider construction of a theory of cylindrical bending of a homogeneous isotropic plate, based on assumption that the transverse strains are not negligible in the expression for the strain energy, and on the assumption that these strains do not vary in the thickness direction. The purpose of this paragraph is to evaluate the accuracy of stresses, obtained from a computational model based on such assumptions, and to determine if the boundary conditions on both the upper and lower surfaces are satisfied exactly by the transverse stresses obtained by integration of the pointwise equilibrium equations. It will be shown also that in this theory the stress boundary conditions on the lateral surfaces are satisfied in the integral sense, i.e. conditions of static equilibrium are satisfied. Using this theory, a problem of a simply supported plate under a uniform loading on the upper surface will be solved and the solution will be compared with the exact elasticity solution. This comparison will enable an assessment of the accuracy of the theory, based on the above mentioned assumptions on the transverse strains.

Cylindrical bending implies the condition of plane strain, i.e.

$$v = 0, \ \frac{\partial u}{\partial y} = 0, \ \frac{\partial w}{\partial y} = 0, \tag{2.1.1}$$

which can occur if the plate's dimension in the y-direction (that will be called width b) is much larger than its dimension in the x-direction (that will be called length L) and the loadings on the upper and lower surfaces of the plate do not vary in the y-direction (figure 2.1). The problem is considered on the basis of **linear elasticity**, i.e. in the general form it is described by the following equations:

equilibrium equations

$$\sigma_{ij,j} = 0; \tag{2.1.2}$$

strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{i,j}\right); \tag{2.1.3}$$

constitutive equations

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{\alpha\alpha}\delta_{ij}\right); \tag{2.1.4}$$

boundary conditions

$$\sigma_{xz} = 0,\ \sigma_{zz} = -\frac{q_l}{b} \text{ at } z = -\frac{h}{2}, \tag{2.1.5}$$

$$\sigma_{xz} = 0,\ \sigma_{zz} = \frac{q_u}{b} \text{ at } z = \frac{h}{2}, \tag{2.1.6}$$

where q_l and q_u are projections on the z-axis of forces per unit length $\left(\frac{df_z}{dx}\right)$ at the lower and upper surfaces correspondingly (by q_l and q_u we denote not absolute values of forces per unit length, but their projections on the z-axis, therefore values q_l and q_u can be positive or negative, depending on direction of the forces);

conditions of static equilibrium [1]:

$$\int_{-h/2}^{h/2} \sigma_{xx}\, dz = 0 \text{ at } x = 0, L\ , \tag{2.1.7}$$

$$\int_{-h/2}^{h/2} \sigma_{xx} z\, dz = 0 \text{ at } x = 0, L\ , \tag{2.1.8}$$

[1] None of the plate theories are capable of providing exact satisfaction of stress boundary conditions at the contour of a plate, i.e. satisfaction of the stress boundary conditions at the contour of a plate at each value of z-coordinate (coordinate in the thickness direction). Therefore, we require satisfaction of the stress boundary conditions only in the integral sense, i.e. conditions of static equilibrium. Our "exact" elasticity solution, the purpose of which is to evaluate the accuracy of the stresses, produced by our plate theory, will also satisfy the stress boundary conditions only in the integral sense, unlike that of Pagano (1969), which satisfies the stress boundary conditions exactly. We chose to require that our "exact" elasticity solution satisfies only the mitigated, integral stress boundary conditions, because such requirement allows one to obtain analytical expressions for stresses. The truly exact solution of Pagano, which satisfies the stress boundary conditions at each point of the plate contour, contains coefficients which can be obtained only numerically, and it is too rigorous for our purposes, because its comparison with the solution, based on the plate theory, will only reveal the fact that the plate theory can not take account of edge effects, that is known in advance.

$$b\int_{-h/2}^{h/2}\left[\sigma_{xz}\left(L\right)-\sigma_{xz}\left(0\right)\right]\,dz=-\int_{0}^{L}\left(q_l+q_u\right)\,dx\;. \tag{2.1.9}$$

In view of plane strain assumptions (2.1.1), the strain-displacement relations (2.1.3), take the form:

$$\varepsilon_{xx}=u_{,x}, \tag{2.1.10}$$

$$\varepsilon_{xz}=\frac{1}{2}\left(u_{,z}+w_{,x}\right), \tag{2.1.11}$$

$$\varepsilon_{zz}=w_{,z}, \tag{2.1.12}$$

$$\varepsilon_{xy}=0,\ \varepsilon_{yy}=0,\ \varepsilon_{yz}=0, \tag{2.1.13}$$

and the constitutive equations (2.1.4) become

$$\sigma_{xx}=\frac{E}{\left(1+\nu\right)\left(1-2\nu\right)}\left[\left(1-\nu\right)\varepsilon_{xx}+\nu\varepsilon_{zz}\right], \tag{2.1.14}$$

$$\sigma_{yy}=\frac{E}{\left(1+\nu\right)}\frac{\nu}{\left(1-2\nu\right)}\left(\varepsilon_{xx}+\varepsilon_{zz}\right), \tag{2.1.15}$$

$$\sigma_{zz}=\frac{E}{(1+\nu)(1-2\nu)}[\nu\varepsilon_{xx}+(1-\nu)\varepsilon_{zz}], \tag{2.1.16}$$

$$\sigma_{xz}=\frac{E}{1+\nu}\varepsilon_{xz}, \tag{2.1.17}$$

$$\sigma_{xy}=0,\ \sigma_{yz}=0. \tag{2.1.18}$$

The equilibrium equations for the plane strain condition have the form

$$\sigma_{xx,x}+\sigma_{xz,z}=0, \tag{2.1.19}$$

$$\sigma_{zx,x}+\sigma_{zz,z}=0. \tag{2.1.20}$$

In order to construct a plate theory, additional simplifying assumptions will be made regarding dependence of the transverse strains, ε_{xz} and ε_{zz}, on the z-coordinate. The purpose of this chapter is

to study the accuracy of a plate theory, based on the assumptions that ε_{xz} and ε_{zz} are independent of the z-coordinate:

$$\varepsilon_{xz} = \varepsilon_{xz}(x), \tag{2.1.21}$$

$$\varepsilon_{zz} = \varepsilon_{zz}(x). \tag{2.1.22}$$

Integration of equation (2.1.12) yields

$$w(x,z) - \underbrace{w\big|_{z=0}}_{w_0(x)} = \int_0^z \frac{\partial w}{\partial z}\, dx = \int_0^z \varepsilon_{zz}(x)\, dz = \varepsilon_{zz}(x)\ z\ ,$$

where

$$w_0(x) \equiv w\big|_{z=0}\,. \tag{2.1.23}$$

Therefore,

$$w(x,z) = w_0(x) + \varepsilon_{zz}(x)\ z. \tag{2.1.24}$$

From equation (2.1.11), we receive

$$\frac{\partial u}{\partial z} = 2\varepsilon_{xz}(x) - \frac{dw(x)}{dx}.$$

Integration of the last equation yields

$$u(x,z) - \underbrace{u\Big|_{z=0}}_{u_0(x)} = \int_0^z \frac{\partial u}{\partial z}\, dz = \int_0^z (2\varepsilon_{xz} - w_{,x})\, dz =$$

$$= \int_0^z (2\varepsilon_{xz} - w_{0,x} - \varepsilon_{zz,x} z)\, dz = (2\varepsilon_{xz} - w_{0,x})z - \varepsilon_{zz,x}\frac{z^2}{2}\ ,$$

where

$$u\big|_{z=0} \equiv u_0(x). \tag{2.1.25}$$

Therefore,

$$u(x,z) = u_0(x) + [2\varepsilon_{xz}(x) - w_{0,x}(x)]z - \frac{1}{2}\varepsilon_{zz,x}(x)z^2. \tag{2.1.26}$$

So, we have **four unknown functions** in this problem:

$$u_0(x),\ w_0(x),\ \varepsilon_{xz}(x),\ \varepsilon_{zz}(x).$$

Let us express strains and stresses in terms of the unknown functions. Strain ε_{xx} can be found by substituting expression (2.1.26) into the strain-displacement relation (2.1.10):

$$\varepsilon_{xx} = u_0' + (2\varepsilon_{xz}' - w_0'')\, z - \frac{1}{2}\varepsilon_{zz}'' z^2 \ . \tag{2.1.27}$$

Here and further primes denote derivatives with respect to x-coordinates. Substitution of expression (2.1.27) into constitutive relation (2.1.14) yields:

$$\sigma_{xx}^H = \frac{E}{(1+\nu)(1-2\nu)} \left\{ (1-\nu) \left[u_0' + (2\varepsilon_{xz}' - w_0'')\, z - \frac{1}{2}\varepsilon_{zz}'' z^2 \right] + \nu\varepsilon_{zz} \right\}. \tag{2.1.28}$$

Here the superscript H means that the stress was obtained from the Hooke's law, as opposed to stresses σ_{xz} and σ_{zz}, which will be obtained from the equilibrium equations. To find expressions for the transverse stresses in terms of the unknown functions, we integrate the equilibrium equations (2.1.19) and (2.1.20), in which for stress σ_{xx} we take its expression (2.1.28), obtained from the Hooke's law:

$$\sigma_{xx,x}^H + \sigma_{xz,z} = 0 \quad , \quad \sigma_{zx,x} + \sigma_{zz,z} = 0.$$

From the first equilibrium equation, we obtain

$$\sigma_{xz} - \underbrace{\sigma_{xz}\big|_{z=-h/2}}_{0} = \int_{-\frac{h}{2}}^{z} \sigma_{xz,z}\, dz = -\int_{-\frac{h}{2}}^{z} \sigma_{xx,x}^H\, dz \ ,$$

where $\sigma_{xz}\big|_{z=-h/2} = 0$ due to the boundary condition (2.1.6). The substitution of expression (2.1.28) for σ_{xx}^H into the last equation yields

$$\sigma_{xz} = -\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left[u_0'' \left(z + \frac{h}{2} \right) + \frac{1}{2} \left(2\varepsilon_{xz}'' - w_0''' \right) \left(z^2 - \frac{h^2}{4} \right) - \frac{1}{6}\varepsilon_{zz}''' \left(z^3 + \frac{h^3}{8} \right) \right] -$$
$$\frac{E\nu}{(1+\nu)(1-2\nu)} \varepsilon_{zz}' \left(z + \frac{h}{2} \right). \tag{2.1.29}$$

From the second equilibrium equation we obtain

$$\sigma_{zz} - \underbrace{\sigma_{zz}\big|_{z=-h/2}}_{\substack{-\frac{q_l}{b} \\ \text{due to BC (2.1.5)}}} = \int_{-h/2}^{z} \sigma_{zz,z}\, dz = -\int_{-h/2}^{z} \sigma_{xz,x}\, dz.$$

Substitution of expression (2.1.29) for σ_{xz} into the last equation gives

$$\sigma_{zz} = -\frac{q_l}{b} + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\frac{1}{8}(2z+h)^2 u_0''' + \frac{1}{24}(z-h)(2z+h)^2\left(2\varepsilon_{xz}''' - w_0^{IV}\right)\right.$$
$$\left.-\frac{1}{384}\left(4z^2-4hz+3h^2\right)(2z+h)^2\varepsilon_{zz}^{IV}\right] + \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{1}{8}(2z+h)^2\varepsilon_{zz}'', \tag{2.1.30}$$

where the superscript IV means the 4-th derivative with respect to x-coordinate.

Expressions for the transverse stresses in terms of the unknown functions can also be obtained from the Hooke's law. Upon substitution of expression (2.1.27) for ε_{xx} into the constitutive equation (2.1.16), we receive

$$\sigma_{zz}^H = \frac{E}{(1+\nu)(1-2\nu)}\left\{\nu\left[u_0' + \left(2\varepsilon_{xz}' - w_0''\right)z - \frac{1}{2}\varepsilon_{zz}''z^2\right] + (1-\nu)\varepsilon_{zz}\right\}. \tag{2.1.31}$$

We will also write the constitutive equation (2.1.17) in the form

$$\sigma_{xz}^H = \frac{E}{1+\nu}\varepsilon_{xz}. \tag{2.1.32}$$

We see that expressions (2.1.29) and (2.1.30) for the transverse stresses σ_{xz} and σ_{zz} in terms of the unknown functions, obtained from the equilibrium equations, are different from the corresponding expressions (2.1.32) and (2.1.31) for σ_{xz}^H and σ_{zz}^H, obtained from the constitutive equations. It was already shown that the transverse stresses σ_{xz} and σ_{zz}, obtained from the equilibrium equations, satisfy the boundary conditions (2.1.5) at the lower surface of the plate, and it will be shown later that they satisfy also the boundary conditions (2.1.6) at the upper surface of the plate. Besides, as it will be shown later, in composite plate theory the transverse stresses obtained from the equilibrium equations (or equations of motion in dynamic problems) can be forced to satisfy also the conditions of continuity of the transverse stresses at the interfaces between the plies with different material properties. On the other hand, the transverse stresses obtained from the constitutive equations, do not satisfy the boundary conditions either at the upper surface or at the lower one, and do not satisfy the conditions of continuity of the transverse stresses at the interfaces between the plies in composite plates. Therefore, the transverse stresses obtained by integrating the equilibrium equations are more accurate.

Now, let us derive differential equations for the unknown functions $u_0(x)$, $w_0(x)$, $\varepsilon_{xz}(x)$, $\varepsilon_{zz}(x)$ and boundary conditions, using the principle of virtual work. The virtual work principle is a convenient way of reducing the three-dimensional continuum mechanics problems to the two-dimensional and one dimensional problems for the following reasons:

1) It allows to formalize the process of derivation of the governing differential equations in terms of the unknown functions and natural boundary conditions, i.e. boundary conditions on the part of the surface, where the displacements are not imposed.
2) The number of boundary conditions, formally derived from the virtual work principle, is equal to the order of the governing differential equations for the unknown functions. This can be not the case if the differential equations for the unknown functions of the plate model are derived by averaging (through the plate's thickness) the pointwise equilibrium equations, due to contradictions between the equations of elasticity, brought about by introducing the simplifying assumptions. An example of such case is the boundary conditions at a free end of a plate in the classical plate theory based on the Kirchhoff – Love assumptions (Saada, 1993). The use of a variational method allowed Kirchhoff to obtain the free-end boundary conditions, consistent with the governing differential equations.
4) The level of accuracy of all equations of a plate theory, derived from the virtual work principle, is the same and is consistent with the simplifying assumptions that lead to the plate theory.
5) The finite element formulation is most easily performed on the basis of the variational formulation.

The virtual work principle is

$$\delta U - \delta' W = 0, \tag{2.1.33}$$

where U is strain energy of the plate and $\delta' W$ is virtual work of external forces, acting on the plate. In the notation $\delta' W$ the prime is used over the δ because in case of nonconservative external loads, the virtual work of external loads is not a variation of some state function W. Here we follow the notation of Washizu (1982).

The expression for the strain energy has the form

$$U = \frac{1}{2}\iiint\limits_{(V)} \left(\sigma_{xx}^H \varepsilon_{xx} + 2\sigma_{xz}^H \varepsilon_{xz} + \sigma_{zz}^H \varepsilon_{zz} + \sigma_{yy}^H \underbrace{\varepsilon_{yy}}_{0} + 2\underbrace{\sigma_{xy}^H}_{0}\underbrace{\varepsilon_{xy}}_{0} + 2\underbrace{\sigma_{yz}^H}_{0}\underbrace{\varepsilon_{yz}}_{0} \right) dV = \tag{2.1.34}$$

$$\frac{b}{2}\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \varepsilon_{zz} \end{Bmatrix}^T \begin{Bmatrix} \sigma_{xx}^H \\ \sigma_{xz}^H \\ \sigma_{zz}^H \end{Bmatrix} dz\, dx.$$

The constitutive equations (2.1.14), (2.1.16) and (2.1.17) can be written in the form

$$\begin{Bmatrix} \sigma_{xx}^H \\ \sigma_{xz}^H \\ \sigma_{zz}^H \end{Bmatrix} = \frac{E}{1+\nu} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & 0 & \frac{\nu}{1-2\nu} \\ 0 & \frac{1}{2} & 0 \\ \frac{\nu}{1-2\nu} & 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \varepsilon_{zz} \end{Bmatrix} dz\ dx. \tag{2.1.35}$$

Substitution of (35) into (34) yields:

$$U = \frac{bE}{2(1+\nu)} \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \varepsilon_{zz} \end{Bmatrix}^T \begin{bmatrix} \frac{1-\nu}{1-2\nu} & 0 & \frac{\nu}{1-2\nu} \\ 0 & \frac{1}{2} & 0 \\ \frac{\nu}{1-2\nu} & 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \varepsilon_{zz} \end{Bmatrix} dz\ dx. \tag{2.1.36}$$

In view of relation (2.1.27), we can write

$$\begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xz} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{bmatrix} 1 & z & -\frac{1}{2}z^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_0' \\ 2\varepsilon_{xz}' - w_0'' \\ \varepsilon_{zz}'' \\ 2\varepsilon_{xz} \\ \varepsilon_{zz} \end{Bmatrix}. \tag{2.1.37}$$

If equation (2.1.37) is substituted into (2.1.36), we receive

$$U = \frac{bE}{2(1+\nu)} \int_0^L \begin{Bmatrix} u_0' \\ 2\varepsilon_{xz}' - w_0'' \\ \varepsilon_{zz}'' \\ 2\varepsilon_{xz} \\ \varepsilon_{zz} \end{Bmatrix}^T [C] \begin{Bmatrix} u_0' \\ 2\varepsilon_{xz}' - w_0'' \\ \varepsilon_{zz}'' \\ 2\varepsilon_{xz} \\ \varepsilon_{zz} \end{Bmatrix} dx\ , \tag{2.1.38}$$

where

$$[C] = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} 1 & 0 & 0 \\ z & 0 & 0 \\ -\frac{1}{2}z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & 0 & \frac{\nu}{1-2\nu} \\ 0 & \frac{1}{2} & 0 \\ \frac{\nu}{1-2\nu} & 0 & \frac{1-\nu}{1-2\nu} \end{bmatrix} \begin{bmatrix} 1 & z & -\frac{1}{2}z^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} dz =$$

$$= \frac{h}{2\nu-1}\begin{bmatrix} \nu-1 & 0 & \frac{1}{24}h^2(1-\nu) & 0 & -\nu \\ 0 & \frac{1}{12}h^2(\nu-1) & 0 & 0 & 0 \\ \frac{1}{24}h^2(1-\nu) & 0 & \frac{1}{320}h^4(\nu-1) & 0 & \frac{1}{24}h^2\nu \\ 0 & 0 & 0 & \nu-\frac{1}{2} & 0 \\ -\nu & 0 & \frac{1}{24}h^2\nu & 0 & \nu-1 \end{bmatrix}. \tag{2.1.39}$$

The substitution of (2.1.39) into (2.1.38) yields

$$U = \int_0^L \hat{U}\ dx\ , \tag{2.1.40}$$

where

$$\hat{U}\left(u_0^{'}, \varepsilon_{xz}, \varepsilon_{xz}^{'}, \varepsilon_{zz}, \varepsilon_{zz}^{''}, w_0^{''}\right) = \frac{bEh}{2(1+\nu)(2\nu-1)}\left[(\nu-1)\left(u_0^{'}\right)^2 + \frac{h^2(1+\nu)}{12}u_0^{'}\varepsilon_{zz}^{''} - \right.$$

$$2\nu u_0^{'}\varepsilon_{zz} + \frac{h^2(\nu-1)}{12}\left(2\varepsilon_{xz}^{'} - w_0^{''}\right)^2 + \frac{h^4(\nu-1)}{320}\left(\varepsilon_{zz}^{''}\right)^2 +$$

$$\left. + \frac{h^2\nu}{12}\varepsilon_{zz}^{''}\varepsilon_{zz} + 4\left(\nu-\frac{1}{2}\right)\varepsilon_{xz}^2 + (\nu-1)\varepsilon_{zz}^2\right]. \tag{2.1.41}$$

So,

$$\delta U = \int_0^L \left(\frac{\partial\hat{U}}{\partial u_0^{'}}\delta u_0^{'} + \frac{\partial\hat{U}}{\partial w_0^{''}}\delta w_0^{''} + \frac{\partial\hat{U}}{\partial\varepsilon_{xz}}\delta\varepsilon_{xz} + \frac{\partial\hat{U}}{\partial\varepsilon_{xz}^{'}}\delta\varepsilon_{xz}^{'} + \frac{\partial\hat{U}}{\partial\varepsilon_{zz}}\delta\varepsilon_{zz} + \frac{\partial\hat{U}}{\partial\varepsilon_{zz}^{''}}\delta\varepsilon_{zz}^{''}\right)\ dx\ . \tag{2.1.42}$$

The integration by parts in the last expression yields

$$\delta U = -\int_0^L \frac{d}{dx}\frac{\partial\hat{U}}{\partial u_0^{'}}\ \delta u_0\ dx + \frac{\partial\hat{U}}{\partial u_0^{'}}\ \delta u_0\Big|_0^L + \int_0^L \frac{d^2}{dx^2}\frac{\partial\hat{U}}{\partial w_0^{''}}\ \delta w_0\ dx + \frac{\partial\hat{U}}{\partial w_0^{''}}\ \delta w_0^{'}\Big|_0^L -$$

$$\frac{d}{dx}\frac{\partial\hat{U}}{\partial w_0^{''}}\ \delta w_0\Big|_0^L + \int_0^L \left(\frac{\partial\hat{U}}{\partial\varepsilon_{xz}} - \frac{d}{dx}\frac{\partial\hat{U}}{\partial\varepsilon_{xz}^{'}}\right)\ \delta\varepsilon_{xz}\ dx + \frac{\partial\hat{U}}{\partial\varepsilon_{xz}^{'}}\ \delta\varepsilon_{xz}\Big|_0^L +$$

$$+\int_0^L \left(\frac{\partial\hat{U}}{\partial\varepsilon_{zz}} + \frac{d^2}{dx^2}\frac{\partial\hat{U}}{\partial\varepsilon_{zz}^{''}}\right)\ \delta\varepsilon_{zz}\ dx + \frac{\partial\hat{U}}{\partial\varepsilon_{zz}^{''}}\ \delta\varepsilon_{zz}^{'}\Big|_0^L - \frac{d}{dx}\frac{\partial\hat{U}}{\partial\varepsilon_{zz}^{''}}\ \delta\varepsilon_{zz}\Big|_0^L\ . \tag{2.1.43}$$

The virtual work of external forces has the form

$$\delta' W = \int_0^L q_u \, \delta w\big|_{z=h/2} \, dx + \int_0^L q_l \, \delta w\big|_{z=-h/2} \, dx \; . \tag{2.1.44}$$

According to equation (2.1.24),

$$\delta w = \delta w_0 + z \, \delta\varepsilon_{zz} \; . \tag{2.1.45}$$

Therefore,

$$\delta' W = \int_0^L q_u \left(\delta w_0 + \frac{h}{2} \, \delta\varepsilon_{zz} \right) \, dx + \int_0^L q_l \left(\delta w_0 - \frac{h}{2} \, \delta\varepsilon_{zz} \right) \, dx. \tag{2.1.46}$$

Upon substitution of expression (2.1.43) and (2.1.46) into the principle of virtual work, $\delta U - \delta' W = 0$, and equating to zero the coefficients of the variations of the unknown functions $u_0, w_0, \varepsilon_{xz}, \varepsilon_{zz}$ and the boundary terms, we receive the following differential equations and boundary conditions:

$$\delta u_0 : \qquad \frac{d}{dx} \frac{\partial \hat{U}}{\partial u_0'} = 0 \quad (0 \le x \le L) \; ,$$

$$\delta w_0 : \qquad \frac{d^2}{dx^2} \frac{\partial \hat{U}}{\partial w_0''} = q_u + q_l \quad (0 \le x \le L) \; ,$$

$$\delta \varepsilon_{xz} : \qquad \frac{\partial \hat{U}}{\partial \varepsilon_{xz}} - \frac{d}{dx} \frac{\partial \hat{U}}{\partial \varepsilon_{xz}'} = 0 \quad (0 \le x \le L) \; ,$$

$$\delta \varepsilon_{zz} : \qquad \frac{\partial \hat{U}}{\partial \varepsilon_{zz}} + \frac{d^2}{dx^2} \frac{\partial \hat{U}}{\partial \varepsilon_{zz}''} = \frac{h}{2} (q_u - q_l) \qquad (0 \le x \le L) \; .$$

$$\text{Either } \frac{\partial \hat{U}}{\partial u_0'} = 0 \text{ or } u_0 \text{ is specified at } x = 0, L \; .$$

$$\text{Either } \frac{\partial \hat{U}}{\partial \varepsilon_{xz}'} = 0 \text{ or } \varepsilon_{xz} \text{ is specified at } x = 0, L \; .$$

$$\text{Either } \frac{\partial \hat{U}}{\partial w_0''} = 0 \text{ or } w_0' \text{ is specified at } x = 0, L \; .$$

$$\text{Either } \frac{d}{dx} \frac{\partial \hat{U}}{\partial w_0''} = 0 \text{ or } w_0 \text{ is specified at } x = 0, L \; .$$

$$\text{Either } \frac{\partial \hat{U}}{\partial \varepsilon_{zz}''} = 0 \text{ or } \varepsilon_{zz}' \text{ is specified at } x = 0, L \ .$$

$$\text{Either } \frac{d}{dx}\frac{\partial \hat{U}}{\partial \varepsilon_{zz}''} = 0 \text{ or } \varepsilon_{zz} \text{ is specified at } x = 0, L \ .$$

The substitution of expression (41) for $\hat{U}$ into these equations yields the following differential equations and boundary conditions:

$$\delta u_0 : \quad (1-\nu)\left(u_0'' - \frac{h^2}{24}\varepsilon_{zz}'''\right) + \nu\varepsilon_{zz}' = 0 \quad (0 \le x \le L) \ , \tag{2.1.47}$$

$$\delta w_0 : \quad \frac{1-\nu}{(1+\nu)(1-2\nu)}\frac{bEh^3}{12}\left(w_0^{IV} - 2\varepsilon_{xz}'''\right) = q_u + q_l \quad (0 \le x \le L) \ , \tag{2.1.48}$$

$$\delta \varepsilon_{xz} : \quad \varepsilon_{xz} + \frac{h^2(1-\nu)}{12(1-2\nu)}\left(w_0''' - 2\varepsilon_{xz}''\right) = 0 \quad (0 \le x \le L) \ , \tag{2.1.49}$$

$$\begin{aligned} \delta \varepsilon_{zz} : \quad & \nu\left(u_0' - \frac{h^2}{24}\varepsilon_{zz}''\right) + (1+\nu)\left[\varepsilon_{zz} + \frac{h^2}{8}\left(\frac{h^2}{40}\varepsilon_{zz}^{IV} - \frac{1}{3}u_0'''\right)\right] \\ & = \frac{(1+\nu)(1-2\nu)}{bE}(q_u - q_l) \quad (0 \le x \le L) \ . \end{aligned} \tag{2.1.50}$$

$$\text{Either } (1-\nu)\left(u_0' - \frac{h^2}{24}\varepsilon_{zz}''\right) + \nu\varepsilon_{zz} = 0 \text{ or } u_0 \text{ is specified at } x = 0, L \ . \tag{2.1.51}$$

$$\text{Either } 2\varepsilon_{xz}' - w_0'' = 0 \text{ or } \varepsilon_{xz} \text{ is specified at } x = 0, L \ . \tag{2.1.52}$$

$$\text{Either } 2\varepsilon_{xz}' - w_0'' = 0 \text{ or } w_0' \text{ is specified at } x = 0, L \ . \tag{2.1.53}$$

$$\text{Either } 2\varepsilon_{xz}'' - w_0''' = 0 \text{ or } w_0 \text{ is specified at } x = 0, L \ . \tag{2.1.54}$$

$$\text{Either } (1-\nu)\left(u_0' - \frac{3}{40}h^2\varepsilon_{zz}''\right) + \nu\varepsilon_{zz} = 0 \text{ or } \varepsilon_{zz}' \text{ is specified at } x = 0, L \ . \tag{2.1.55}$$

$$\text{Either } (1-\nu)\left(u_0'' - \frac{3}{40}h^2\varepsilon_{zz}'''\right) + \nu\varepsilon_{zz}' = 0 \text{ or } \varepsilon_{zz} \text{ is specified at } x = 0, L \ . \tag{2.1.56}$$

Equations (2.1.47) and (2.1.48) can be derived, also, by substituting expressions (2.1.29) and (2.1.30) for stresses σ_{xz} and σ_{zz}, obtained from the equilibrium equations, into the boundary conditions (2.1.6) at the upper surface of the beam. Indeed,

$$0 = \sigma_{xz}\big|_{z=h/2} = -\frac{Eh}{(1+\nu)(1-2\nu)}\left[(1-\nu)\left(u_0^{''} - \frac{h^2}{24}\varepsilon_{zz}^{'''}\right) + \nu\varepsilon_{zz}^{'}\right] , \tag{2.1.57}$$

$$\frac{q_u}{b} = \sigma_{zz}\big|_{z=h/2} = -\frac{q_l}{b} + \frac{Eh^2}{2(1+\nu)(1-2\nu)}\underbrace{\left[(1-\nu)\left(u_0^{'''} - \frac{h^2}{24}\varepsilon_{zz}^{IV}\right) + \nu\varepsilon_{zz}^{''}\right]}_{\substack{0\\ \text{because of eqn. (56)}}} +$$

$$\frac{Eh^3}{12}\frac{1-\nu}{(1+\nu)(1-2\nu)}\left(w_0^{IV} - 2\varepsilon_{xz}^{'''}\right) . \tag{2.1.58}$$

Equations (2.1.57) and (2.1.58) are the same as equations (2.1.47) and (2.1.48), derived from the principle of total potential energy. So, the principle of total potential energy produces differential equations for the unknown functions such that their solution guarantees that the expressions for the transverse stresses σ_{xz} and σ_{zz} (obtained from the equilibrium equations), in terms of the unknown functions, satisfy the boundary conditions (2.1.6) at the upper surface of the plate. Satisfaction of the boundary conditions (2.1.5) at the lower surface of the plate by the stresses σ_{xz} and σ_{zz} (obtained from the equilibrium equations) is guaranteed by the fact that these conditions were used in the process of deriving expressions for σ_{xz} and σ_{zz} from the equilibrium equations.

Let us express the conditions of static equilibrium (2.1.7)–(2.1.9) in terms of the unknown functions u_0, w_0, ε_{xz}, ε_{zz}. The substitution of expression (2.1.28) for σ_{xx}^H into the conditions of static equilibrium (2.1.7) and (2.1.9) yields

$$(1-\nu)\,u_0^{'} - \frac{h^2}{24}\varepsilon_{zz}^{''} + \nu\varepsilon_{zz} = 0 \text{ at } x = 0, L , \tag{2.1.59}$$

$$2\varepsilon_{xz}^{'} - w_0^{''} = 0 \text{ at } x = 0, L . \tag{2.1.60}$$

Let us substitute expression (2.1.29) for σ_{xz} into the left-hand side of static equilibrium condition (2.1.9)

$$b\int_{-h/2}^{h/2}\left[\sigma_{xz}(L) - \sigma_{xz}(0)\right]\,dz = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{bh^3}{12}\left(2\varepsilon_{xz}^{''} - w_0^{'''}\right)\Big|_0^L -$$

$$\frac{Ebh^2}{2(1+\nu)(1-2\nu)}\underbrace{\left[(1-\nu)\left(u_0^{''} - \frac{h^2}{24}\varepsilon_{zz}^{'''}\right) + \nu\varepsilon_{zz}^{'}\right]}_{\substack{0\\ \text{due to diff. eqn. (47)}}}\Bigg|_0^L .$$

Using differential equation (2.1.48), we can write the right-hand side of the static equilibrium condition (2.1.9) in the form

$$-\int_0^L (q_u + q_l)\ dx = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{bh^3}{12}\left(2\varepsilon_{xz}^{''} - w_0^{'''}\right)\Big|_0^L .$$

From the last two equations it follows that the left-hand side $b\int_{-h/2}^{h/2}[\sigma_{xz}(L) - \sigma_{xz}(0)]\ dz$ of the static equilibrium equation (2.1.9) is identically equal to its right-hand side $-\int_0^L (q_u + q_l)\ dx$. Since the static equilibrium conditions (2.1.7) and (2.1.8), being expressed in terms of the unknown functions of our plate theory (equations (2.1.59) and (2.1.60)), are the same as the natural boundary conditions of the plate theory (equations (2.1.51), (2.1.52) and (2.1.53)), and since the static equilibrium condition (2.1.9), being expressed in terms of the unknown functions of our plate theory, is an identity, we make a conclusion that our plate theory guarantees satisfaction of all the static equilibrium conditions (2.1.7)–(2.1.9).

Now, let us solve a problem of **cylindrical bending of a plate, simply supported at the edges $x = 0, L$, under a uniform load** q_u, applied to the upper surface (figure 2.2). The results will be compared with the exact elasticity solution.

The boundary conditions (2.1.51)–(2.1.56) for this problem take the form

$$(1-\nu)\left(u_0^{'} - \frac{h^2}{24}\varepsilon_{zz}^{''}\right) + \nu\varepsilon_{zz} = 0 \text{ at } x = 0, L \ , \tag{2.1.61}$$

$$2\varepsilon_{xz}^{'} - w_0^{''} = 0 \text{ at } x = 0, L \ , \tag{2.1.62}$$

$$w_0 = 0 \text{ at } x = 0, L \ , \tag{2.1.63}$$

$$(1-\nu)\left(u_0^{'} - \frac{3}{40}h^2\varepsilon_{zz}^{''}\right) + \nu\varepsilon_{zz} = 0 \text{ at } x = 0, L \ , \tag{2.1.64}$$

$$(1-\nu)\left(u_0^{''} - \frac{3}{40}h^2\varepsilon_{zz}^{'''}\right) + \nu\varepsilon_{zz}^{'} = 0 \text{ at } x = 0, L \ . \tag{2.1.65}$$

In addition, due to symmetry of the problem,

$$\varepsilon_{xz}(0) = -\varepsilon_{xz}(L), \quad u\left(\frac{L}{2}\right) = 0 \ . \tag{2.1.66}$$

Differential equations (2.1.47)–(2.1.50) can be written as the following independent sets of equations:

1) equation

$$\frac{hbE}{1+\nu}\varepsilon'_{xz} = -q_u \quad (0 \le x \le L) \tag{2.1.67}$$

with symmetry condition

$$\varepsilon_{xz}(0) = -\varepsilon_{xz}(L) \ ;$$

2) equation

$$\frac{1-\nu}{(1+\nu)(1-2\nu)}\frac{bEh^3}{12}\left(w_0^{IV} - 2\varepsilon_{xz}^{'''}\right) = q_u \quad (0 \le x \le L)$$

with boundary conditions

$$2\varepsilon_{xz}^{'} - w_0^{''} = 0, \ w_0 = 0 \text{ at } x = 0, L \ ;$$

3) equations

$$(1-\nu)\left(u_0^{''} - \frac{h^2}{24}\varepsilon_{zz}^{'''}\right) + \nu\varepsilon_{zz}^{'} = 0 \quad (0 \le x \le L)$$

and

$$\nu\left(u_0^{'} - \frac{h^2}{24}\varepsilon_{zz}^{''}\right) + (1-\nu)\left[\varepsilon_{zz} + \frac{h^2}{8}\left(\frac{h^2}{40}\varepsilon_{zz}^{IV} - \frac{1}{3}u_0^{'''}\right)\right] = \frac{(1+\nu)(1-2\nu)}{bE}q_u \quad (0 \le x \le L)$$

with boundary conditions

$$(1-\nu)\left(u_0^{'} - \frac{h^2}{24}\varepsilon_{zz}^{''}\right) + \nu\varepsilon_{zz} = 0 \text{ at } x = 0, L \ ,$$

$$(1-\nu)\left(u_0^{''} - \frac{h^2}{24}\varepsilon_{zz}^{'''}\right) + \nu\varepsilon_{zz}^{'} = 0 \text{ at } x = 0, L$$

and symmetry condition

$$u\left(\frac{L}{2}\right) = 0 \ .$$

For $q_u = const$ these equations have the following solution

$$u_0 = -\nu(1+\nu)\frac{q_u}{bE}\left(x - \frac{L}{2}\right) \ , \tag{2.1.68}$$

$$w_0 = \frac{q_u(1+\nu)}{2bh^3(1-\nu)E}x(x-L)\left[(1-2\nu)\left(x^2 - Lx - L^2\right) - 2h^2(1-\nu)\right] \ , \tag{2.1.69}$$

$$\varepsilon_{xz} = \frac{q_u (1+\nu)}{2bEh} (L - 2x) \ , \tag{2.1.70}$$

$$\varepsilon_{zz} = (1+\nu)(1-\nu) \frac{q_u}{bE} \ . \tag{2.1.71}$$

Substitution of expressions (2.1.68)-(2.1.71) into the expressions (2.1.28), (2.1.29) and (2.1.30) for $\sigma_{xx}^H, \sigma_{xz}, \sigma_{zz}$ yields

$$\sigma_{xx}^H = -\frac{6q_u}{bh^3} x (x - L) z, \tag{2.1.72}$$

$$\sigma_{xz} = \frac{6q_u}{bh^3} \left(x - \frac{L}{2}\right) \left(z^2 - \frac{h^2}{4}\right), \tag{2.1.73}$$

$$\sigma_{zz} = -\frac{q_u}{2bh^3} (2z + h)^2 (z - h) . \tag{2.1.74}$$

It can be verified that the conditions of static equilibrium

$$b \int_{-\frac{h}{2}}^{\frac{h}{2}} [\sigma_{xz}(L) - \sigma_{xz}(0)] \ dz \ = - \int_0^L q_u \ dx, \tag{2.1.75}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}^H \ dz = 0 \text{ at } x = 0, L, \tag{2.1.76}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}^H \ z \ dz = 0 \text{ at } x = 0, L \tag{2.1.77}$$

are satisfied by the found stresses (2.1.72)–(2.1.74). The expressions (2.1.72)–(2.1.74) for the stresses satisfy the equilibrium equations

$$\left. \begin{array}{l} \sigma_{xx,x} + \sigma_{xz,z} = 0 \\ \sigma_{xz,x} + \sigma_{zz,z} = 0 \end{array} \right\}, \tag{2.1.78}$$

the boundary conditions

$$\left. \begin{array}{r} \sigma_{xz} = 0, \ \sigma_{zz} = -\frac{q_l}{b} \ \text{ at } z = -\frac{h}{2} \\ \sigma_{xz} = 0, \ \sigma_{zz} = \frac{q_u}{b} \ \text{ at } z = \frac{h}{2} \end{array} \right\} \tag{2.1.79}$$

and the conditions of static equilibrium (2.1.75)–(2.1.77). But the equation of compatibility in terms of stress

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)(\sigma_{xx}+\sigma_{zz})=0 \tag{2.1.80}$$

is not satisfied by the expressions (2.1.72) and (2.1.74), obtained from the plate theory. Therefore, the expressions for stresses (2.1.72)-(2.1.74), obtained from the plate theory, are not exact.

The **exact elasticity solution** (within a framework of linear elasticity) for the plate in cylindrical bending (in plane strain condition), which satisfies the equilibrium equations (2.1.78), boundary conditions (2.1.79), conditions of static equilibrium (2.1.75)-(2.1.77) **and** equation of compatibility in terms of stress (2.1.80), is derived in Appendix 2-A. This solution is:

$$\sigma_{xx}=-\frac{6}{h^3}\frac{q_u}{b}x\,(x-L)\,z+\frac{4q_u}{b}\frac{z^3}{h^3}-\frac{3}{5}\frac{q_u}{b}\frac{z}{h}, \tag{2.1.81}$$

$$\sigma_{xz}=\frac{6}{h^3}\frac{q_u}{b}\left(x-\frac{L}{2}\right)\left(z^2-\frac{h^2}{4}\right), \tag{2.1.82}$$

$$\sigma_{zz}=-\frac{1}{2h^3}\frac{q_u}{b}\,(2z+h)^2\,(z-h)\,. \tag{2.1.83}$$

Comparison of formulas (2.1.72)–(2.1.74) and (2.1.81)–(2.1.83) shows that the theory of a homogeneous plate, based on assumption $\varepsilon_{xz}=\varepsilon_{xz}(x), \varepsilon_{zz}=\varepsilon_{zz}(x)$, produces exact expressions for the transverse stresses σ_{xz} and σ_{zz}, if these stresses are calculated by integration of equilibrium equations (not from Hooke's law). But expression (2.1.72) for the in-plane stress σ_{xx}^H, calculated from the plate theory, differs from the corresponding exact stress (expression (2.1.81)).

Let us compare the exact stress σ_{xx} with the one obtained from the plate theory at a point $x=\frac{L}{2}$, $z=\frac{h}{2}$, i.e. at a point, where, according to the plate theory, the stress σ_{xx} is the highest. From formulas (2.1.70) and (2.1.79) we find

$$\sigma_{xx}^{(plate\ theory)}=\frac{3}{4}\frac{q}{b}\left(\frac{L}{h}\right)^2 \text{ at } x=\frac{L}{2},\, z=\frac{h}{2},$$

$$\sigma_{xx}^{(exact)}=\frac{3}{4}\frac{q}{b}\left(\frac{L}{h}\right)^2+\frac{1}{5}\frac{q}{b} \text{ at } x=\frac{L}{2},\, z=\frac{h}{2}.$$

So, a relative error in computation of σ_{xx}, produced by the plate theory, is

$$\frac{\sigma_{xx}^{(exact)}-\sigma_{xx}^{(plate\ theory)}}{\sigma_{xx}^{(exact)}}=\frac{1}{\frac{15}{4}\left(\frac{L}{h}\right)^2+1}.$$

Therefore, in order for relative error of the plate theory not to exceed 5%, the height to length ratio of the plate must not exceed 0.44426 :

$$\frac{h}{L} \leq 0.44426.$$

This condition is met for the problem, which is the topic of the dissertation. So, a theory of homogeneous plates, based on assumption that the transverse strains do not vary in the thickness direction, produces sufficiently accurate values of all stresses, both in-plane and transverse. The transverse strains, as unknown functions of the problem, which, according to the assumptions (2.1.21) and (2.1.22), do not vary in the thickness direction, were found to be expressed by the formulas (2.1.70) and (2.1.71). This is the first form of the transverse strains:

$$(\varepsilon_{xz})^{(I)} = \frac{q_u (1+\nu)}{2bEh} (L - 2x), \tag{2.1.84}$$

$$(\varepsilon_{zz})^{(I)} = (1+\nu)(1-\nu)\frac{q_u}{bE}. \tag{2.1.85}$$

The more accurate expressions for the transverse strains (second form of the transverse strains) can be found by substitution of the transverse stresses σ_{xz} and σ_{zz}, obtained from the equilibrium equations (expressions (2.1.73) and (2.1.74)), into the strain-stress relations

$$(\varepsilon_{xz})^{(II)} = \frac{1+\nu}{E}\sigma_{xz}, \tag{2.1.86}$$

$$(\varepsilon_{zz})^{(II)} = \frac{1-\nu^2}{E}\left(\sigma_{zz} - \frac{\nu}{1-\nu}\sigma_{xx}^H\right). \tag{2.1.87}$$

The substitution yields:

$$(\varepsilon_{xz})^{(II)} = \frac{1+\nu}{E}\frac{6q_u}{bh^3}\left(x - \frac{L}{2}\right)\left(z^2 - \frac{h^2}{4}\right), \tag{2.1.88}$$

$$(\varepsilon_{zz})^{(II)} = \frac{q_u}{2Ebh^3}\left[(\nu-1)(2z+h)^2(z-h) + 6\nu x (x-L) z\right]. \tag{2.1.89}$$

For consistency of nomenclature, the stresses σ_{xz}^H and σ_{zz}^H, obtained from the Hooke's law, must be called the first forms of transverse stresses:

$$\sigma_{xz}^H \equiv (\sigma_{xz})^{(I)} \ , \ \sigma_{zz}^H \equiv (\sigma_{zz})^{(I)},$$

and the stresses σ_{xz} and σ_{zz}, obtained from the equilibrium equations, must be called the second forms of the transverse stresses:

$$\sigma_{xz} \equiv (\sigma_{xz})^{(II)} \ , \ \sigma_{zz} \equiv (\sigma_{zz})^{(II)}.$$

In-plane strain ε_{xx} and in-plane stress σ_{xx} have only one form. The second forms of the transverse strains and stresses are more accurate than the first forms. Stress σ_{yy} can be found by substituting expressions for strains ε_{xx} and $(\varepsilon_{zz})^{(II)}$into the constitutive equation (2.1.15).

In this section we came to the conclusion that a theory of a homogeneous plate, based on the assumption that the transverse stresses do not vary in the thickness direction, leads to sufficiently accurate results if the thickness of a plate is much smaller than its length and width. In the next section we will consider construction of a layerwise theory of cylindrical bending of a sandwich plate, based on the similar assumptions: the assumed transverse strains (i.e. the first forms of the transverse strains) do not vary in the thickness direction within a layer (a face sheet or a core) of a sandwich plate, but can be different in different layers. Then we will consider a problem of cylindrical bending of a simply supported sandwich plate under a constant load and compare a solution of the problem, based on the plate theory, with the exact elasticity solution. This will enable us to evaluate the validity of the assumptions on the transverse strains. In order to avoid the excessive complexity of the problem, we will consider the material of the face sheets and the core to be isotropic.

2.2 Cylindrical Bending of a Sandwich Isotropic Plate

2.2.1 Formulation of the Problem Based on Linear Elasticity

Let us consider cylindrical bending of a wide sandwich plate with isotropic face sheets and the core (Figure 2.3). The upper and lower surfaces of the plate are under loads with intensity (force per unit length) q_u and q_l. By q_u and q_l we denote not absolute values of the load intensities, but projections of the load intensities on the z-axis, i.e. q_u and q_l can be positive or negative, depending on direction of the load.

We will denote a number of a layer of the plate by a superscript k ($k = 1, 2, 3$).

The equations of linear elasticity, as applied to this problem, have the form: equilibrium equations:

$$\sigma^{(k)}_{xx,x} + \sigma^{(k)}_{xz,z} = 0, \tag{2.2.1}$$

$$\sigma^{(k)}_{xz,x} + \sigma^{(k)}_{zz,z} = 0; \tag{2.2.2}$$

Strain-displacement relations for plane strain are:

$$\varepsilon^{(k)}_{xx} = u^{(k)}_{,x}, \tag{2.2.3}$$

$$\varepsilon^{(k)}_{zz} = w^{(k)}_{,z}, \tag{2.2.4}$$

$$\varepsilon^{(k)}_{xz} = \frac{1}{2}\left(u^{(k)}_{,z} + w^{(k)}_{,x}\right), \tag{2.2.5}$$

$$\varepsilon^{(k)}_{yy} = \varepsilon^{(k)}_{yz} = \varepsilon^{(k)}_{xy} = 0; \tag{2.2.6}$$

The constitutive relations for plane strain can be stated as:

$$\sigma^{(k)}_{xx} = \frac{E^{(k)}}{\left(1+\nu^{(k)}\right)\left(1-2\nu^{(k)}\right)}\left[\left(1-\nu^{(k)}\right)\varepsilon^{(k)}_{xx} + \nu^{(k)}\varepsilon^{(k)}_{zz}\right]; \tag{2.2.7}$$

$$\sigma^{(k)}_{zz} = \frac{E^{(k)}}{\left(1+\nu^{(k)}\right)\left(1-2\nu^{(k)}\right)}\left[\left(1-\nu^{(k)}\right)\varepsilon^{(k)}_{zz} + \nu^{(k)}\varepsilon^{(k)}_{xx}\right]; \tag{2.2.8}$$

$$\sigma^{(k)}_{yy} = \frac{E^{(k)}}{\left(1+\nu^{(k)}\right)\left(1-2\nu^{(k)}\right)}\left(\varepsilon^{(k)}_{xx} + \varepsilon^{(k)}_{zz}\right) = \nu^{(k)}\left(\sigma^{(k)}_{xx} + \sigma^{(k)}_{zz}\right); \tag{2.2.9}$$

$$\sigma_{xz}^{(k)} = \frac{E^{(k)}}{\left(1+\nu^{(k)}\right)}\varepsilon_{xz}^{(k)}; \tag{2.2.10}$$

$$\sigma_{xy}^{(k)} = \sigma_{yz}^{(k)} = 0; \tag{2.2.11}$$

or, in the inverse form

$$\varepsilon_{xx}^{(k)} = \frac{1-\left(\nu^{(k)}\right)^2}{E^{(k)}}\left(\sigma_{xx}^{(k)} - \frac{\nu^{(k)}}{1-\nu^{(k)}}\sigma_{zz}^{(k)}\right); \tag{2.2.12}$$

$$\varepsilon_{zz}^{(k)} = \frac{1-\left(\nu^{(k)}\right)^2}{E^{(k)}}\left(\sigma_{zz}^{(k)} - \frac{\nu^{(k)}}{1-\nu^{(k)}}\sigma_{xx}^{(k)}\right); \tag{2.2.13}$$

$$\varepsilon_{xz}^{(k)} = \frac{1+\nu^{(k)}}{E^{(k)}}\sigma_{xz}^{(k)}; \tag{2.2.14}$$

$$\varepsilon_{yy}^{(k)} = \varepsilon_{xy}^{(k)} = \varepsilon_{yz}^{(k)} = 0; \tag{2.2.15}$$

The boundary conditions at the upper and lower surfaces are

$$\sigma_{xz}^{(1)} = 0, \sigma_{zz}^{(1)} = -\frac{q_l}{b} \text{ at } z = -\frac{h}{2} = z_1; \tag{2.2.16}$$

$$\sigma_{xz}^{(3)} = 0, \sigma_{zz}^{(3)} = \frac{q_u}{b} \text{ at } z = \frac{h}{2} = z_4; \tag{2.2.17}$$

The continuity of displacements and stresses at the interfaces between the core and the face sheets can be stated as:

$$u^{(1)} = u^{(2)}, w^{(1)} = w^{(2)}, \sigma_{xz}^{(1)} = \sigma_{xz}^{(2)}, \sigma_{zz}^{(1)} = \sigma_{zz}^{(2)} \text{ at } z = -\frac{t}{2} = z_2, \tag{2.2.18}$$

$$u^{(2)} = u^{(3)}, w^{(2)} = w^{(3)}, \sigma_{xz}^{(2)} = \sigma_{xz}^{(3)}, \sigma_{zz}^{(2)} = \sigma_{zz}^{(3)} \text{ at } z = \frac{t}{2} = z_3. \tag{2.2.19}$$

The conditions of static equilibrium yield:

$$b\int_{-h/2}^{h/2}\left(\sigma_{xz}|_{x=L} - \sigma_{xz}|_{x=0}\right)\,dz = \int_0^L (q_l+q_u)\,dx,$$

or

$$\int_{-h/2}^{-t/2}\left(\sigma_{xz}^{(1)}\Big|_0^L\right)\,dz + \int_{-t/2}^{t/2}\left(\sigma_{xz}^{(2)}\Big|_0^L\right)\,dz + \int_{t/2}^{h/2}\left(\sigma_{xz}^{(3)}\Big|_0^L\right)\,dz = -\frac{1}{b}\int_0^L (q_l+q_u)\,dx\,. \tag{2.2.20}$$

The formulation of the problem includes also the boundary conditions at $x = 0, L$. For example, for a plate, simply supported along the edges $x = 0, L$, the boundary conditions have the form: mitigated (integral) stress boundary conditions, that can also be looked upon as conditions of static equilibrium

$$\left.\begin{array}{l} \displaystyle\int_{-h/2}^{-t/2} \sigma_{xx}^{(1)}\, dz = 0 \text{ at } x = 0, L \\ \displaystyle\int_{-t/2}^{t/2} \sigma_{xx}^{(2)}\, dz = 0 \text{ at } x = 0, L \\ \displaystyle\int_{t/2}^{h/2} \sigma_{xx}^{(3)}\, dz = 0 \text{ at } x = 0, L \end{array}\right\} ; \tag{2.2.21}$$

$$\left.\begin{array}{c} \displaystyle\int_{-h/2}^{h/2} \sigma_{xx}\, z\, dz = 0 \text{ at } x = 0, L \\ \text{or} \\ \displaystyle\int_{-h/2}^{-t/2} \sigma_{xx}^{(1)}\, z\, dz + \int_{-t/2}^{t/2} \sigma_{xx}^{(2)}\, z\, dz + \int_{t/2}^{h/2} \sigma_{xx}^{(3)}\, z\, dz = 0 \text{ at } x = 0, L \end{array}\right\} \tag{2.2.22}$$

and the displacement boundary conditions

$$w = 0 \text{ at } x = 0, L \text{ and } z = 0. \tag{2.2.23}$$

If the boundary conditions and the load are symmetric with respect to the plane $x = \frac{L}{2}$, then we also have a symmetry condition

$$u\left(\frac{L}{2}\right) = 0. \tag{2.2.24}$$

2.2.2 Construction of a Plate Theory for Cylindrical Bending of an Isotropic Sandwich Plate, Based on Linear Elasticity

In order to construct a plate theory, we make an assumption that the transverse strains do not vary in the thickness direction within a layer (a face sheet or a core) of a sandwich plate, but can be

different in different layers:

$$\varepsilon_{xz}^{(k)} = \varepsilon_{xz}^{(k)}(x), \ \varepsilon_{zz}^{(k)} = \varepsilon_{zz}^{(k)}(x) \qquad (k = 1, 2, 3). \tag{2.2.25}$$

These are the first forms of the transverse strains. To indicate that the assumed transverse strains of equations (2.2.25) are the first forms of the strains, we will also use another notation:

$$\varepsilon_{xz}^{(k)} \equiv \left(\varepsilon_{xz}^{(k)}\right)^{(I)}, \quad \varepsilon_{zz}^{(k)} \equiv \left(\varepsilon_{zz}^{(k)}\right)^{(I)}. \tag{2.2.26}$$

The notation (2.2.26), with the second upper superscript, will be used only when it is necessary to distinguish between the first and the second forms of the transverse strains.

The unknown functions of the problem are

$$u_0(x) = u^{(2)}\Big|_{z=0} = u|_{z=0}, \ w_0(x) = w^{(2)}\Big|_{z=0} = w|_{z=0}, \ \varepsilon_{xz}^{(k)}(x), \ \varepsilon_{zz}^{(k)}(x) \quad (k = 1, 2, 3). \tag{2.2.27}$$

So, there are 8 unknown functions in this theory of cylindrical bending of a sandwich plate.

Expressions for displacements $u(x,z),\ w(x,z)$ in terms of the unknown functions $u_0, w_0, \varepsilon_{xz}^{(k)}, \varepsilon_{zz}^{(k)}$ $(k=1,2,3)$

Let us integrate strain-displacement relations (2.2.4)

$$\varepsilon_{zz}^{(k)} = w_{,z}^{(k)}.$$

For the core of the sandwich plate (k=2), which contains plane z=0, we receive

$$w^{(2)}(x,z) - \underbrace{w^{(2)}\Big|_{z=0}}_{w_0(x)} = \int_0^z \frac{\partial w^{(2)}}{\partial z}\, dz = \int_0^z \varepsilon_{zz}^{(2)}(x,z)\ dz \qquad (z_2 \le z \le z_3),$$

or

$$w^{(2)}(x,z) = w_0(x) + \int_0^z \varepsilon_{zz}^{(2)}(x)\ dz \qquad (z_2 \le z \le z_3). \tag{2.2.28}$$

From equation (2.2.28) it follows

$$w^{(2)}\Big|_{z=z_2} = w_0 + \int_0^{z_2} \varepsilon_{zz}^{(2)}\ dz. \tag{2.2.29}$$

Integration of equation $\varepsilon_{zz}^{(1)} = \frac{\partial w^{(1)}}{\partial z}$ from z_2 to z, where z belongs to the region of the lower face sheet $(z_1 \le z \le z_2)$, yields

$$w^{(1)} - w^{(1)}\Big|_{z=z_2} = \int_{z_2}^z \frac{\partial w^{(1)}}{\partial z}\, dz = \int_{z_2}^z \varepsilon_{zz}^{(1)}\ dz \qquad (z_1 \le z \le z_2). \tag{2.2.30}$$

or, due to continuity condition $w^{(1)}\big|_{z=z_2} = w^{(2)}\big|_{z=z_2}$,

$$w^{(1)} = w^{(2)}\Big|_{z=z_2} + \int_{z_2}^z \varepsilon_{zz}^{(1)}\ dz. \tag{2.2.31}$$

If we substitute expression (2.2.29) for $w^{(2)}\big|_{z=z_2}$ into (2.2.31), we receive

$$w^{(1)} = w_0 + \int_0^{z_2} \varepsilon_{zz}^{(2)}\ dz + \int_{z_2}^z \varepsilon_{zz}^{(1)}\ dz \qquad (z_1 \le z \le z_2). \tag{2.2.32}$$

Analogously, if we integrate equation $\varepsilon_{zz}^{(3)} = \frac{\partial w^{(3)}}{\partial z}$ and satisfy the continuity condition at the interface between the second and the third zone, $w^{(3)}\big|_{z=z_3} = w^{(2)}\big|_{z=z_3}$, we receive

$$w^{(3)} = w_0 + \int_0^{z_3} \varepsilon_{zz}^{(2)}\ dz + \int_{z_3}^z \varepsilon_{zz}^{(3)}\ dz \qquad (z_3 \le z \le z_4). \tag{2.2.33}$$

Integration in equations (2.2.28), (2.2.32) and (2.2.33) yields

$$w^{(1)} = w_0 + \varepsilon_{zz}^{(2)} z_2 + \varepsilon_{zz}^{(1)} (z - z_2) \quad (z_1 \le z \le z_2)\,, \tag{2.2.34}$$

$$w^{(2)} = w_0 + \varepsilon_{zz}^{(2)}\, z \qquad (z_2 \le z \le z_3)\,, \tag{2.2.35}$$

$$w^{(3)} = w_0 + \varepsilon_{zz}^{(2)} z_3 + \varepsilon_{zz}^{(3)} (z - z_3) \quad (z_3 \le z \le z_4)\,. \tag{2.2.36}$$

Now, let us find expressions for displacements $u^{(1)}, u^{(2)}, u^{(3)}$ in terms of the unknown functions. From the strain-displacement relations (2.2.5) we receive

$$u_{,z}^{(k)} = 2\varepsilon_{xz}^{(k)} - w_{,x}^{(k)}. \tag{2.2.37}$$

Integration of equation (2.2.37) yields

$$u^{(2)}(x,z) - \underbrace{u^{(2)}\Big|_{z=0}}_{u_0(x)} = \int_0^z \frac{\partial u^{(2)}}{\partial z}\, dz = \int_0^z \left(2\varepsilon_{xz}^{(2)} - w_{,x}^{(2)}\right) dz \qquad (z_2 \le z \le z_3)\,, \tag{2.2.38}$$

$$u^{(1)}(x,z) - u^{(1)}\Big|_{z=z_2} = \int_{z_2}^z \frac{\partial u^{(1)}}{\partial z}\, dz = \int_{z_2}^z \left(2\varepsilon_{xz}^{(1)} - w_{,x}^{(1)}\right) dz \qquad (z_1 \le z \le z_2)\,, \tag{2.2.39}$$

$$u^{(3)}(x,z) - u^{(3)}\Big|_{z=z_3} = \int_{z_3}^z \frac{\partial u^{(1)}}{\partial z}\, dz = \int_{z_3}^z \left(2\varepsilon_{xz}^{(1)} - w_{,x}^{(1)}\right) dz \qquad (z_3 \le z \le z_4)\,. \tag{2.2.40}$$

When we substitute expressions (2.2.34) - (2.2.36) for $w^{(1)}, w^{(2)}, w^{(3)}$ into expressions (2.2.38) - (2.2.40), perform the integration in the resulting expressions and find the constants of integration from the conditions of continuity of displacements u at the interfaces between the zones,

$$u^{(1)}\Big|_{z=z_2} = u^{(2)}\Big|_{z=z_2}\,, \quad u^{(2)}\Big|_{z=z_3} = u^{(3)}\Big|_{z=z_3}\,,$$

we receive expressions for displacements $u^{(1)}, u^{(2)}, u^{(3)}$ in terms of the unknown functions $u_0(x)$, $w_0(x)$, $\varepsilon_{xz}^{(k)}(x)$, $\varepsilon_{zz}^{(k)}(x)$:

$$\begin{aligned} u^{(1)} = {} & u_0 + \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right) z_2 - \frac{1}{2}\varepsilon_{zz,x}^{(2)}\, z_2^2 + \left(2\varepsilon_{xz}^{(1)} - w_{0,x} - \varepsilon_{zz,x}^{(2)}\, z_2\right)(z - z_2) \\ & - \frac{1}{2}\varepsilon_{zz,x}^{(1)} (z - z_2)^2 \qquad (z_1 \le z \le z_2)\,, \end{aligned} \tag{2.2.41}$$

$$u^{(2)} = u_0 + \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right) z - \frac{1}{2}\varepsilon_{zz,x}^{(2)}\, z^2 \qquad (z_2 \le z \le z_3)\,, \tag{2.2.42}$$

$$\begin{aligned} u^{(3)} = u_0 &+ \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right) z_3 - \frac{1}{2}\varepsilon_{zz,x}^{(2)}\, z_3^2 + \left(2\varepsilon_{xz}^{(3)} - w_{0,x} - \varepsilon_{zz,x}^{(2)} z_3\right)(z - z_3) \\ &- \frac{1}{2}\varepsilon_{zz,x}^{(3)} (z - z_3)^2 \qquad (z_3 \le z \le z_4)\,. \end{aligned} \tag{2.2.43}$$

Expressions (2.2.41)–(2.2.43) can be written in the form

$$u^{(1)} = \psi_{u0}^{(1)} + \psi_{u1}^{(1)} z + \psi_{u2}^{(1)} z^2, \tag{2.2.44}$$

$$u^{(2)} = \psi_{u0}^{(2)} + \psi_{u1}^{(2)} z + \psi_{u2}^{(2)} z^2, \tag{2.2.45}$$

$$u^{(3)} = \psi_{u0}^{(3)} + \psi_{u1}^{(3)} z + \psi_{u2}^{(3)} z^2, \tag{2.2.46}$$

where

$$\psi_{u0}^{(1)} = u_0 + 2z_2\left(\varepsilon_{xz}^{(2)} - \varepsilon_{xz}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(1)}\right),$$

$$\psi_{u1}^{(1)} = 2\varepsilon_{xz}^{(1)} - w_{0,x} + z_2\left(\varepsilon_{zz,x}^{(1)} - \varepsilon_{zz,x}^{(2)}\right),$$

$$\psi_{u2}^{(1)} = -\frac{1}{2}\varepsilon_{zz,x}^{(1)},$$

$$\psi_{u0}^{(2)} = u_0,$$

$$\psi_{u1}^{(2)} = 2\varepsilon_{xz}^{(2)} - w_{0,x},$$

$$\psi_{u2}^{(2)} = -\frac{1}{2}\varepsilon_{zz,x}^{(2)},$$

$$\psi_{u0}^{(3)} = u_0 + 2z_3\left(\varepsilon_{xz}^{(2)} - \varepsilon_{xz}^{(3)}\right) + \frac{1}{2}z_3^2\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(3)}\right),$$

$$\psi_{u1}^{(3)} = 2\varepsilon_{xz}^{(3)} - w_{0,x} + z_3\left(\varepsilon_{zz,x}^{(3)} - \varepsilon_{zz,x}^{(2)}\right),$$

$$\psi_{u2}^{(3)} = -\frac{1}{2}\varepsilon_{zz,x}^{(3)}.$$

In-Plane Strains $\varepsilon_{xx}^{(1)}, \varepsilon_{xx}^{(2)}, \varepsilon_{xx}^{(3)}$ in Terms of the Unknown Functions $u_0, w_0, \varepsilon_{xz}^{(k)}, \varepsilon_{zz}^{(k)}$ $(k = 1,2,3)$

Substitution of the expressions (2.2.44)–(2.2.46) into the strain-displacement relations $\varepsilon_{xx}^{(k)} = \frac{\partial u^{(k)}}{\partial x}$ yields

$$\varepsilon_{xx}^{(1)} = \varphi_{xx0}^{(1)} + \varphi_{xx1}^{(1)}\ z + \varphi_{xx2}^{(1)}\ z^2, \tag{2.2.47}$$

$$\varepsilon_{xx}^{(2)} = \varphi_{xx0}^{(2)} + \varphi_{xx1}^{(2)}\ z + \varphi_{xx2}^{(2)}\ z^2, \tag{2.2.48}$$

$$\varepsilon_{xx}^{(3)} = \varphi_{xx0}^{(3)} + \varphi_{xx1}^{(3)}\ z + \varphi_{xx2}^{(3)}\ z^2, \tag{2.2.49}$$

where

$$\varphi_{xx0}^{(1)} = u_{0,x} + 2z_2\left(\varepsilon_{xz,x}^{(2)} - \varepsilon_{xz,x}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,xx}^{(2)} - \varepsilon_{zz,xx}^{(1)}\right), \tag{2.2.50}$$

$$\varphi_{xx1}^{(1)} = 2\varepsilon_{xz,x}^{(1)} - w_{0,xx} + z_2\left(\varepsilon_{zz,xx}^{(1)} - \varepsilon_{zz,xx}^{(2)}\right), \tag{2.2.51}$$

$$\varphi_{xx2}^{(1)} = -\frac{1}{2}\varepsilon_{zz,xx}^{(1)}, \tag{2.2.52}$$

$$\varphi_{xx0}^{(2)} = u_{0,x}, \tag{2.2.53}$$

$$\varphi_{xx1}^{(2)} = 2\varepsilon_{xz,x}^{(2)} - w_{0,xx}, \tag{2.2.54}$$

$$\varphi_{xx2}^{(2)} = -\frac{1}{2}\varepsilon_{zz,xx}^{(2)}, \tag{2.2.55}$$

$$\varphi_{xx0}^{(3)} = u_{0,x} + 2z_3\left(\varepsilon_{xz,x}^{(2)} - \varepsilon_{xz,x}^{(3)}\right) + \frac{1}{2}z_3^2\left(\varepsilon_{zz,xx}^{(2)} - \varepsilon_{zz,xx}^{(3)}\right), \tag{2.2.56}$$

$$\varphi_{xx1}^{(3)} = 2\varepsilon_{xz,x}^{(3)} - w_{0,xx} + z_3\left(\varepsilon_{zz,xx}^{(3)} - \varepsilon_{zz,xx}^{(2)}\right), \tag{2.2.57}$$

$$\varphi_{xx2}^{(3)} = -\frac{1}{2}\varepsilon_{zz,xx}^{(3)}. \tag{2.2.58}$$

Using the found expression for the in-plane strains in terms of the unknown functions, we can write the following matrix relations, which will be useful in writing the expression for strain energy in terms of the unknown functions

$$\underset{(3\times 1)}{\left\{\varepsilon^{(k)}(x,z)\right\}} = \underset{(3\times 5)}{[Z(z)]}\underset{(5\times 1)}{\left\{f^{(k)}(x)\right\}} \qquad (k=1,2,3)\,, \tag{2.2.59}$$

where

$$\left\{\varepsilon^{(k)}(x,z)\right\} = \begin{Bmatrix} \varepsilon_{xx}^{(k)} \\ 2\varepsilon_{xz}^{(k)} \\ \varepsilon_{zz}^{(k)} \end{Bmatrix}, \tag{2.2.60}$$

$$[Z(z)] = \begin{bmatrix} 1 & z & z^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.2.61}$$

$$\left\{f^{(k)}(x)\right\} = \begin{Bmatrix} \varphi_{xx0}^{(k)} \\ \varphi_{xx1}^{(k)} \\ \varphi_{xx2}^{(k)} \\ 2\varepsilon_{xz}^{(k)} \\ \varepsilon_{zz}^{(k)} \end{Bmatrix}. \tag{2.2.62}$$

Expressions for In-Plane Stresses and the First Forms of Transverse Stresses in Terms of the Unknown Functions $u_0, w_0, \varepsilon_{xz}^{(k)}, \varepsilon_{zz}^{(k)}$ $(k=1,2,3)$.

We will distinguish between the two forms of expressions for the transverse stresses in terms of the unknown functions: the first forms, ${}^{H}\sigma_{xz}^{(k)} \equiv {}^{(I)}\sigma_{xz}^{(k)}$ and ${}^{H}\sigma_{zz}^{(k)} \equiv {}^{(I)}\sigma_{zz}^{(k)}$, obtained from the Hooke's law by substituting into the stress-strain relations the assumed transverse strains (2.2.25), (which we also called the first forms of the transverse strains and denoted as $\varepsilon_{xz}^{(k)} \equiv \left(\varepsilon_{xz}^{(k)}\right)^{(I)}$, $\varepsilon_{zz}^{(k)} \equiv \left(\varepsilon_{zz}^{(k)}\right)^{(I)}$), and the second forms of transverse stresses, obtained from the equilibrium equations (2.2.1) and (2.2.2), which will be denoted as $\sigma_{xz}^{(k)} \equiv {}^{(II)}\sigma_{xz}^{(k)}$ and $\sigma_{zz}^{(k)} \equiv {}^{(II)}\sigma_{zz}^{(k)}$. We showed in the first section of this chapter that in homogeneous plates the second forms of expressions for the transverse stresses satisfy the stress boundary conditions at the upper and lower surfaces of the plate. We will show

later that the same is true for the second forms of the transverse stresses in the sandwich plates. Besides, the second forms of the transverse stresses in the sandwich plate satisfy the conditions of continuity of the transverse stresses at the interfaces between the layers with different material properties. The first form of the transverse stresses can not satisfy the mentioned boundary and continuity conditions. Therefore, the second form of the transverse stresses is more accurate than the first one. The expressions for the in-plane stresses $\sigma_{xx}^{(k)}$ in terms of the unknown functions will be determined only from the Hooke's law and, therefore, these expressions will be denoted by ${}^{H}\sigma_{xx}^{(k)}$. Constitutive relations (2.27), (2.2.8) and (2.2.10) can be written in matrix form as follows

$$\underset{(3\times1)}{\left\{{}^{H}\sigma^{(k)}\right\}} = \underset{(3\times3)}{\left[C^{(k)}\right]}\underset{(3\times1)}{\left\{\varepsilon^{(k)}\right\}} \quad (k = 1, 2, 3)\,, \tag{2.2.63}$$

where

$$\left\{{}^{H}\sigma^{(k)}\right\} = \left\{\begin{array}{c} {}^{H}\sigma_{xx}^{(k)} \\ {}^{H}\sigma_{xz}^{(k)} \\ {}^{H}\sigma_{zz}^{(k)} \end{array}\right\}, \tag{2.2.64}$$

$$\left[C^{(k)}\right] = \frac{E^{(k)}}{1+\nu}\begin{bmatrix} \frac{1-\nu^{(k)}}{1-2\nu^{(k)}} & 0 & \frac{\nu^{(k)}}{1-2\nu^{(k)}} \\ 0 & \frac{1}{2} & 0 \\ \frac{\nu^{(k)}}{1-2\nu^{(k)}} & 0 & \frac{1-\nu^{(k)}}{1-2\nu^{(k)}} \end{bmatrix}, \tag{2.2.65}$$

$$\left\{\varepsilon^{(k)}\right\} = \left\{\begin{array}{c} \varepsilon_{xx}^{(k)} \\ 2\varepsilon_{xz}^{(k)} \\ \varepsilon_{zz}^{(k)} \end{array}\right\}. \tag{2.2.66}$$

Using equation (2.2.59) we can write

$$\underset{(3\times1)}{\left\{{}^{H}\sigma^{(k)}\right\}} = \underset{(3\times3)}{\left[C^{(k)}\right]}\underset{(3\times5)}{\left[Z(z)\right]}\underset{(5\times1)}{\left\{f^{(k)}(x)\right\}}. \tag{2.2.67}$$

Strain Energy of the Sandwich Plate

Strain energy of the sandwich plates consists of strain energies of the face sheets and the core.

Therefore, it can be written as follows:

$$U = \frac{1}{2}\iiint\limits_{(V_1)} \left({}^H\sigma_{xx}^{(1)}\varepsilon_{xx}^{(1)} + 2\,{}^H\sigma_{xz}^{(1)}\varepsilon_{xz}^{(1)} + {}^H\sigma_{zz}^{(1)}\varepsilon_{zz}^{(1)} + {}^H\sigma_{yy}^{(1)}\underbrace{\varepsilon_{yy}^{(1)}}_{0} + 2\,\underbrace{{}^H\sigma_{xy}^{(1)}}_{0}\underbrace{\varepsilon_{xy}^{(1)}}_{0} + 2\,\underbrace{{}^H\sigma_{yz}^{(1)}}_{0}\underbrace{\varepsilon_{yz}^{(1)}}_{0} \right) dV +$$

$$+\frac{1}{2}\iiint\limits_{(V_2)} \left({}^H\sigma_{xx}^{(2)}\varepsilon_{xx}^{(2)} + 2\,{}^H\sigma_{xz}^{(2)}\varepsilon_{xz}^{(2)} + {}^H\sigma_{zz}^{(2)}\varepsilon_{zz}^{(2)} + {}^H\sigma_{yy}^{(2)}\underbrace{\varepsilon_{yy}^{(2)}}_{0} + 2\,\underbrace{{}^H\sigma_{xy}^{(2)}}_{0}\underbrace{\varepsilon_{xy}^{(2)}}_{0} + 2\,\underbrace{{}^H\sigma_{yz}^{(2)}}_{0}\underbrace{\varepsilon_{yz}^{(2)}}_{0} \right) dV +$$

$$+\frac{1}{2}\iiint\limits_{(V_3)} \left({}^H\sigma_{xx}^{(3)}\varepsilon_{xx}^{(3)} + 2\,{}^H\sigma_{xz}^{(3)}\varepsilon_{xz}^{(3)} + {}^H\sigma_{zz}^{(3)}\varepsilon_{zz}^{(3)} + {}^H\sigma_{yy}^{(3)}\underbrace{\varepsilon_{yy}^{(3)}}_{0} + 2\,\underbrace{{}^H\sigma_{xy}^{(3)}}_{0}\underbrace{\varepsilon_{xy}^{(3)}}_{0} + 2\,\underbrace{{}^H\sigma_{yz}^{(3)}}_{0}\underbrace{\varepsilon_{yz}^{(3)}}_{0} \right) dV\ ,$$

where V_1, V_2, V_3 are volumes of the lower face sheet, core and upper face sheet. The underbraced terms in the above expression are equal to zero due to the condition of plane strain. Using definitions (2.2.60) and (2.2.64), we can write the expression for the strain energy in the form

$$U = \frac{1}{2}b\int_0^L\int_{z_1}^{z_2} \left\{\varepsilon^{(1)}\right\}^T \left\{{}^H\sigma^{(1)}\right\}\, dz\, dx + \frac{1}{2}b\int_0^L\int_{z_2}^{z_3} \left\{\varepsilon^{(2)}\right\}^T \left\{{}^H\sigma^{(2)}\right\}\, dz\, dx +$$

$$+\frac{1}{2}b\int_0^L\int_{z_3}^{z_4} \left\{\varepsilon^{(3)}\right\}^T \left\{{}^H\sigma^{(3)}\right\}\, dz\, dx =$$

$$= \frac{1}{2}b\int_0^L\int_{z_1}^{z_2} \left\{\varepsilon^{(1)}\right\}^T \left[C^{(1)}\right]\left\{\varepsilon^{(1)}\right\}\, dz\, dx + \frac{1}{2}b\int_0^L\int_{z_2}^{z_3} \left\{\varepsilon^{(2)}\right\}^T \left[C^{(2)}\right]\left\{\varepsilon^{(2)}\right\}\, dz\, dx +$$

$$+\frac{1}{2}b\int_0^L\int_{z_3}^{z_4} \left\{\varepsilon^{(3)}\right\}^T \left[C^{(3)}\right]\left\{\varepsilon^{(3)}\right\}\, dz\, dx.$$

One can substitute expression (2.2.59) into the last expression yielding

$$U = \frac{1}{2}b\int_0^L \underset{(1\times5)}{\left\{f^{(1)}(x)\right\}^T} \left(\int_{z_1}^{z_2} \underset{(5\times3)}{[Z(z)]^T} \underset{(3\times3)}{\left[C^{(1)}\right]} \underset{(3\times5)}{[Z(z)]}\, dz\right) \underset{(5\times1)}{\left\{f^{(1)}(x)\right\}}\, dx +$$

$$+\frac{1}{2}b\int_0^L \underset{(1\times5)}{\left\{f^{(2)}(x)\right\}^T} \left(\int_{z_2}^{z_3} \underset{(5\times3)}{[Z(z)]^T} \underset{(3\times3)}{\left[C^{(2)}\right]} \underset{(3\times5)}{[Z(z)]}\, dz\right) \underset{(5\times1)}{\left\{f^{(2)}(x)\right\}}\, dx +$$

$$+\frac{1}{2}b\int_0^L \underset{(1\times5)}{\left\{f^{(3)}(x)\right\}^T} \left(\int_{z_3}^{z_4} \underset{(5\times3)}{[Z(z)]^T} \underset{(3\times3)}{\left[C^{(3)}\right]} \underset{(3\times5)}{[Z(z)]}\, dz\right) \underset{(5\times1)}{\left\{f^{(3)}(x)\right\}} dx\ ,$$

or

$$U=\frac{1}{2}b\int_{0}^{L}\left(\left\{\underset{(1\times5)}{f^{(1)}(x)}\right\}^{T}\left[\underset{(5\times5)}{D^{(1)}}\right]\left\{\underset{(5\times1)}{f^{(1)}(x)}\right\}+\left\{\underset{(1\times5)}{f^{(2)}(x)}\right\}^{T}\left[\underset{(5\times5)}{D^{(2)}}\right]\left\{\underset{(5\times1)}{f^{(2)}(x)}\right\}+\right.$$
$$\left.+\left\{\underset{(1\times5)}{f^{(3)}(x)}\right\}^{T}\left[\underset{(5\times5)}{D^{(3)}}\right]\left\{\underset{(5\times1)}{f^{(3)}(x)}\right\}\right)dx\,, \tag{2.2.68}$$

where

$$\left[D^{(1)}\right]=\int_{z_1}^{z_2}\underset{(5\times3)}{[Z(z)]^{T}}\left[\underset{(3\times3)}{C^{(1)}}\right]\underset{(3\times5)}{[Z(z)]}\,dz=$$

$$=\frac{E_1}{1+\nu_1}\begin{bmatrix}(1-\nu_1)\frac{z_1-z_2}{2\nu_1-1} & \frac{1}{2}(1-\nu_1)\frac{z_1^2-z_2^2}{2\nu_1-1} & \frac{1}{3}(1-\nu_1)\frac{z_1^3-z_2^3}{2\nu_1-1} & 0 & \nu_1\frac{z_1-z_2}{2\nu_1-1}\\ \frac{1}{2}(1-\nu_1)\frac{z_1^2-z_2^2}{2\nu_1-1} & \frac{1}{3}(1-\nu_1)\frac{z_1^3-z_2^3}{2\nu_1-1} & \frac{1}{4}(1-\nu_1)\frac{z_1^4-z_2^4}{2\nu_1-1} & 0 & \frac{1}{2}\nu_1\frac{z_1^2-z_2^2}{2\nu_1-1}\\ \frac{1}{3}(1-\nu_1)\frac{z_1^3-z_2^3}{2\nu_1-1} & \frac{1}{4}(1-\nu_1)\frac{z_1^4-z_2^4}{2\nu_1-1} & \frac{1}{5}(1-\nu_1)\frac{z_1^5-z_2^5}{2\nu_1-1} & 0 & \frac{1}{3}\nu_1\frac{z_1^3-z_2^3}{2\nu_1-1}\\ 0 & 0 & 0 & \frac{1}{2}z_2-\frac{1}{2}z_1 & 0\\ \nu_1\frac{z_1-z_2}{2\nu_1-1} & \frac{1}{2}\nu_1\frac{z_1^2-z_2^2}{2\nu_1-1} & \frac{1}{3}\nu_1\frac{z_1^3-z_2^3}{2\nu_1-1} & 0 & (1-\nu_1)\frac{z_1-z_2}{2\nu_1-1}\end{bmatrix}, \tag{2.2.69}$$

$$\left[D^{(2)}\right]=\int_{z_2}^{z_3}\underset{(5\times3)}{[Z(z)]^{T}}\left[\underset{(3\times3)}{C^{(2)}}\right]\underset{(3\times5)}{[Z(z)]}\,dz=$$

$$=\frac{E_2}{1+\nu_2}\begin{bmatrix}(1-\nu_2)\frac{z_2-z_3}{2\nu_2-1} & \frac{1}{2}(1-\nu_2)\frac{z_2^2-z_3^2}{2\nu_2-1} & \frac{1}{3}(1-\nu_2)\frac{z_2^3-z_3^3}{2\nu_2-1} & 0 & \nu_2\frac{z_2-z_3}{2\nu_2-1}\\ \frac{1}{2}(1-\nu_2)\frac{z_2^2-z_3^2}{2\nu_2-1} & \frac{1}{3}(1-\nu_2)\frac{z_2^3-z_3^3}{2\nu_2-1} & \frac{1}{4}(1-\nu_2)\frac{z_2^4-z_3^4}{2\nu_2-1} & 0 & \frac{1}{2}\nu_2\frac{z_2^2-z_3^2}{2\nu_2-1}\\ \frac{1}{3}(1-\nu_2)\frac{z_2^3-z_3^3}{2\nu_2-1} & \frac{1}{4}(1-\nu_2)\frac{z_2^4-z_3^4}{2\nu_2-1} & \frac{1}{5}(1-\nu_2)\frac{z_2^5-z_3^5}{2\nu_2-1} & 0 & \frac{1}{3}\nu_2\frac{z_2^3-z_3^3}{2\nu_2-1}\\ 0 & 0 & 0 & \frac{1}{2}z_3-\frac{1}{2}z_2 & 0\\ \nu_2\frac{z_2-z_3}{2\nu_2-1} & \frac{1}{2}\nu_2\frac{z_2^2-z_3^2}{2\nu_2-1} & \frac{1}{3}\nu_2\frac{z_2^3-z_3^3}{2\nu_2-1} & 0 & (1-\nu_2)\frac{z_2-z_3}{2\nu_2-1}\end{bmatrix}, \tag{2.2.70}$$

$$\left[D^{(3)}\right]=\int_{z_3}^{z_4}\underset{(5\times3)}{[Z(z)]^{T}}\left[\underset{(3\times3)}{C^{(3)}}\right]\underset{(3\times5)}{[Z(z)]}\,dz=$$

$$
= \frac{E_3}{1+\nu_3}\begin{bmatrix}
(1-\nu_3)\frac{z_3-z_4}{2\nu_3-1} & \frac{1}{2}(1-\nu_3)\frac{z_3^2-z_4^2}{2\nu_3-1} & \frac{1}{3}(1-\nu_3)\frac{z_3^3-z_4^3}{2\nu_3-1} & 0 & \nu_3\frac{z_3-z_4}{2\nu_3-1} \\
\frac{1}{2}(1-\nu_3)\frac{z_3^2-z_4^2}{2\nu_3-1} & \frac{1}{3}(1-\nu_3)\frac{z_3^3-z_4^3}{2\nu_3-1} & \frac{1}{4}(1-\nu_3)\frac{z_3^4-z_4^4}{2\nu_3-1} & 0 & \frac{1}{2}\nu_3\frac{z_3^2-z_4^2}{2\nu_3-1} \\
\frac{1}{3}(1-\nu_3)\frac{z_3^3-z_4^3}{2\nu_3-1} & \frac{1}{4}(1-\nu_3)\frac{z_3^4-z_4^4}{2\nu_3-1} & \frac{1}{5}(1-\nu_3)\frac{z_3^5-z_4^5}{2\nu_3-1} & 0 & \frac{1}{3}\nu_3\frac{z_3^3-z_4^3}{2\nu_3-1} \\
0 & 0 & 0 & \frac{1}{2}z_4-\frac{1}{2}z_3 & 0 \\
\nu_3\frac{z_3-z_4}{2\nu_3-1} & \frac{1}{2}\nu_3\frac{z_3^2-z_4^2}{2\nu_3-1} & \frac{1}{3}\nu_3\frac{z_3^3-z_4^3}{2\nu_3-1} & 0 & (1-\nu_3)\frac{z_3-z_4}{2\nu_3-1}
\end{bmatrix}.
\tag{2.2.71}
$$

The expression (2.2.68) for the potential energy can be written in the form

$$
U = \frac{1}{2}b\int_0^L \underset{(1\times 15)}{\begin{Bmatrix} \{f^{(1)}\} \\ \{f^{(2)}\} \\ \{f^{(3)}\} \end{Bmatrix}^T} \underset{(15\times 1)}{\begin{Bmatrix} [D^{(1)}]\{f^{(1)}\} \\ [D^{(2)}]\{f^{(2)}\} \\ [D^{(3)}]\{f^{(3)}\} \end{Bmatrix}} dx =
$$

$$
= \frac{1}{2}b\int_0^L \underset{(1\times 15)}{\begin{Bmatrix} \{f^{(1)}\} \\ \{f^{(2)}\} \\ \{f^{(3)}\} \end{Bmatrix}^T} \underset{(15\times 15)}{\begin{bmatrix} \underset{(5\times 5)}{[D^{(1)}]} & \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[0]} \\ \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[D^{(2)}]} & \underset{(5\times 5)}{[0]} \\ \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[D^{(3)}]} \end{bmatrix}} \underset{(15\times 1)}{\begin{Bmatrix} \{f^{(1)}\} \\ \{f^{(2)}\} \\ \{f^{(3)}\} \end{Bmatrix}} dx,
$$

or

$$
U = \frac{1}{2}b\int_0^L \underset{(1\times 15)}{\{f\}^T}\ \underset{(15\times 15)}{[D]}\ \underset{(15\times 1)}{\{f\}}\ dx,
\tag{2.2.72}
$$

where

$$
\underset{(1\times 15)}{\{f\}^T} = \begin{Bmatrix} \{f^{(1)}\} \\ \{f^{(2)}\} \\ \{f^{(3)}\} \end{Bmatrix},
\tag{2.2.73}
$$

$$
\underset{(15\times 15)}{[D]} = \begin{bmatrix} \underset{(5\times 5)}{[D^{(1)}]} & \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[0]} \\ \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[D^{(2)}]} & \underset{(5\times 5)}{[0]} \\ \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[0]} & \underset{(5\times 5)}{[D^{(3)}]} \end{bmatrix}.
\tag{2.2.74}
$$

Virtual Work of External Forces in Terms of Variations of the Unknown Functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$

Virtual work of loads on the upper and lower surfaces, q_u and q_l correspondingly, is

$$\delta' W = \int_0^L \left(q_l \ \ \delta w|_{z=z_1} \ + q_u \ \ \delta w|_{z=z_4} \right) \ dx =$$

$$= \int_0^L \left(q_l \ \ \delta w^{(1)}\Big|_{z=z_1} \ + q_u \ \ \delta w^{(3)}\Big|_{z=z_4} \right) \ dx \ . \qquad (2.2.75)$$

In notation $\delta' W$ the prime is used because in case of nonconservative external loads, the virtual work $\delta' W$ is not a variation of some state function W.

If equations (2.2.34) and (2.2.36) are used then

$$\begin{aligned} \delta w^{(1)}\Big|_{z=z_1} &= \delta w_0 + z_2 \ \delta \varepsilon_{zz}^{(2)} + (z_1 - z_2) \ \ \delta \varepsilon_{zz}^{(1)} \ , \\ \delta w^{(3)}\Big|_{z=z_4} &= \delta w_0 + z_3 \ \delta \varepsilon_{zz}^{(2)} + (z_4 - z_3) \ \ \delta \varepsilon_{zz}^{(3)} \ . \end{aligned} \qquad (2.2.76)$$

The results of (2.2.76) can be substituted into (2.2.75) yielding

$$\begin{aligned} \delta' W = & \int_0^L q_l \left[\delta w_0 + z_2 \ \delta \varepsilon_{zz}^{(2)} + (z_1 - z_2) \ \ \delta \varepsilon_{zz}^{(1)} \right] \ dx \ + \\ & + \int_0^L q_u \left[\delta w_0 + z_3 \ \delta \varepsilon_{zz}^{(2)} + (z_4 - z_3) \ \ \delta \varepsilon_{zz}^{(3)} \right] \ dx \ . \end{aligned} \qquad (2.2.77)$$

Finite Element Formulation for Static Problem of Cylindrical Bending of the Sandwich Isotropic Plate

The column-matrices $\{f^{(k)}\}$, defined by equation (2.2.62), can be written in the form

$$\{f^{(1)}\} \equiv \left\{ \begin{array}{c} \varphi_{xx0}^{(1)} \\ \varphi_{xx1}^{(1)} \\ \varphi_{xx2}^{(1)} \\ 2\varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\} \equiv \left\{ \begin{array}{c} u_{0,x} + 2z_2 \left(\varepsilon_{xz,x}^{(2)} - \varepsilon_{xz,x}^{(1)} \right) + \frac{1}{2} z_2^2 \left(\varepsilon_{zz,xx}^{(2)} - \varepsilon_{zz,xx}^{(1)} \right) \\ 2\varepsilon_{xz,x}^{(1)} - w_{0,xx} + z_2 \left(\varepsilon_{zz,xx}^{(1)} - \varepsilon_{zz,xx}^{(2)} \right) \\ -\frac{1}{2} \varepsilon_{zz,xx}^{(1)} \\ 2\varepsilon_{xz}^{(1)} \\ \varepsilon_{zz}^{(1)} \end{array} \right\} =$$

$$
= \begin{bmatrix} \frac{d}{dx} & 0 & -2z_2\frac{d}{dx} & -\frac{1}{2}z_2^2\frac{d^2}{dx^2} & 2z_2\frac{d}{dx} & \frac{1}{2}z_2^2\frac{d^2}{dx^2} & 0 & 0 \\ 0 & -\frac{d^2}{dx^2} & 2\frac{d}{dx} & z_2\frac{d^2}{dx^2} & 0 & -z_2\frac{d^2}{dx^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_0 \\ w_0 \\ \varepsilon_{xz}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ \varepsilon_{xz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{Bmatrix}, \tag{2.2.78}
$$

$$
\{f^{(2)}\} \equiv \begin{Bmatrix} \varphi_{xx0}^{(2)} \\ \varphi_{xx1}^{(2)} \\ \varphi_{xx2}^{(2)} \\ 2\varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{Bmatrix} \equiv \begin{Bmatrix} u_{0,x} \\ 2\varepsilon_{xz,x}^{(2)} - w_{0,xx} \\ -\frac{1}{2}\varepsilon_{zz,xx}^{(2)} \\ 2\varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{Bmatrix} =
$$

$$
= \begin{bmatrix} \frac{d}{dx} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{d^2}{dx^2} & 0 & 0 & 2\frac{d}{dx} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_0 \\ w_0 \\ \varepsilon_{xz}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ \varepsilon_{xz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{Bmatrix}, \tag{2.2.79}
$$

$$
\{f^{(3)}\} \equiv \begin{Bmatrix} \varphi_{xx0}^{(3)} \\ \varphi_{xx1}^{(3)} \\ \varphi_{xx2}^{(3)} \\ 2\varepsilon_{xz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{Bmatrix} \equiv \begin{Bmatrix} u_{0,x} + 2z_3\left(\varepsilon_{xz,x}^{(2)} - \varepsilon_{xz,x}^{(3)}\right) + \frac{1}{2}z_3^2\left(\varepsilon_{zz,xx}^{(2)} - \varepsilon_{zz,xx}^{(3)}\right) \\ 2\varepsilon_{xz,x}^{(3)} - w_{0,xx} + z_3\left(\varepsilon_{zz,xx}^{(3)} - \varepsilon_{zz,xx}^{(2)}\right) \\ -\frac{1}{2}\varepsilon_{zz,xx}^{(3)} \\ 2\varepsilon_{xz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{Bmatrix} =
$$

$$\begin{bmatrix} \frac{d}{dx} & 0 & 0 & 0 & 2z_3\frac{d}{dx} & \frac{1}{2}z_3^2\frac{d^2}{dx^2} & -2z_3\frac{d}{dx} & -\frac{1}{2}z_3^2\frac{d^2}{dx^2} \\ 0 & -\frac{d^2}{dx^2} & 0 & 0 & 0 & -z_3\frac{d^2}{dx^2} & 2\frac{d}{dx} & z_3\frac{d^2}{dx^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_0 \\ w_0 \\ \varepsilon_{xz}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ \varepsilon_{xz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{Bmatrix} . \tag{2.2.80}$$

Equations (2.2.78)-(2.2.80) can be written briefly in the form

$$\underset{(5\times1)}{\{f^{(1)}\}} = \underset{(5\times8)}{[\partial_1]}\,\underset{(8\times1)}{\{F\}} ,$$

$$\underset{(5\times1)}{\{f^{(2)}\}} = \underset{(5\times8)}{[\partial_2]}\,\underset{(8\times1)}{\{F\}} ,$$

$$\underset{(5\times1)}{\{f^{(3)}\}} = \underset{(5\times8)}{[\partial_3]}\,\underset{(8\times1)}{\{F\}} ,$$

or

$$\begin{Bmatrix} \underset{(5\times1)}{\{f^{(1)}\}} \\ \\ \underset{(5\times1)}{\{f^{(2)}\}} \\ \\ \underset{(5\times1)}{\{f^{(3)}\}} \end{Bmatrix} = \begin{bmatrix} \underset{(5\times8)}{[\partial_1]} \\ \\ \underset{(5\times8)}{[\partial_2]} \\ \\ \underset{(5\times8)}{[\partial_3]} \end{bmatrix} \underset{(8\times1)}{\{F\}} , \tag{2.2.81}$$

where

$$\underset{(8\times1)}{\{F\}} \equiv \begin{Bmatrix} u_0 \\ w_0 \\ \varepsilon_{xz}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ \varepsilon_{xz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{Bmatrix} , \tag{2.2.82}$$

is column-matrix of the unknown functions of the problem and

$$\underset{(5\times 8)}{[\partial_1]} \equiv \begin{bmatrix} \frac{d}{dx} & 0 & -2z_2\frac{d}{dx} & -\frac{1}{2}z_2^2\frac{d^2}{dx^2} & 2z_2\frac{d}{dx} & \frac{1}{2}z_2^2\frac{d^2}{dx^2} & 0 & 0 \\ 0 & -\frac{d^2}{dx^2} & 2\frac{d}{dx} & z_2\frac{d^2}{dx^2} & 0 & -z_2\frac{d^2}{dx^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.2.83}$$

$$\underset{(5\times 8)}{[\partial_2]} \equiv \begin{bmatrix} \frac{d}{dx} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{d^2}{dx^2} & 0 & 0 & 2\frac{d}{dx} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{2.2.84}$$

$$\underset{(5\times 8)}{[\partial_3]} \equiv \begin{bmatrix} \frac{d}{dx} & 0 & 0 & 0 & 2z_3\frac{d}{dx} & \frac{1}{2}z_3^2\frac{d^2}{dx^2} & -2z_3\frac{d}{dx} & -\frac{1}{2}z_3^2\frac{d^2}{dx^2} \\ 0 & -\frac{d^2}{dx^2} & 0 & 0 & 0 & -z_3\frac{d^2}{dx^2} & 2\frac{d}{dx} & z_3\frac{d^2}{dx^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.2.85}$$

Thus, from the notation (2.2.73)

$$\underset{(15\times 1)}{\{f\}} \equiv \begin{Bmatrix} \{f^{(1)}\} \\ \{f^{(2)}\} \\ \{f^{(3)}\} \end{Bmatrix}$$

and the notation

$$\underset{(15\times 8)}{[\partial]} \equiv \begin{bmatrix} \underset{(5\times 8)}{[\partial_1]} \\ \underset{(5\times 8)}{[\partial_2]} \\ \underset{(5\times 8)}{[\partial_3]} \end{bmatrix}, \tag{2.2.86}$$

One can write equation (2.2.81) in the form

$$\underset{(15\times1)}{\{f\}} = \underset{(15\times8)}{[\partial]}\underset{(8\times1)}{\{F\}}. \tag{2.2.87}$$

The substitution of expression (2.2.87) into expression (2.2.72) for the strain energy yields

$$U = \frac{1}{2}b\int_0^L \left(\underset{(15\times8)}{[\partial]}\underset{(8\times1)}{\{F\}}\right)^T \underset{(15\times15)}{[D]}\underset{(15\times8)}{[\partial]}\underset{(8\times1)}{\{F\}}\, dx. \tag{2.2.89}$$

Strain energy of a finite element is

$$\overline{U} = \frac{1}{2}b\int_{x_1}^{x_2} \left(\underset{(15\times8)}{[\partial]}\underset{(8\times1)}{\{F\}}\right)^T \underset{(15\times15)}{[D]}\underset{(15\times8)}{[\partial]}\underset{(8\times1)}{\{F\}}\, dx =$$

$$= \frac{1}{2}b\int_0^l \left(\underset{(15\times8)}{[\partial]}\underset{(8\times1)}{\{F\}}\right)^T \underset{(15\times15)}{[D]}\underset{(15\times8)}{[\partial]}\underset{(8\times1)}{\{F\}}\, d\overline{x}, \tag{2.2.90}$$

where x_1 and x_2 are coordinates of the end-points of a finite element in a global coordinate system; $\overline{x}$ is an x-coordinate in a local, element coordinate system (figure 2.4); $l = x_2 - x_1$ is a length of a finite element.

According to equation (2.2.77), virtual work of external forces, acting on a finite element of the plate, is

$$\overline{\delta' W} = \int_0^l \left[(q_l + q_u)\,\delta w_0 + (z_1 - z_2)\; q_l\; \delta\varepsilon_{zz}^{(1)} + (z_2\, q_l + z_3\, q_u)\;\delta\varepsilon_{zz}^{(2)} + (z_4 - z_3)\; q_u\; \delta\varepsilon_{zz}^{(3)}\right] d\overline{x} =$$

$$= \int_0^l \left\{\begin{array}{c} \delta u_0 \\ \delta w_0 \\ \delta\varepsilon_{xz}^{(1)} \\ \delta\varepsilon_{zz}^{(1)} \\ \delta\varepsilon_{xz}^{(2)} \\ \delta\varepsilon_{zz}^{(2)} \\ \delta\varepsilon_{xz}^{(3)} \\ \delta\varepsilon_{zz}^{(3)} \end{array}\right\}^T \left\{\begin{array}{c} 0 \\ q_l + q_u \\ 0 \\ (z_1 - z_2)q_l \\ 0 \\ (z_2 q_l + z_3 q_u) \\ 0 \\ (z_4 - z_3)q_u \end{array}\right\} d\overline{x} = \int_0^l \left(\delta\underset{(8\times1)}{\{F\}}\right)^T \underset{(8\times1)}{\{q\}}\, d\overline{x}\,, \tag{2.2.91}$$

where $\{F\}$ is defined by equation (2.2.82), and

$$\underset{(8\times1)}{\{q\}} \equiv \left\lfloor\; 0 \quad (q_l + q_u) \quad 0 \quad (z_1 - z_2)q_l \quad 0 \quad (z_2 q_l + z_3 q_u) \quad 0 \quad (z_4 - z_3)q_u \;\right\rfloor^T. \tag{2.2.92}$$

So, the principle of total potential energy for a finite element, $\delta\overline{U} - \overline{\delta' W} = 0$, takes the form

$$\frac{1}{2} b\, \delta \int_0^l \left(\underset{(15\times 8)}{[\partial]} \underset{(8\times 1)}{\{F\}} \right)^T \underset{(15\times 15)}{[D]} \underset{(15\times 8)}{[\partial]} \underset{(8\times 1)}{\{F\}}\, d\overline{x} - \int_0^l \left(\delta \underset{(8\times 1)}{\{F\}} \right)^T \underset{(8\times 1)}{\{q\}}\, d\overline{x} = 0. \tag{2.2.93}$$

Now, we need to represent the unknown functions $u_0, w_0, \varepsilon_{xz}^{(k)}, \varepsilon_{zz}^{(k)}$ by interpolation polynomials. The maximum order of derivatives of u_0 and of $\varepsilon_{xz}^{(k)}$ $(k = 1, 2, 3)$, entering into the virtual work principle (2.2.93), is 1, as observed from investigating equations (2.2.78)–(2.2.80). Therefore, interpolation polynomials for u_0 and $\varepsilon_{xz}^{(k)}$ must be of at least first degree, and across boundaries between elements there must be continuity of, at least, u_0 and $\varepsilon_{xz}^{(k)}$ (continuity of derivatives of u_0 and $\varepsilon_{xz}^{(k)}$ is not required). Therefore, we choose the first degree Lagrange polynomials to interpolate u_0 and $\varepsilon_{xz}^{(k)}$ $(k = 1, 2, 3)$ as functions of $\overline{x}$:

$$u_0 = \lfloor M \rfloor \{\bar{u}\} = \lfloor M_1\ M_2 \rfloor \{\bar{u}\}\ , \tag{2.2.94}$$

$$\varepsilon_{xz}^{(k)} = \lfloor M \rfloor \left\{ \bar{\epsilon}^{(k)} \right\} = \lfloor M_1\ M_2 \rfloor \left\{ \bar{\epsilon}^{(k)} \right\}\ , \tag{2.2.95}$$

where

$$M_1 = 1 - \frac{\overline{x}}{l},\ M_2 = \frac{\overline{x}}{l}, \tag{2.2.96}$$

$$\{\bar{u}\} = \left\{ \begin{array}{c} u_0(0) \\ u_0(l) \end{array} \right\}, \tag{2.2.97}$$

$$\left\{ \bar{\epsilon}^{(k)} \right\} = \left\{ \begin{array}{c} \varepsilon_{xz}^{(k)}(0) \\ \varepsilon_{xz}^{(k)}(l) \end{array} \right\}. \tag{2.2.98}$$

In the same fashion, the maximum order of derivatives of w_0 and $\varepsilon_{zz}^{(k)}$ is 2. Therefore, interpolation polynomials for w_0 and $\varepsilon_{zz}^{(k)}$ must be of at least second degree and must have derivatives, continuous at the element boundaries up to the first order (i.e. w_0, $\frac{dw_0}{dx}, \varepsilon_{zz}^{(k)}$ and $\frac{d\varepsilon_{zz}^{(k)}}{dx}$ must be continuous). Therefore, we choose the Hermit polynomial of the third degree to interpolate w_0 and $\varepsilon_{zz}^{(k)}$ (the lowest degree of the Hermit polynomials is three):

$$w_0 = \lfloor N \rfloor \{\bar{w}\} = \lfloor N_1\ N_2\ N_3\ N_4 \rfloor \{\bar{w}\}\,, \tag{2.2.99}$$

$$\varepsilon_{zz}^{(k)} = \lfloor N \rfloor \left\{ \bar{\varepsilon}^{(k)} \right\} = \lfloor N_1\ N_2\ N_3\ N_4 \rfloor \left\{ \bar{\varepsilon}^{(k)} \right\}, \tag{2.2.100}$$

where

$$N_1 = 1 - \frac{3\overline{x}^2}{l^2} + \frac{2\overline{x}^3}{l^3},\ N_2 = \overline{x} - \frac{2\overline{x}^2}{l} + \frac{\overline{x}^3}{l^2},\ N_3 = \frac{3\overline{x}^2}{l^2} - \frac{2\overline{x}^3}{l^3},\ N_4 = -\frac{\overline{x}^2}{l} + \frac{\overline{x}^3}{l^2}, \tag{2.2.101}$$

$$\{\bar{w}\} = \begin{Bmatrix} w_0(0) \\ w_0'(0) \\ w_0(l) \\ w_0'(l) \end{Bmatrix}, \tag{2.2.102}$$

$$\left\{\bar{\varepsilon}^{(k)}\right\} = \begin{Bmatrix} \varepsilon_{zz}^{(k)}(0) \\ \frac{d\varepsilon_{zz}^{(k)}}{dx}(0) \\ \varepsilon_{zz}^{(k)}(l) \\ \frac{d\varepsilon_{zz}^{(k)}}{dx}(l) \end{Bmatrix}. \tag{2.2.103}$$

The column-matrix $\{F\}$ of the unknown functions of the problem, defined by equation (82), now can be written in the form:

$$\{F\} \equiv \begin{Bmatrix} u_0 \\ w_0 \\ \varepsilon_{xz}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ \varepsilon_{xz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{Bmatrix} = \begin{Bmatrix} \lfloor M \rfloor \{\bar{u}\} \\ \lfloor N \rfloor \{\bar{w}\} \\ \lfloor M \rfloor \left\{\bar{\varepsilon}^{(1)}\right\} \\ \lfloor N \rfloor \left\{\bar{\varepsilon}^{(1)}\right\} \\ \lfloor M \rfloor \left\{\bar{\varepsilon}^{(2)}\right\} \\ \lfloor N \rfloor \left\{\bar{\varepsilon}^{(2)}\right\} \\ \lfloor M \rfloor \left\{\bar{\varepsilon}^{(3)}\right\} \\ \lfloor N \rfloor \left\{\bar{\varepsilon}^{(3)}\right\} \end{Bmatrix} =$$

$$= \begin{bmatrix} \lfloor M \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor N \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor M \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor N \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor M \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor N \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor M \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor N \rfloor \end{bmatrix} \begin{Bmatrix} \{\overline{u}\} \\ \{\overline{w}\} \\ \left\{\bar{\varepsilon}^{(1)}\right\} \\ \left\{\bar{\varepsilon}^{(1)}\right\} \\ \left\{\bar{\varepsilon}^{(2)}\right\} \\ \left\{\bar{\varepsilon}^{(2)}\right\} \\ \left\{\bar{\varepsilon}^{(3)}\right\} \\ \left\{\bar{\varepsilon}^{(3)}\right\} \end{Bmatrix}, \tag{2.2.104}$$

or

$$\underset{(8\times1)}{\{F\}} = \underset{(8\times24)}{[Q]}\underset{(24\times1)}{\{d\}} \,, \tag{2.2.105}$$

where

$$\underset{(8\times24)}{[Q]} \equiv \begin{bmatrix} \underset{(1\times2)}{\lfloor M \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \underset{(1\times4)}{\lfloor N \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times2)}{\lfloor M \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times4)}{\lfloor N \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times2)}{\lfloor M \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times4)}{\lfloor N \rfloor} & \lfloor 0 \rfloor & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times2)}{\lfloor M \rfloor} & \lfloor 0 \rfloor \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \lfloor 0 \rfloor & \underset{(1\times4)}{\lfloor N \rfloor} \end{bmatrix} \tag{2.2.106}$$

is a matrix of shape functions, and

$$\underset{(24\times1)}{\{d\}} \equiv \left\{ \begin{array}{c} \underset{(2\times1)}{\{\overline{u}\}} \\ \\ \underset{(4\times1)}{\{\overline{w}\}} \\ \\ \underset{(2\times1)}{\left\{\overline{\epsilon}^{(1)}\right\}} \\ \\ \underset{(4\times1)}{\left\{\overline{\varepsilon}^{(1)}\right\}} \\ \\ \underset{(2\times1)}{\left\{\overline{\epsilon}^{(2)}\right\}} \\ \\ \underset{(4\times1)}{\left\{\overline{\varepsilon}^{(2)}\right\}} \\ \underset{(2\times1)}{\left\{\overline{\epsilon}^{(3)}\right\}} \\ \\ \underset{(4\times1)}{\left\{\overline{\varepsilon}^{(3)}\right\}} \end{array} \right\} = \left\{ \begin{array}{c} d_1 \\ d_2 \\ . \\ . \\ . \\ d_{24} \end{array} \right\} \tag{2.2.107}$$

is a vector of nodal degrees of freedom of an element. In equation (2.2.107)

$$d_1 = u_0\,(0)\,,\ d_2 = u_0\,(l)\,,\ d_3 = w_0\,(0)\,,\ d_4 = w_0^{'}\,(0)\,,\ d_5 = w_0\,(l)\,,\ d_6 = w_0^{'}\,(l)\,,$$

$$d_7 = \varepsilon_{xz}^{(1)}\,(0)\,,\ d_8 = \varepsilon_{xz}^{(1)}\,(l)\,,\ d_9 = \varepsilon_{zz}^{(1)}\,(0)\,,\ d_{10} = \frac{d\varepsilon_{zz}^{(1)}}{dx}\,(0)\,,\ d_{11} = \varepsilon_{zz}^{(1)}\,(l)\,,$$

$$d_{12} = \frac{d\varepsilon_{zz}^{(1)}}{dx}\,(l)\,,\ d_{13} = \varepsilon_{xz}^{(2)}\,(0)\,,\ d_{14} = \varepsilon_{xz}^{(2)}\,(l)\,,\ d_{15} = \varepsilon_{zz}^{(2)}\,(0)\,,\ d_{16} = \frac{d\varepsilon_{zz}^{(2)}}{dx}\,(0)\,,$$

$$d_{17} = \varepsilon_{zz}^{(2)}\,(l)\,,\ d_{18} = \frac{d\varepsilon_{zz}^{(2)}}{dx}\,(l)\,,\ d_{19} = \varepsilon_{xz}^{(3)}\,(0)\,,\ d_{20} = \varepsilon_{xz}^{(3)}\,(l)\,,\ d_{21} = \varepsilon_{zz}^{(3)}\,(0)\,,$$

$$d_{22} = \frac{d\varepsilon_{zz}^{(3)}}{dx}\,(0)\,,\ d_{23} = \varepsilon_{zz}^{(3)}\,(l)\,,\ d_{24} = \frac{d\varepsilon_{zz}^{(3)}}{dx}\,(l)\ . \tag{2.2.108}$$

These are the **nodal degrees of freedom of an element**.

Let us write expression (2.2.90) for the strain energy of a finite element in terms of the nodal degrees of freedom:

$$U=\frac{1}{2}b\int_0^l\left(\underset{(15\times8)}{[\partial]}\ \underset{(8\times1)}{\{F(\overline{x})\}}\right)^T\ \underset{(15\times15)}{[D]}\underset{(15\times8)}{[\partial]}\ \underset{(8\times1)}{\{F(\overline{x})\}}\ d\overline{x}=$$

$$=\frac{1}{2}b\int_0^l\left(\underset{(15\times8)}{[\partial]}\ \underset{(8\times24)}{[Q(\overline{x})]}\ \underset{(24\times1)}{\{d\}}\right)^T\ \underset{(15\times15)}{[D]}\underset{(15\times8)}{[\partial]}\ \underset{(8\times24)}{[Q(\overline{x})]}\ \underset{(24\times1)}{\{d\}}\ d\overline{x}=$$

$$=\frac{1}{2}b\int_0^l\underset{(1\times24)}{\{d\}^T}\underset{(24\times15)}{\left(\underset{(15\times8)}{[\partial]}\ \underset{(8\times24)}{[Q(\overline{x})]}\right)^T}\ \underset{(15\times15)}{[D]}\underset{(15\times8)}{[\partial]}\ \underset{(8\times24)}{[Q(\overline{x})]}\ \underset{(24\times1)}{\{d\}}\ d\overline{x}=$$

$$=\frac{1}{2}\underset{(1\times24)}{\{d\}^T}\left(b\int_0^l\underset{(24\times15)}{\left(\underset{(15\times8)}{[\partial]}\ \underset{(8\times24)}{[Q(\overline{x})]}\right)^T}\ \underset{(15\times15)}{[D]}\underset{(15\times8)}{[\partial]}\ \underset{(8\times24)}{[Q(\overline{x})]}\ d\overline{x}\right)\underset{(24\times1)}{\{d\}}\ ,$$

or

$$\overline{U}=\frac{1}{2}\underset{(1\times24)}{\{d\}^T}\ \underset{(24\times24)}{\left[\widetilde{k}\right]}\ \underset{(24\times1)}{\{d\}}\ , \tag{2.2.109}$$

where

$$\underset{(24\times24)}{\left[\widetilde{k}\right]}=b\int_0^l\underset{(24\times15)}{\left(\underset{(15\times8)}{[\partial]}\ \underset{(8\times24)}{[Q(\overline{x})]}\right)^T}\ \underset{(15\times15)}{[D]}\underset{(15\times8)}{[\partial]}\ \underset{(8\times24)}{[Q(\overline{x})]}\ d\overline{x}\ . \tag{2.2.110}$$

Let us write expression (2.2.91) for the virtual work of external forces, acting on a finite element of the plate, in terms of variations of the nodal degrees of freedom:

$$\overline{\delta' W}=\int_0^l\left(\delta\underset{(8\times1)}{\{F(x)\}}\right)^T\ \underset{(8\times1)}{\{q(\overline{x})\}}\ d\overline{x}\ =\int_0^l\left(\underset{(8\times24)}{[Q(\overline{x})]}\ \underset{(24\times1)}{\delta\{d\}}\right)^T\ \underset{(8\times1)}{\{q(x)\}}\ d\overline{x}=$$

$$=\ \underset{(1\times24)}{\delta\{d\}^T}\int_0^l\underset{(24\times8)}{[Q(\overline{x})]^T}\underset{(8\times1)}{\{q(\overline{x})\}}\ d\overline{x}\ ,$$

or

$$\overline{\delta' W}=\ \underset{(1\times24)}{\delta\{d\}^T}\ \underset{(24\times1)}{\{r\}}\ , \tag{2.2.111}$$

where

$$\underset{(24\times1)}{\{\widetilde{r}\}}\ =\int_0^l\underset{(24\times8)}{[Q(\overline{x})]^T}\underset{(8\times1)}{\{q(\overline{x})\}}\ d\overline{x}\ . \tag{2.2.112}$$

Let us substitute expressions (2.2.109) and (2.2.111) into the principle of total potential energy for a finite element, $\delta\overline{U} - \overline{\delta' W} = 0$:

$$0 = \delta\left(\frac{1}{2}\underset{(1\times24)}{\{d\}^T}\underset{(24\times24)}{\left[\widetilde{k}\right]}\underset{(24\times1)}{\{d\}}\right) - \underset{(1\times24)}{\delta\{d\}^T}\underset{(24\times1)}{\{r\}} =$$

$$= \frac{1}{2}\left(\delta\{d\}^T\right)\left[\widetilde{k}\right]\{d\} + \frac{1}{2}\{d\}^T\left[\widetilde{k}\right]\delta\{d\} - \left(\delta\{d\}^T\right)\{r\} \ . \tag{2.2.113}$$

But

$$\left(\delta\{d\}^T\right)\left[\widetilde{k}\right]\{d\} = \{d\}^T\left[\widetilde{k}\right]\delta\{d\} \ ,$$

therefore, equation (2.2.113) takes the form

$$\left(\delta\{d\}^T\right)\left(\left[\widetilde{k}\right]\{d\} - \{r\}\right) = 0 \ ,$$

or

$$\underset{(24\times24)}{\left[\widetilde{k}\right]}\underset{(24\times1)}{\{d\}} = \underset{(24\times1)}{\{\widetilde{r}\}} \ . \tag{2.2.114}$$

This is equilibrium equation for a finite element in terms of the nodal degrees of freedom. For convenience of representation of a load, acting on a wide plate in cylindrical bending, let us divide the left-hand and the right-hand sides of equation (2.2.114) by b:

$$\frac{1}{b}\underset{(24\times24)}{\left[\widetilde{k}\right]}\underset{(24\times1)}{\{d\}} = \frac{1}{b}\underset{(24\times1)}{\{\widetilde{r}\}} \ ,$$

or

$$\boxed{\underset{(24\times24)}{[k]}\underset{(24\times1)}{\{d\}} = \underset{(24\times1)}{\{r\}}} \ , \tag{2.2.115}$$

where

$$\boxed{\underset{(24\times24)}{[k]} = \frac{1}{b}\underset{(24\times24)}{\left[\widetilde{k}\right]} = \int_0^l \underset{(24\times15)}{\left(\underset{(15\times8)}{[\partial]}\underset{(8\times24)}{[Q(\overline{x})]}\right)^T}\underset{(15\times15)}{[D]}\underset{(15\times8)}{[\partial]}\underset{(8\times24)}{[Q(\overline{x})]}\,d\overline{x}} \ , \tag{2.2.116}$$

$$\boxed{\underset{(24\times1)}{\{r\}} = \frac{1}{b}\underset{(24\times1)}{\{\widetilde{r}\}} = \frac{1}{b}\int_0^l \underset{(24\times8)}{[Q(\overline{x})]^T}\underset{(8\times1)}{\{q(\overline{x})\}}\,d\overline{x}} \ . \tag{2.2.117}$$

Matrices $\underset{(24\times 24)}{[k]}$ and $\underset{(24\times 1)}{\{r\}}$ are the **stiffness matrix and load vector** of a finite element. In equations (2.2.116) and (2.2.117) matrix $[\partial]$ is defined by equation (2.2.86), matrix $[Q]$-by equation (2.2.106), matrix $[D]$-by equation (2.2.74), matrix $\{q\}$-by equation (2.2.92).

The components of the element stiffness matrix were computed analytically, with the help of a program for symbolic computation. Some components of the stiffness matrix are shown in Appendix 2-B.

Second forms of expressions for the transverse stresses in terms of u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$

After computing the unknown functions $u_0(x)$, $w_0(x)$, $\varepsilon_{xz}^{(k)}(x)$, $\varepsilon_{zz}^{(k)}(x)$ $(k = 1,2,3)$ as a result of solving the finite element equations, we can find displacements, strains and stresses in the plate as functions of x- and z-coordinates (there is no dependence on the y-coordinate because we consider cylindrical bending). The displacements can be computed by formulas (2.2.34)-(2.2.36) and (2.2.41)-(2.2.42), the in-plane strains $\varepsilon_{xx}^{(1)}$, $\varepsilon_{xx}^{(2)}$, $\varepsilon_{xx}^{(3)}$ - by formulas (2.2.47)-(2.2.49), the in-plane stresses $\sigma_{xx}^{(1)}$, $\sigma_{xx}^{(2)}$, $\sigma_{xx}^{(3)}$ - by formulas (2.2.67). The first forms of expressions for the transverse stresses in terms of $u_0(x)$, $w_0(x)$, $\varepsilon_{xz}^{(k)}(x)$, $\varepsilon_{zz}^{(k)}(x)$ (equations (2.2.67)), i.e. expressions for the transverse stresses obtained from the constitutive relations, were used only for the purpose of expressing the strain energy in terms of the unknown functions, which was used for the finite element formulation and can also be used for deriving differential equilibrium equations in terms of the unknown functions. In order to compute the transverse stresses, we will use the second forms of expressions for the transverse stresses in terms of $u_0(x)$, $w_0(x)$, $\varepsilon_{xz}^{(k)}(x)$, $\varepsilon_{zz}^{(k)}(x)$ (denoted as $\sigma_{xz}^{(k)} \equiv \left(\sigma_{xz}^{(k)}\right)^{(II)}$, $\sigma_{zz}^{(k)} \equiv \left(\sigma_{zz}^{(k)}\right)^{(II)}$), obtained from the equilibrium equations (2.2.1) and (2.2.2). As it was mentioned previously, the second forms of the transverse stresses are more accurate than the first forms.

First, let us write expressions (2.2.67) for the in-plane stresses ${}^{H}\sigma_{xx}^{(1)}$, ${}^{H}\sigma_{xx}^{(2)}$, ${}^{H}\sigma_{xx}^{(3)}$ in expanded form:

$$
{}^{H}\sigma_{xx}^{(1)} = \frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[u_{0,x} + 2z_2\left(\varepsilon_{xz,x}^{(2)} - \varepsilon_{xz,x}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,xx}^{(2)} - \varepsilon_{zz,xx}^{(1)}\right)\right] +
$$

$$
+\frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}}\varepsilon_{zz}^{(1)} +
$$

$$
+\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[2\varepsilon_{xz,x}^{(1)} - w_{0,xx} + z_2\left(\varepsilon_{zz,xx}^{(1)} - \varepsilon_{zz,xx}^{(2)}\right)\right]\ z -
$$

$$\frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\varepsilon^{(1)}_{zz,xx}\ z^2\ , \tag{2.2.118}$$

$$^{H}\sigma^{(2)}_{xx}=\frac{E^{(2)}}{1+\nu^{(2)}}\frac{1-\nu^{(2)}}{1-2\nu^{(2)}}\ u_{0,x}+\frac{E^{(2)}}{1+\nu}\frac{\nu^{(2)}}{1-2\nu^{(2)}}\ \varepsilon^{(2)}_{zz}+$$

$$+\frac{E^{(2)}}{1+\nu^{(2)}}\frac{1-\nu^{(2)}}{1-2\nu^{(2)}}\left(2\varepsilon^{(2)}_{xz,x}-w_{0,xx}\right)z-\frac{1}{2}\frac{E^{(2)}}{1+\nu^{(2)}}\frac{1-\nu^{(2)}}{1-2\nu^{(2)}}\varepsilon^{(2)}_{zz,xx}\ z^2\ , \tag{2.2.119}$$

$$^{H}\sigma^{(3)}_{xx}=\frac{E^{(3)}}{1+\nu^{(3)}}\frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\left[u_{0,x}+2z_3\left(\varepsilon^{(2)}_{xz,x}-\varepsilon^{(3)}_{xz,x}\right)+\frac{1}{2}z_3^2\left(\varepsilon^{(2)}_{zz,xx}-\varepsilon^{(3)}_{zz,xx}\right)\right]+$$

$$+\frac{E^{(3)}}{1+\nu^{(3)}}\frac{\nu^{(3)}}{1-2\nu^{(3)}}\varepsilon^{(3)}_{zz}+$$

$$+\frac{E^{(3)}}{1+\nu^{(3)}}\frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\left[2\varepsilon^{(3)}_{xz,x}-w_{0,xx}+z_3\left(\varepsilon^{(3)}_{zz,xx}-\varepsilon^{(2)}_{zz,xx}\right)\right]\ z-$$

$$\frac{1}{2}\frac{E^{(3)}}{1+\nu^{(3)}}\frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\varepsilon^{(3)}_{zz,xx}\ z^2\ . \tag{2.2.120}$$

Now, let us find expressions for $\sigma^{(k)}_{xz,z}$ and $\sigma^{(k)}_{zz,z}$ by integration of equilibrium equations (2.2.1) and (2.2.2). Performing integration of the first equilibrium equation for the lower face sheet of the sandwich plate (k=1),

$$\sigma^{(1)}_{xx,x}+\sigma^{(1)}_{xz,z}=0,$$

with respect to z in the direction from the lower surface of the plate to its upper surface, we receive

$$\sigma^{(1)}_{xz}=\underbrace{\sigma^{(1)}_{xz}\Big|_{z=z_1}}_{0}-\int_{z_1}^{z}{}^{H}\sigma^{(1)}_{xx,x}\ dz\qquad(z_1\le z\le z_2)\ , \tag{2.2.121}$$

where $\sigma^{(1)}_{xz}\Big|_{z=z_1}=0$ due to the first boundary condition (2.2.20). From (2.2.121) it follows that

$$\sigma^{(1)}_{xz}\Big|_{z=z_2}=-\int_{z_1}^{z_2}{}^{H}\sigma^{(1)}_{xx,x}\ dz\ . \tag{2.2.122}$$

Integration of the first equilibrium equation for the core of the sandwich plate (k=2),

$$\sigma^{(2)}_{xx,x} + \sigma^{(2)}_{xz,z} = 0 \ ,$$

from z_2 to z, where $z_2 \le z \le z_3$, yields

$$\sigma^{(2)}_{xz} = \sigma^{(2)}_{xz}\Big|_{z=z_2} - \int_{z_2}^{z} {}^{H}\sigma^{(2)}_{xx,x}\, dz \ . \tag{2.2.123}$$

According to the continuity conditions (2.2.23) between the plies with different material properties and according to equation (2.2.122), we have

$$\sigma^{(2)}_{xz}\Big|_{z=z_2} = \sigma^{(1)}_{xz}\Big|_{z=z_2} = -\int_{z_1}^{z_2} {}^{H}\sigma^{(1)}_{xx,x}\, dz \ . \tag{2.2.124}$$

Substitution of (2.2.124) into (2.2.123) yields:

$$\sigma^{(2)}_{xz} = -\int_{z_1}^{z_2} {}^{H}\sigma^{(1)}_{xx,x}\, dz - \int_{z_2}^{z} {}^{H}\sigma^{(2)}_{xx,x}\, dz \qquad (z_2 \le z \le z_3) \ . \tag{2.2.125}$$

For the upper face sheet (k=3) we receive analogously

$$\sigma^{(3)}_{xz} = -\int_{z_1}^{z_2} {}^{H}\sigma^{(1)}_{xx,x}\, dz - \int_{z_2}^{z_3} {}^{H}\sigma^{(2)}_{xx,x}\, dz - \int_{z_3}^{z} {}^{H}\sigma^{(3)}_{xx,x}\, dz \qquad (z_3 \le z \le z_4) \ . \tag{2.2.126}$$

Substitution of expressions (2.2.118)–(2.2.120) into expressions (2.2.121), (2.2.125) and (2.2.126) yields the required second forms of expressions for the transverse stresses $\sigma^{(k)}_{xz}$ in terms of the functions u_0, w_0, $\varepsilon^{(k)}_{xz}$, $\varepsilon^{(k)}_{zz}$:

$$\left(\sigma^{(1)}_{xz}\right)^{(II)} \equiv \sigma^{(1)}_{xz} =$$

$$= \frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[u_{0,xx} + 2z_2\left(\varepsilon^{(2)}_{xz,xx} - \varepsilon^{(1)}_{xz,xx}\right) + \frac{1}{2}z_2^2\left(\varepsilon^{(2)}_{zz,xxx} - \varepsilon^{(1)}_{zz,xxx}\right)\right](z_1 - z) +$$

$$+\frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}}\varepsilon^{(1)}_{zz,x}(z_1 - z) +$$

$$+\frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[2\varepsilon^{(1)}_{xz,xx} - w_{0,xxx} + z_2\left(\varepsilon^{(1)}_{zz,xxx} - \varepsilon^{(2)}_{zz,xxx}\right)\right]\left(z_1^2 - z^2\right) -$$

$$\frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\varepsilon^{(1)}_{zz,xxx}\left(z_1^3-z^3\right) \ , \tag{2.2.127}$$

$$\left(\sigma^{(2)}_{xz}\right)^{(II)} \equiv \sigma^{(2)}_{xz} =$$

$$= \frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[u_{0,xx}+2z_2\left(\varepsilon^{(2)}_{xz,xx}-\varepsilon^{(1)}_{xz,xx}\right)+\frac{1}{2}z_2^2\left(\varepsilon^{(2)}_{zz,xxx}-\varepsilon^{(1)}_{zz,xxx}\right)\right]\ (z_1-z_2)+$$

$$+\frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}}\varepsilon^{(1)}_{zz,x}\left(z_1-z_2\right)+$$

$$+\frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[2\varepsilon^{(1)}_{xz,xx}-w_{0,xxx}+z_2\left(\varepsilon^{(1)}_{zz,xxx}-\varepsilon^{(2)}_{zz,xxx}\right)\right]\left(z_1^2-z_2^2\right)-$$

$$\frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\varepsilon^{(1)}_{zz,xxx}\left(z_1^3-z_2^3\right)+$$

$$+\frac{E^{(2)}}{\left(1-2\nu^{(2)}\right)\left(1+\nu^{(2)}\right)}\left[\left(1-\nu^{(2)}\right)\ u_{0,xx}+\nu^{(2)}\ \varepsilon^{(2)}_{zz,x}\right]\left(z_2-z\right)+$$

$$+\frac{1}{2}\frac{E^{(2)}}{1+\nu^{(2)}}\frac{1-\nu^{(2)}}{1-2\nu^{(2)}}\left[\left(2\varepsilon^{(2)}_{xz,xx}-w_{0,xxx}\right)\left(z_2^2-z^2\right)-\frac{1}{3}\varepsilon^{(2)}_{zz,xxx}\ \left(z_2^3-z^3\right)\right]\ , \tag{2.2.128}$$

$$\left(\sigma^{(3)}_{xz}\right)^{(II)} \equiv \sigma^{(3)}_{xz} =$$

$$
\begin{aligned}
&= \frac{E^{(1)}}{1+\nu^{(1)}} \frac{1-\nu^{(1)}}{1-2\nu^{(1)}} \left[u_{0,xx} + 2z_2 \left(\varepsilon^{(2)}_{xz,xx} - \varepsilon^{(1)}_{xz,xx} \right) + \frac{1}{2} z_2^2 \left(\varepsilon^{(2)}_{zz,xxx} - \varepsilon^{(1)}_{zz,xxx} \right) \right] (z_1 - z_2) + \\
&+ \frac{E^{(1)}}{1+\nu^{(1)}} \frac{\nu^{(1)}}{1-2\nu^{(1)}} \varepsilon^{(1)}_{zz,x} (z_1 - z_2) + \\
&+ \frac{1}{2} \frac{E^{(1)}}{1+\nu^{(1)}} \frac{1-\nu^{(1)}}{1-2\nu^{(1)}} \left[2\varepsilon^{(1)}_{xz,xx} - w_{0,xxx} + z_2 \left(\varepsilon^{(1)}_{zz,xxx} - \varepsilon^{(2)}_{zz,xxx} \right) \right] \left(z_1^2 - z_2^2 \right) - \\
&\frac{1}{6} \frac{E^{(1)}}{1+\nu^{(1)}} \frac{1-\nu^{(1)}}{1-2\nu^{(1)}} \varepsilon^{(1)}_{zz,xxx} \left(z_1^3 - z_2^3 \right) + \\
&+ \frac{E^{(2)}}{\left(1-2\nu^{(2)}\right)\left(1+\nu^{(2)}\right)} \left[\left(1 - \nu^{(2)} \right) u_{0,xx} + \nu^{(2)} \varepsilon^{(2)}_{zz,x} \right] (z_2 - z_3) + \\
&+ \frac{1}{2} \frac{E^{(2)}}{1+\nu^{(2)}} \frac{1-\nu^{(2)}}{1-2\nu^{(2)}} \left[\left(2\varepsilon^{(2)}_{xz,xx} - w_{0,xxx} \right) \left(z_2^2 - z_3^2 \right) - \frac{1}{3} \varepsilon^{(2)}_{zz,xxx} \left(z_2^3 - z_3^3 \right) \right] + \\
&+ \frac{E^{(3)}}{1+\nu^{(3)}} \frac{1-\nu^{(3)}}{1-2\nu^{(3)}} \left[u_{0,xx} + 2z_3 \left(\varepsilon^{(2)}_{xz,xx} - \varepsilon^{(3)}_{xz,xx} \right) + \frac{1}{2} z_3^2 \left(\varepsilon^{(2)}_{zz,xxx} - \varepsilon^{(3)}_{zz,xxx} \right) \right] (z_3 - z) + \\
&+ \frac{E^{(3)}}{1+\nu^{(3)}} \frac{\nu^{(3)}}{1-2\nu^{(3)}} \varepsilon^{(3)}_{zz,x} (z_3 - z) + \\
&+ \frac{1}{2} \frac{E^{(3)}}{1+\nu^{(3)}} \frac{1-\nu^{(3)}}{1-2\nu^{(3)}} \left[2\varepsilon^{(3)}_{xz,xx} - w_{0,xxx} + z_3 \left(\varepsilon^{(3)}_{zz,xxx} - \varepsilon^{(2)}_{zz,xxx} \right) \right] \left(z_3^2 - z^2 \right) \\
&- \frac{1}{6} \frac{E^{(3)}}{1+\nu^{(3)}} \frac{1-\nu^{(3)}}{1-2\nu^{(3)}} \varepsilon^{(3)}_{zz,xxx} \left(z_3^3 - z^3 \right) . \qquad (2.2.129)
\end{aligned}
$$

Integration of equilibrium equations

$$\sigma^{(k)}_{xz,x} + \sigma^{(k)}_{zz,z} = 0 \qquad (k = 1, 2, 3)$$

yields

$$\sigma^{(1)}_{zz} = \underbrace{\sigma^{(1)}_{zz}\Big|_{z=z_1}}_{-\frac{q_l}{b}} - \int_{z_1}^{z} \sigma^{(1)}_{xz,x} \, dz \qquad (z_1 \le z \le z_2) \ , \tag{2.2.130}$$

where $\sigma^{(1)}_{zz}\Big|_{z=z_1} = -\frac{q_l}{b}$ due to a boundary condition (2.2.20),

$$\sigma^{(2)}_{zz} = -\frac{q_l}{b} - \int_{z_1}^{z_2} \sigma^{(1)}_{xz,x} \, dz - \int_{z_2}^{z} \sigma^{(2)}_{xz,x} \, dz \qquad (z_2 \le z \le z_3) \ , \tag{2.2.131}$$

$$\sigma^{(3)}_{zz} = -\frac{q_l}{b} - \int_{z_1}^{z_2} \sigma^{(1)}_{xz,x} \, dz - \int_{z_2}^{z_3} \sigma^{(2)}_{xz,x} \, dz - \int_{z_3}^{z} \sigma^{(3)}_{xz,x} \, dz \qquad (z_3 \le z \le z_4) \ . \tag{2.2.132}$$

Substitution of (2.2.127)-(2.2.129) into (2.2.130)-(2.2.132) yields

$$\left(\sigma_{zz}^{(1)}\right)^{(II)} \equiv \sigma_{zz}^{(1)} = -\frac{q_l}{b} +$$
$$\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left(z-z_1\right)^2\left[\frac{d^3u_0}{dx^3} + 2z_2\left(\frac{d^3\varepsilon_{xz}^{(2)}}{dx^3} - \frac{d^3\varepsilon_{xz}^{(1)}}{dx^3}\right) + \frac{1}{4}z_2^2\left(\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4} - \frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}\right)\right] +$$
$$\frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}}\left(z-z_1\right)^2\frac{d^2\varepsilon_{zz}^{(1)}}{dx^2} +$$
$$\frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left(z+2z_1\right)\left(z-z_1\right)^2\left[2\frac{d^3\varepsilon_{xz}^{(1)}}{dx^3} - \frac{d^4w_0}{dx^4} + z_2\left(\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4} - \frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}\right)\right] -$$
$$\frac{1}{24}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left(z^2+2z_1z+3z_1^2\right)\left(z-z_1\right)^2\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}, \tag{2.2.133}$$

$$\left(\sigma_{zz}^{(2)}\right)^{(II)} \equiv \sigma_{zz}^{(2)} = -\frac{q_l}{b} +$$
$$\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\ (z_2-z_1)^2\left[\frac{d^3u_0}{dx^3}+2z_2\left(\frac{d^3\varepsilon_{xz}^{(2)}}{dx^3}-\frac{d^3\varepsilon_{xz}^{(1)}}{dx^3}\right)+\frac{1}{4}z_2^2\left(\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}\right)\right]+$$
$$\frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}}(z_2-z_1)^2\frac{d^2\varepsilon_{zz}^{(1)}}{dx^2}+$$
$$\frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}(z_2+2z_1)(z_2-z_1)^2\left[2\frac{d^3\varepsilon_{xz}^{(1)}}{dx^3}-\frac{d^4w_0}{dx^4}+z_2\left(\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}\right)\right]-$$
$$\frac{1}{24}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left(z_2^2+2z_1z_2+3z_1^2\right)(z_2-z_1)^2\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}+$$
$$+\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[\frac{d^3u_0}{dx^3}+2z_2\left(\varepsilon_{xz,xx}^{(2)}-\varepsilon_{xz,xx}^{(1)}\right)+\right.$$
$$\left.\frac{1}{2}z_2^2\left(\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}\right)\right]\ (z_1-z_2)(z_2-z)+$$
$$\frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}}\frac{d^2\varepsilon_{zz}^{(1)}}{dx^2}(z_1-z_2)(z_2-z)+$$
$$\frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[2\frac{d^3\varepsilon_{xz}^{(1)}}{dx^3}-\frac{d^4w_0}{dx^4}+z_2\left(\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}\right)\right]\left(z_1^2-z_2^2\right)(z_2-z)-$$
$$\frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}\left(z_1^3-z_2^3\right)(z_2-z)+$$
$$\frac{E^{(2)}}{\left(1-2\nu^{(2)}\right)\left(1+\nu^{(2)}\right)}\left[\left(1-\nu^{(2)}\right)\frac{d^3u_0}{dx^3}+\nu^{(2)}\ \frac{d^2\varepsilon_{zz}^{(2)}}{dx^2}\right]\frac{1}{2}(z_2-z)^2+$$
$$\frac{1}{2}\frac{E^{(2)}}{1+\nu^{(2)}}\frac{1-\nu^{(2)}}{1-2\nu^{(2)}}\left[\left(2\frac{d^3\varepsilon_{xz}^{(2)}}{dx^3}-\frac{d^4w_0}{dx^4}\right)\frac{1}{3}(2z_2+z)(z_2-z)^2-\right.$$
$$\left.\frac{1}{3}\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}\ \frac{1}{4}\left(3z_2^2+2z_2z+z^2\right)(z_2-z)^2\right]\ , \qquad (2.2.134)$$

$$\left(\sigma_{zz}^{(3)}\right)^{(II)} \equiv \sigma_{zz}^{(3)} = -\frac{q_{\iota}}{b} +$$

$$\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left(z_2-z_1\right)^2\left[\frac{d^3u_0}{dx^3}+2z_2\left(\frac{d^3\varepsilon_{xz}^{(2)}}{dx^3}-\frac{d^3\varepsilon_{xz}^{(1)}}{dx^3}\right)+\frac{1}{4}z_2^2\left(\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}\right)\right]+$$

$$\frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}}\left(z_2-z_1\right)^2\frac{d^2\varepsilon_{zz}^{(1)}}{dx^2}+$$

$$\frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left(z_2+2z_1\right)\left(z_2-z_1\right)^2\left[2\frac{d^3\varepsilon_{xz}^{(1)}}{dx^3}-\frac{d^4w_0}{dx^4}+z_2\left(\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}\right)\right]-$$

$$\frac{1}{24}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left(z_2^2+2z_1z_2+3z_1^2\right)\left(z_2-z_1\right)^2\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}+$$

$$+\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[\frac{d^3u_0}{dx^3}+2z_2\left(\varepsilon_{xz,xx}^{(2)}-\varepsilon_{xz,xx}^{(1)}\right)+\right.$$

$$\left.\frac{1}{2}z_2^2\left(\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}\right)\right]\left(z_1-z_2\right)\left(z_2-z_3\right)+$$

$$\frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}}\frac{d^2\varepsilon_{zz}^{(1)}}{dx^2}\left(z_1-z_2\right)\left(z_2-z_3\right)+$$

$$\frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[2\frac{d^3\varepsilon_{xz}^{(1)}}{dx^3}-\frac{d^4w_0}{dx^4}+z_2\left(\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}\right)\right]\left(z_1^2-z_2^2\right)\left(z_2-z_3\right)-$$

$$\frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}\left(z_1^3-z_2^3\right)\left(z_2-z_3\right)+$$

$$+\frac{E^{(2)}}{\left(1-2\nu^{(2)}\right)\left(1+\nu^{(2)}\right)}\left[\left(1-\nu^{(2)}\right)\frac{d^3u_0}{dx^3}+\nu^{(2)}\frac{d^2\varepsilon_{zz}^{(2)}}{dx^2}\right]\frac{1}{2}\left(z_2-z_3\right)^2+$$

$$\frac{1}{2}\frac{E^{(2)}}{1+\nu^{(2)}}\frac{1-\nu^{(2)}}{1-2\nu^{(2)}}\left[\left(2\frac{d^3\varepsilon_{xz}^{(2)}}{dx^3}-\frac{d^4w_0}{dx^4}\right)\frac{1}{3}\left(2z_2+z_3\right)\left(z_2-z_3\right)^2-\right.$$

$$\left.\frac{1}{3}\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}\frac{1}{4}\left(3z_2^2+2z_2z_3+z_3^2\right)\left(z_2-z_3\right)^2\right]+$$

$$\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[\frac{d^3u_0}{dx^3}+2z_2\left(\frac{d^3\varepsilon_{xz}^{(2)}}{dx^3}-\frac{d^3\varepsilon_{xz}^{(1)}}{dx^3}\right)+\frac{1}{2}z_2^2\left(\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}\right)\right]\left(z_1-z_2\right)\left(z_3-z\right)+$$
$$\frac{E^{(1)}}{1+\nu^{(1)}}\frac{\nu^{(1)}}{1-2\nu^{(1)}}\frac{d^2\varepsilon_{zz}^{(1)}}{dx^2}\left(z_1-z_2\right)\left(z_3-z\right)+$$
$$\frac{1}{2}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\left[2\frac{d^3\varepsilon_{xz}^{(1)}}{dx^3}-\frac{d^4w_0}{dx^4}+z_2\left(\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}\right)\right]\left(z_1^2-z_2^2\right)\left(z_3-z\right)-$$
$$\frac{1}{6}\frac{E^{(1)}}{1+\nu^{(1)}}\frac{1-\nu^{(1)}}{1-2\nu^{(1)}}\frac{d^4\varepsilon_{zz}^{(1)}}{dx^4}\left(z_1^3-z_2^3\right)\left(z_3-z\right)+$$
$$+\frac{E^{(2)}}{\left(1-2\nu^{(2)}\right)\left(1+\nu^{(2)}\right)}\left[\left(1-\nu^{(2)}\right)\frac{d^3u_0}{dx^3}+\nu^{(2)}\frac{d^2\varepsilon_{zz}^{(2)}}{dx^2}\right]\left(z_2-z_3\right)\left(z_3-z\right)+$$
$$\frac{1}{2}\frac{E^{(2)}}{1+\nu^{(2)}}\frac{1-\nu^{(2)}}{1-2\nu^{(2)}}\left[\left(2\frac{d^3\varepsilon_{xz}^{(2)}}{dx^3}-\frac{d^4w_0}{dx^4}\right)\left(z_2^2-z_3^2\right)-\frac{1}{3}\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}\left(z_2^3-z_3^3\right)\right]\left(z_3-z\right)+$$
$$\frac{E^{(3)}}{1+\nu^{(3)}}\frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\left[\frac{d^3u_0}{dx^3}+2z_3\left(\frac{d^3\varepsilon_{xz}^{(2)}}{dx^3}-\frac{d^3\varepsilon_{xz}^{(3)}}{dx^3}\right)+\frac{1}{2}z_3^2\left(\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(3)}}{dx^4}\right)\right]\frac{1}{2}\left(z_3-z\right)^2+$$
$$\frac{E^{(3)}}{1+\nu^{(3)}}\frac{\nu^{(3)}}{1-2\nu^{(3)}}\frac{d^2\varepsilon_{zz}^{(3)}}{dx^2}\frac{1}{2}\left(z_3-z\right)^2+$$
$$\frac{1}{2}\frac{E^{(3)}}{1+\nu^{(3)}}\frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\left[2\frac{d^3\varepsilon_{xz}^{(3)}}{dx^3}-\frac{d^4w_0}{dx^4}+z_3\left(\frac{d^4\varepsilon_{zz}^{(3)}}{dx^4}-\frac{d^4\varepsilon_{zz}^{(2)}}{dx^4}\right)\right]\frac{1}{3}\left(2z_3+z\right)\left(z_3-z\right)^2$$
$$-\frac{1}{6}\frac{E^{(3)}}{1+\nu^{(3)}}\frac{1-\nu^{(3)}}{1-2\nu^{(3)}}\frac{d^4\varepsilon_{zz}^{(3)}}{dx^4}\frac{1}{4}\left(3z_3^2+2z_3z+z^2\right)\left(z_3-z\right)^2\ . \tag{2.2.135}$$

Second forms of expressions for the transverse strains in terms of the unknown functions

The first forms of the transverse strains $\varepsilon_{xz}^{(k)}, \varepsilon_{zz}^{(k)}$ $(k=1,2,3)$ are the unknown functions of the problem, that can be found directly from the finite element solution, as the nodal variables. The more accurate values of the transverse strains, the second forms of the transverse strains, can be computed by substituting the second forms of the transverse stresses, formulas (2.2.127)–(2.2.129) and (2.2.130)–(2.2.133) into the strain-stress relations (2.2.13) and (2.2.14):

$$\left(\varepsilon_{zz}^{(k)}\right)^{(II)}=\frac{1-\left(\nu^{(k)}\right)^2}{E^{(k)}}\left[\left(\sigma_{zz}^{(k)}\right)^{(II)}-\frac{\nu^{(k)}}{1-\nu^{(k)}}\ {}^H\sigma_{xx}^{(k)}\right], \tag{2.2.136}$$

$$\left(\varepsilon_{xz}^{(k)}\right)^{(II)}=\frac{1+\nu^{(k)}}{E^{(k)}}\left(\sigma_{xz}^{(k)}\right)^{(II)} \tag{2.2.137}$$

$$(k = 1, 2, 3)\,.$$

The in-plane stresses ${}^{H}\sigma_{xx}^{(k)}$, which enter into these formulas, are computed by formulas (2.2.118)–(2.2.120).

Satisfaction of stress boundary conditions on the upper surface

In the process of derivation of the second forms of expressions for the transverse stresses in terms of the functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ (equations (2.2.127)–(2.2.129) and (2.2.133)–(2.2.135)), we used stress boundary conditions at the lower surface and the conditions of continuity of the transverse stresses at the interfaces of the layers of sandwich plate:

$$\sigma_{xz}^{(3)} = 0,\ \sigma_{zz}^{(3)} = -\frac{q_l}{b} \text{ at } z = -\frac{h}{2} = z_1;$$

$$\sigma_{xz}^{(1)} = \sigma_{xz}^{(2)},\ \sigma_{zz}^{(1)} = \sigma_{zz}^{(2)} \text{ at } z = -\frac{t}{2} = z_2;$$

$$\sigma_{xz}^{(2)} = \sigma_{xz}^{(3)},\ \sigma_{zz}^{(2)} = \sigma_{zz}^{(3)} \text{ at } z = \frac{t}{2} = z_3;$$

Therefore, the second forms of the transverse stresses satisfy these boundary and continuity conditions. When we considered a homogeneous plate, we showed that the second forms of the transverse stresses satisfy also the boundary conditions at the upper surface of the plate. Now, let us show that the same is true for the sandwich plate in cylindrical bending, i.e. the second forms of the transverse stresses satisfy the boundary conditions at the upper surface. These boundary conditions, written here again, are

$$\sigma_{xz}^{(3)} = 0 \quad \text{at } z = \frac{h}{2} = z_4, \tag{2.2.138}$$

$$\sigma_{zz}^{(3)} = \frac{q_u}{b} \quad \text{at } z = \frac{h}{2} = z_4. \tag{2.2.139}$$

Like in the case of homogeneous plates, this can be proven by showing that the differential equations for the unknown functions, that result from substitution of the second forms of the transverse stresses into the boundary conditions on the upper surface (equations (2.2.138) and (2.2.139)), are the same

equations that follow from the virtual work principle[2]. But in case of the sandwich plates, or laminated composite plates, such a proof requires very voluminous derivations. Therefore, for the sandwich plates the same thing will be shown in slightly different way: it will be shown that the differential equations in terms of force and moment resultants, that are derived by requiring that the second forms of the transverse stresses on the upper surface are equal to the externally applied loads on the upper surface, are the same equations that follow from the virtual work principle.

Substitution of expression (2.2.126) into the boundary condition (2.2.138) yields

$$\int_{z_1}^{z_2} {}^H\sigma_{xx,x}^{(1)}\, dz + \int_{z_2}^{z_3} {}^H\sigma_{xx,x}^{(2)}\, dz + \int_{z_3}^{z_4} {}^H\sigma_{xx,x}^{(3)}\, dz = 0, \tag{2.2.140}$$

or

$$\frac{d}{dx}\int_{z_1}^{z_4} {}^H\sigma_{xx}\, dz = 0\ , \tag{2.2.141}$$

where

$${}^H\sigma_{xx} \equiv \begin{cases} {}^H\sigma_{xx}^{(1)} \text{ in } z_1 \le z \le z_2 \\ {}^H\sigma_{xx}^{(2)} \text{ in } z_2 \le z \le z_3 \\ {}^H\sigma_{xx}^{(3)} \text{ in } z_3 \le z \le z_4 \end{cases} . \tag{2.2.142}$$

Introducing an in-plane force resultant, defined as

$$N_{xx} \equiv \int_0^l {}^H\sigma_{xx}\, dz = \sum_{k=1}^{3}\int_{z_k}^{z_{k+1}} {}^H\sigma_{xx}^{(k)}\, dz\ , \tag{2.2.143}$$

[2]Therefore, the system of differential equations for the unknown functions, that is derived from the virtual work principle, contains those differential equations that can be derived also by substituting the second forms of the transverse stresses into the boundary conditions (2.2.138) and (2.2.139) on the upper surface. Therefore the solution of this system of differential equations for the unknown functions, derived from the virtual work principle, is such, that being substituted into the expressions for the second forms of the transverse stresses in terms of the unknown functions (field variables), this solution guarantees that the second forms of the transverse stresses satisfy the boundary conditions on the upper surface. More generally, the fact that the same differential equations for the unknown functions (but not all of them) can be derived both from the boundary conditions on the upper surface and from the virtual work principle, means that the virtual work principle contains information that the second forms of the transverse stresses satisfy the boundary conditions on the upper surface. Therefore, the finite element formulation, based on the virtual work principle, leads to the finite element solution for the field variables that guarantees the approximate equality of transverse stresses (written in terms of those field variables) on the upper surface to the external loads (per unit area) on the upper surface.

we can write equation (2.2.141) in the form

$$\frac{dN_{xx}}{dx} = 0. \tag{2.2.144}$$

Substitution of expression (2.2.132) into the boundary condition (2.2.139) yields

$$-\frac{q_l}{b} - \int_{z_1}^{z_2} \sigma^{(1)}_{xz,x}\, dz - \int_{z_2}^{z_3} \sigma^{(2)}_{xz,x}\, dz - \int_{z_3}^{z_4} \sigma^{(3)}_{xz,x}\, dz = \frac{q_u}{b} \ , \tag{2.2.145}$$

or

$$\frac{d}{dx}\int_{z_1}^{z_4} \sigma_{xz}\, dz + \frac{q_u + q_l}{b} = 0 \ , \tag{2.2.146}$$

where

$$\sigma_{xz} \equiv \begin{cases} \sigma^{(1)}_{xz} \text{ in } z_1 \leq z \leq z_2 \\ \sigma^{(2)}_{xz} \text{ in } z_2 \leq z \leq z_3 \\ \sigma^{(3)}_{xz} \text{ in } z_3 \leq z \leq z_4 \end{cases} . \tag{2.2.147}$$

Using definition of a transverse force resultant

$$Q_{xz} \equiv \int_{z_1}^{z_4} \sigma_{xz}\, dz = \sum_{k=1}^{3} \int_{z_k}^{z_{k+1}} \sigma^{(k)}_{xz}\, dz = \sum_{k=1}^{3} Q^{(k)}_{xz} \ , \tag{2.2.148}$$

where

$$Q^{(k)}_{xz} \equiv \int_{z_k}^{z_{k+1}} \sigma^{(k)}_{xz}\, dz,$$

we can write equation (2.2.148) in the form

$$\frac{dQ_{xz}}{dx} + \frac{q_u + q_l}{b} = 0 \ . \tag{2.2.149}$$

Differential equations (2.2.144) and (2.2.149) are the stress boundary conditions at the upper surface of the plate in cylindrical bending, expressed in terms of the force resultants. Equations (2.2.144) and (2.2.149) express the statement that the second forms of transverse stresses[3] satisfy the boundary conditions at the upper surface. The same equations follow from the principle of virtual work (Appendix 2-C). Therefore, the virtual work principle contains information that the second forms

[3] obtained from the pointwise equilibrium equations (2.2.1) and (2.2.2)

of the transverse stresses σ_{xz}, σ_{zz} satisfy the boundary conditions (2.2.138) and (2.2.139) on the upper surface of the layered plate. Therefore, the finite element formulation, based on the principle of virtual work, guarantees that the second forms of the transverse stresses (expressions (2.2.127)–(2.2.129) and (2.2.133)–(2.2.135)), satisfy approximately the boundary conditions (2.2.138) and (2.2.139) on the upper surface of the plate.

2.3 Comparison of Results of the Plate Theory with Exact Elasticity Solution for a Simply Supported Isotropic Sandwich Plate in Cylindrical Bending under a Uniform Load on the Upper Surface

Let us consider cylindrical bending of a symmetric sandwich plate with isotropic face sheets and the core (Figure 2.3). The upper surface of the plate is under a uniform load with intensity (force per unit length) q_u. By q_u we denote not an absolute value of the load intensity, but a projection of the load intensity on the z-axis, therefore q can be positive or negative, depending on direction of the load. Along the edges $x = 0, L$ the plate is simply supported. The Young's moduli of the face sheets are equal and will be denoted by E_1 and the Young's modulus of the core will be denoted by E_2. We will consider the Poisson ratio ν to be the same for all layers.

A load vector of a finite element is defined by equation (2.2.117), written here again:

$$\underset{(24\times 1)}{\{r\}} = \frac{1}{b}\int_0^l \underset{(24\times 8)}{[Q(\overline{x})]^T} \underset{(8\times 1)}{\{q(\overline{x})\}}\, d\overline{x}\ , \tag{2.3.1}$$

where $[Q]$ is defined by equation (2.2.106), and $\{q\}$ is defined by equation (2.2.92). Computations give the following result for the load vector:

$r_1 = 0,\ r_2 = 0,\ r_3 = \frac{1}{2}l\frac{q_u}{b},\ r_4 = \frac{1}{12}l^2\frac{q_u}{b},\ r_5 = \frac{1}{2}l\frac{q_u}{b},\ r_6 = -\frac{1}{12}l^2\frac{q_u}{b},\ r_7 = 0,$
$r_8 = 0,\ r_9 = 0,\ r_{10} = 0,\ r_{11} = 0,\ r_{12} = 0,\ r_{13} = 0,\ r_{14} = 0,\ r_{15} = \frac{1}{2}lz_3\frac{q_u}{b},$

$r_{16} = \frac{1}{12}l^2\frac{q_u}{b}z_3,\ r_{17} = \frac{1}{2}lz_3\frac{q_u}{b},\ r_{18} = -\frac{1}{12}l^2\frac{q_u}{b}z_3,\ r_{19} = 0,\ r_{20} = 0,$
$r_{21} = \frac{1}{2}l\frac{q_u}{b}(z_4 - z_3),\ r_{22} = \frac{1}{12}l^2\frac{q_u}{b}(z_4 - z_3),\ r_{23} = \frac{1}{2}l\frac{q_u}{b}(z_4 - z_3),$

$r_{24} = -\frac{1}{12}l^2\frac{q_u}{b}(z_4 - z_3).$

As an example, let us consider a sandwich plate with steel face sheets and an isotropic core, made of foam. We assume the following properties of the face sheets and the core:

core: Young's modulus $E_2 = 1.0192\times 10^8\frac{N}{m^2}$, $\nu = 0.3$, thickness $t = 2\times 10^{-2}m$, mass density $\rho_c = 2\times 10^2\frac{kg}{m^3}$;

face sheets: Young's modulus $E_1 = 1.9796\times 10^{11}\frac{N}{m^2}$, Poisson ratio $\nu = 0.3$, thickness of each face sheet $\frac{h}{2} - \frac{t}{2} = 1\times 10^{-3}m$, mass density $\rho_1 = 7.8\times 10^3\frac{kg}{m^3}$.

The total thickness of the plate is $h = 2.2 \times 10^{-2}m$. We will consider the lengths L of the plate, varying in the range from $0.05m$ to $1.2m$. In order to provide the condition of cylindrical bending, we assume that the width b of the plate is much higher than its length L. The plate is under the load $\frac{q_u}{b} = -1 \times 10^5 \frac{N}{m^2}$ (directed downward, in the negative direction of z-axis). In this example problem the plate is weightless, i.e. the intensity of gravity field is considered to be equal to zero.

We will compare the stress σ_{xx}, obtained from the finite element solution, based on the plate theory, and from the exact solution, presented in Appendix 2-E. The stresses will be evaluated at $x = \frac{L}{2}$ and at various values of z-coordinate. In this linear static problem, the transverse stresses σ_{xz} and σ_{zz} are obtained by substituting the stress σ_{xx} into the equilibrium equations $\sigma_{xx,x} + \sigma_{xz,z} = 0$, $\sigma_{zx,x} + \sigma_{zz,z} = 0$ and integrating these equilibrium equations. Therefore, if the in-plane stress σ_{xx} is accurate, the transverse stresses σ_{xz} and σ_{zz} must be accurate too, if the numerical procedures of integrating the equilibrium equations are correct. Therefore, in this chapter, the purpose of which is to evaluate the quality of the simplifying assumptions on which our plate theory is based, it is sufficient to compare only the in-plane stress σ_{xx}, obtained from the finite element analysis, with that of exact elasticity solution.

The tables below show the results of comparison.

Table 2.1: Comparison of exact and finite element solutions for stress σ_{xx} in a simply supported uniformly loaded sandwich plate with homogeneous isotropic face sheets and the core. Stress σ_{xx} is computed at $x = \frac{L}{2}$, thickness of the plate is $h = 0.022m$, thickness of each face sheet is $0.001m$, length L of the plate varies

L (m)	$\frac{h}{L}$	σ_{xx} at $z = -\frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{z_3+z_4}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$	
		exact	plate theory	exact	plate theory	exact	plate theory
0.05	0.44	1.556	1.555 error0.06 %	−1.484	−1.481 error 0.2 %	−1.556	−1.555 error0.06 %
0.1	0.22	6.222	6.221 error 0.02%	−5.938	−5.922 error 0.3 %	−6.222	6.221 error 0.02%
0.2	0.11	24.887	24.875 error 0.05%	−23.75	−23.69 error 0.25 %	−24.887	−24.875 error 0.05%
0.3	0.07	55.99	55.97 error0.04 %	−53.45	−53.23 error 0.4 %	−55.99	−55.97 error 0.04 %
0.4	0.055	99.54	99.49 error 0.05 %	−95.02	−94.64 error 0.4 %	−99.54	−99.49 error 0.05 %
0.5	0.044	155.5	155.4 error 0.06%	−148.5	−147.91 error 0.4 %	−155.5	−155.4 error 0.06 %
0.6	0.037	223.97	223.75 error 0.1 %	−213.8	−212.74 error 0.5 %	−223.97	−223.75 error 0.1 %
0.7	0.031	304.85	304.69 error 0.05 %	−291.0	−289.3 error 0.6 %	−304.85	−304.69 error 0.05 %
0.8	0.0275	398.2	399.18 error 0.2 %	−380.1	−378.3 error 0.5 %	−398.2	399.18 error 0.2 %
0.9	0.024	503.9	504.5 error 0.1 %	−481.0	−477.5 error 0.7 %	−503.9	504.5 error 0.1 %
1	0.022	622.1	624.4 error 0.4 %	−593.9	−587.55 error 1.1 %	−622.1	−624.4 error 0.4 %
1.1	0.02	752.8	756.6 error 0.5 %	−718.58	−698.7 error 2.8 %	−752.8	756.6 error 0.5 %
1.2	0.018	895.9	873.2 error 2.5 %	−855.2	−790.85 error 7.5 %	−895.9	873.2 error 2.5 %

Table 2.2: Comparison of exact and finite element solutions for stress σ_{xx} in a simply supported uniformly loaded sandwich plate with homogeneous isotropic face sheets and the core. Stress σ_{xx} is computed at $x = \frac{L}{2}$ $(L = 0.5m)$, thickness of the plate is h=0.022m, thickness of the face sheet τ varies

τ (m)	$\frac{\tau}{h}$	σ_{xx} at $z = -\frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{z_3+z_4}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$	
		exact	plate theory	exact	plate theory	exact	plate theory
0.001	0.045	155.5	155.4 error 0.06 %	−148.5	−147.8 error 0.5 %	−155.5	−155.4 error 0.06 %
0.002	0.09	85.60	85.48 error 0.1 %	−77.82	−77.57 error 0.3 %	−85.60	−85.48 error 0.1 %
0.003	0.14	62.94	62.83 error 0.17 %	−54.35	−54.23 error 0.2 %	−62.94	−62.83 error 0.17 %
0.004	0.18	52.18	52.09 error 0.2 %	−42.69	−42.56 error 0.3 %	−52.18	−52.09 error 0.2 %
0.005	0.18	46.245	46.18 error 0.14 %	−35.728	−35.67 error 0.2 %	−46.245	−46.18 error 0.14 %
0.006	0.27	42.76	42.67 error 0.2 %	−31.09	−30.98 error 0.35 %	−42.76	−42.67 error 0.2 %
0.010	0.45	38.78	38.69 error 0.2 %	−21.14	−21.09 error 0.2 %	−38.78	−38.69 error 0.2 %

So, we see, that the layerwise theory of the sandwich plates, based on assumptions of non-zero, constant (in the thickness direction) transverse strains in the face sheets and the core, leads to highly accurate values of the in-plane stresses. Therefore, the high accuracy of the transverse stresses can also be achieved, if they are computed by integration of equilibrium equations (or equations of motion in dynamic case), in which the in-plane stresses are substituted. But this approach to construction of the sandwich plate theory leads to the finite element formulation with many degrees of freedom per element: 24 degrees of freedom for a one-dimensional element for cylindrical bending. Therefore, in the next section a simplified approach to construction of the sandwich plate theory, with fewer degrees of freedom in the finite element formulation will be considered.

2.4 Simplified theory of a sandwich plate in cylindrical bending

If the thickness of the face sheets is much lower than the thickness of the core, then we can consider the face sheets on the basis of the classical plate theory, i.e. set the first forms of the transverse

strains (assumed transverse strains) in the face sheets equal to zero:

$$\varepsilon_{xz}^{(1)} = 0,\ \varepsilon_{zz}^{(1)} = 0,\ \varepsilon_{xz}^{(3)} = 0,\ \varepsilon_{zz}^{(3)} = 0, \tag{2.4.1}$$

The accuracy of analysis with these additional assumptions will be verified in the end of this section by comparing results of the finite element analysis, based on assumptions (2.4.1), with the corresponding exact elasticity solutions. The assumptions (2.4.1) do not mean that the transverse strains and stresses in the face sheets are completely ignored in this computational model. In the post-process stage, the second form of the transverse stresses is computed by substitution of the in-plane stress σ_{xx} into the pointwise equilibrium equations $\sigma_{xx,x} + \sigma_{xz,z} = 0$, $\sigma_{xz,x} + \sigma_{zz,z} = 0$, and by integration of these equations. Then the second form of the transverse strains can be obtained by substitution of the second form of the transverse stresses into strain-stress relations. So, the assumed transverse strains, defined by equations (2.4.1), are used only in the expression for the strain energy, that is used for the finite element formulation. If one needs the values of the transverse stresses in the face sheets that counteract the external forces, and the corresponding transverse strains, one has to use the second form of these strains and stresses.

The similar approaches to analysis of the sandwich plates with thin face sheets, in which either transverse strains or transverse stresses in the face sheets are assumed to be equal to zero, are adopted, for example, by Mead (1972), Markus and Nanashi (1981), Whitney (1987), Al-Qarra (1988), Yu (1997) and other authors.

Besides, according to assumptions (2.2.25), we have

$$\varepsilon_{xz}^{(2)} = \varepsilon_{xz}^{(2)}(x)\,,\ \varepsilon_{zz}^{(2)} = \varepsilon_{zz}^{(2)}(x). \tag{2.4.2}$$

If there are no external in-plane forces, applied to the plate, then, due to the fact that the Poisson's ratio of the core is usually small, we can set

$$u_0 = 0. \tag{2.4.3}$$

So, the **unknown functions of the problem** in our simplified theory of cylindrical bending of sandwich plates are

$$w_0(x)\,,\ \varepsilon_{xz}^{(2)}(x)\,,\ \varepsilon_{zz}^{(2)}(x)\,.$$

In an example problem we will show that this simplified approach to the analysis of the sandwich plates, based on the additional assumptions (2.4.1) and (2.4.3) does not lead to a significant loss of accuracy of stress computation if the face sheets are thin as compared to the core.

In this simplified computational model all of the formulas of section 2.2 are applicable, if according to the assumptions (2.4.1) and (2.4.3), we set $\varepsilon_{xz}^{(1)} = 0$, $\varepsilon_{zz}^{(1)} = 0$, $\varepsilon_{xz}^{(3)} = 0$, $\varepsilon_{zz}^{(3)} = 0$, $u_0 = 0$. In the finite element formulation of the nonsimplified model, presented in section 2.2 of this chapter, the nodal variables are (Figure 2.4):

$$u_0,\, w_0,\, \frac{dw_0}{dx},\, \varepsilon_{xz}^{(1)},\, \varepsilon_{zz}^{(1)},\, \frac{d\varepsilon_{zz}^{(1)}}{dx},\, \varepsilon_{xz}^{(2)},\, \varepsilon_{zz}^{(2)},\, \frac{d\varepsilon_{zz}^{(2)}}{dx},\, \varepsilon_{xz}^{(3)},\, \varepsilon_{zz}^{(3)},\, \frac{d\varepsilon_{zz}^{(3)}}{dx}. \tag{2.4.4}$$

In the simplified model, the nodal variables, associated with the unknown functions $\varepsilon_{xz}^{(1)}$, $\varepsilon_{zz}^{(1)}$, $\varepsilon_{xz}^{(3)}$, $\varepsilon_{zz}^{(3)}$, u_0 are to be set equal to zero:

$$u_0 = 0,\, \varepsilon_{xz}^{(1)} = 0,\, \varepsilon_{zz}^{(1)} = 0,\, \frac{d\varepsilon_{zz}^{(1)}}{dx} = 0,\, \varepsilon_{xz}^{(3)} = 0,\, \varepsilon_{zz}^{(3)} = 0,\, \frac{d\varepsilon_{zz}^{(3)}}{dx} = 0 \tag{2.4.5}$$

So, the nodal variables of the simplified model of the sandwich pate in cylindrical bending are

$$w_0,\, \frac{dw_0}{dx},\, \varepsilon_{xz}^{(2)},\, \varepsilon_{zz}^{(2)},\, \frac{d\varepsilon_{zz}^{(2)}}{dx}. \tag{2.4.6}$$

In order to find the accuracy of stress computation by the simplified model of the sandwich plates, presented in this section, let us consider the same numerical example as in section 2.3 (page 2-71) and compare the results with the exact elasticity solution (Appendix 2-E). The tables of comparison are given below.

Table 2.3: Comparison of exact and finite element solutions, based on the simplified model, for stress σ_{xx} in a simply supported uniformly loaded sandwich plate with homogeneous isotropic face sheets and the core. Stress σ_{xx} is computed at $x = \frac{L}{2}$, thickness of the plate is $h = 0.022m$, thickness of each face sheet is $0.001m$, length L of the plate varies

L (m)	$\frac{h}{L}$	σ_{xx} at $z = -\frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{z_3+z_4}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$	
		exact	plate theory	exact	plate theory	exact	plate theory
0.05	0.44	1.556	1.555 error 0.06%	−1.484	−1.476 error 0.5%	−1.556	−1.555 error 0.06%
0.1	0.22	6.222	6.219 error 0.05%	−5.938	−5.906 error 0.5%	−6.222	−6.219 error 0.05%
0.2	0.11	24.887	24.865 error 0.09%	−23.75	−23.63 error 0.5%	−24.887	−24.865 error 0.09%
0.3	0.07	55.99	55.92 error 0.125%	−53.45	−53.17 error 0.5%	−55.99	−55.92 error 0.125%
0.4	0.055	99.54	99.38 error 0.16%	−95.02	−94.52 error 0.5%	−99.54	−99.38 error 0.16%
0.5	0.044	155.5	155.3 error 0.13%	−148.5	−147.68 error 0.55%	−155.5	−155.3 error 0.13%
0.6	0.037	223.97	223.64 error 0.15%	−213.8	−212.57 error 0.58%	−223.97	−223.64 error 0.15%
0.7	0.031	304.85	304.58 error 0.09%	−291.0	−289.1 error 0.65%	−304.85	−304.58 error 0.09%
0.8	0.0275	398.2	400.015 error 0.46%	−380.1	−377.1 error 0.8%	−398.2	−400.015 error 0.46%
0.9	0.024	503.9	505.0 error 0.2%	−481.0	−476.4 error 0.96%	−503.9	−505.0 error 0.2%
1	0.022	622.1	625.1 error 0.48%	−593.9	−586.55 error 1.2%	−622.1	−625.1 error 0.48%
1.1	0.02	752.8	744.5 error 1.1%	−718.58	−691.1 error 3.8%	−752.8	−744.5 error 1.1%
1.2	0.018	895.9	837.5 error 6.5%	−855.2	−760.76 error 11%	−895.9	−837.5 error 6.5%

Table 2.4: Comparison of exact and finite element solutions for stress σ_{xx} in a simply supported uniformly loaded sandwich plate with homogeneous isotropic face sheets and the core for stress σ_{xx} at $x = \frac{L}{2}$ $(L = 0.5m)$. Thickness of the plate is h=0.022m, thickness of the face sheet τ varies

for stress σ_{xx} at $x = \frac{L}{2}$ $(L = 0.5m)$, thickness of the plate is h=0.022m, thickness of the face sheet τ varies

τ (m)	$\frac{\tau}{h}$	σ_{xx} at $z = -\frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{z_3+z_4}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$	
		exact	plate theory	exact	plate theory	exact	plate theory
0.001	0.045	155.5	155.3 error 0.13%	−148.5	−147.68 error 0.55%	−155.5	−155.3 error 0.13%
0.002	0.09	85.60	85.39 error 0.2%	−77.82	−77.48 error 0.4%	−85.60	−85.39 error 0.2%
0.003	0.14	62.94	62.78 error 0.25%	−54.35	−54.14 error 0.4%	−62.94	−62.78 error 0.25%
0.004	0.18	52.18	52.04 error 0.27%	−42.69	−42.52 error 0.4%	−52.18	−52.04 error 0.27%
0.005	0.18	46.245	46.12 error 0.27%	−35.728	−35.64 error 0.25%	−46.245	−46.12 error 0.27%
0.006	0.27	42.76	42.63 error 0.3%	−31.09	−30.97 error 0.4%	−42.76	−42.63 error 0.3%
0.010	0.45	38.78	38.64 error 0.4%	−21.14	−21.07 error 0.3%	−38.78	−38.64 error 0.4%

We see that with a simplified approach to construction of the sandwich plate theory, we have achieved an accuracy of the stresses that is quite acceptable for practical analysis of thick sandwich plates, though slightly worse than the accuracy of the stresses obtained with the non-simplified approach, i.e. with non-zero assumed stresses in the face sheets. The advantage of the simplified model of the sandwich pate, presented in this section, is a lower number of degrees of freedom in finite element models. This conclusion allows to apply the similar simplified approach to modeling the sandwich plates with the laminated composite face sheets and anisotropic core. The finite element program for analysis of the sandwich cargo platforms, dropped on the ground, with account of damage progression, presented in the chapter 5, is based on the simplified theory presented in this section.

2.5 Appendix 2-A

Exact solution for a simply supported homogeneous plate in cylindrical bending under a uniform load on the upper surface

This problem is solved in order to compare the stresses obtained from exact solution with the stresses obtained from the plate theory, based on assumed transverse strains, presented in chapter 2, equations (2.1.72)–(2.1.74). The exact solution for a wide simply supported uniformly loaded plate in cylindrical bending (which is a plane strain problem with respect to the y-direction) presented in this Appendix, is similar to the exact solution for a narrow rectangular simply supported uniformly loaded beam (which is a plane stress problem with respect to the y-direction) presented in the book of Saada (1993).

Let us consider the problem of cylindrical bending of a plate of length L, height h and width b. Cylindrical bending implies that $b \gg h$. The plate is under the uniform load, acting on the upper surface with intensity (force per unit length) q_u (Figure 2.2). By q_u we denoted not an absolute value of the load intensity, but a projection of the load intensity on the z-axis, i.e. q_u can be positive or negative, depending on the direction of the load. The sides $x = 0, L$ are acted upon by reaction forces $\frac{q_u L}{2}$, and the longitudinal forces and moments at these edges are equal to zero. So, the boundary conditions for this problem can be written in the form:

$$\sigma_{xz} = 0 \text{ and } \sigma_{zz} = \frac{q_u}{b} \text{ at } z = \frac{h}{2}, \tag{2-A.1}$$

$$\sigma_{xz} = 0 \text{ and } \sigma_{zz} = 0 \text{ at } z = -\frac{h}{2}, \tag{2-A.2}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}\, dz = 0 \text{ and } \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}\, z\, dz = 0 \text{ at } x = 0,\ L. \tag{2-A.3}$$

The boundary conditions for the edges $x = 0, L$ are written on the basis of Saint-Venant principle, according to which the substitution of the actual load by the statically equivalent load influences the distribution of stresses only in the limited area around the place of application of the external load.

Let us write the equilibrium equations and the equation of compatibility in terms of stress:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0, \tag{2-A.4}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0\ , \tag{2-A.5}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)(\sigma_{xx} + \sigma_{zz}) = 0\ . \tag{2-A.6}$$

As a first approximation, we will assume that the stresses σ_{xx}, σ_{zz}, σ_{xz} are defined by the known expressions for beams from Mechanics of Materials courses. Then we will add to these expressions some unknown functions and find these functions by requiring that the expressions for the stresses satisfy the equilibrium equations (4) and the compatibility equation (5). The first approximation for the stresses is

$$\sigma_{xx} = \frac{M(x)}{I_y} z = \frac{q_u x (L - x)}{2 I_y} z\ ,$$

$$\sigma_{zz} = 0\ ,$$

$$\sigma_{xz} = \frac{QS}{I_y b},$$

where $I_y = \frac{1}{12} b h^3$ is a moment of inertia of rectangular cross-section with respect to y-axis, $Q = \frac{dM}{dx}$ is a shear force, $S = \int\limits_{-b/2}^{b/2} \int\limits_{z}^{h/2} z\, dz\, dy = -\frac{1}{2} b \left(z^2 - \frac{h^2}{4}\right)$ is the first moment of rectangular cross section above a line $z = const$. So, the first approximation for the stresses has the form:

$$\sigma_{xx} = -\frac{6}{h^3} \frac{q_u}{b} x (x - L) z\ , \tag{2-A.7}$$

$$\sigma_{xz} = \frac{6}{h^3} \frac{q_u}{b} \left(x - \frac{L}{2}\right)\left(z^2 - \frac{h^2}{4}\right), \tag{2-A.8}$$

$$\sigma_{zz} = 0. \tag{2-A.9}$$

These expressions for stresses do not satisfy the equilibrium equation (5). In order to satisfy the equilibrium equation (5), let us find σ_{zz} from this equation :

$$\sigma_{zz} - \underbrace{\sigma_{zz}\big|_{z=h/2}}_{\frac{q_u}{b}} = \int\limits_{h/2}^{z} \frac{\partial \sigma_{zz}}{\partial z}\, dz = -\int\limits_{h/2}^{z} \frac{\partial \sigma_{xz}}{\partial x}\, dz,$$

$$\sigma_{zz} = -\frac{1}{2h^3}\frac{q_u}{b}(2z+h)^2(z-h). \tag{2-A.10}$$

Expression (10) satisfies the boundary conditions (1) and (2) for σ_{zz}.

The equilibrium equation (4) is satisfied by the first approximations of σ_{xx} and σ_{xz} (expressions (7) and (8)), but the compatibility equation (6) is not satisfied by the first approximation of σ_{xx} and σ_{zz} (expressions (7) and (10)). To satisfy the compatibility equation (6) we use the fact that the equilibrium equation (4) will still be satisfied, if we add to the expression (7) for σ_{xx} some function of z:

$$\sigma_{xx} = -\frac{6}{h^3}\frac{q_u}{b}x(x-L)z + f(z). \tag{2-A.11}$$

If we substitute expressions (10) and (11) for σ_{zz} and σ_{xx} into the equation of compatibility (6), we receive the differential equation

$$-\frac{24}{h^3}\frac{q_u}{b}z + \frac{d^2 f(z)}{dz^2} = 0 \tag{2-A.12}$$

the solution of which is

$$f(z) = \frac{4}{h^3}\frac{q_u}{b}z^3 + C_1 z + C_2. \tag{2-A.13}$$

So, expression (11) for σ_{xx} takes the form:

$$\sigma_{xx} = -\frac{6}{h^3}\frac{q_u}{b}x(x-L)z + \frac{4}{h^3}\frac{q_u}{b}z^3 + C_1 z + C_2. \tag{2-A.14}$$

The constants of integration C_1 and C_2 must be found from the conditions (3). From (14) it follows that

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx}\, dz = C_2 h,$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x\, z\, dz = \frac{1}{20}\frac{q_u}{b}h^2 + \frac{1}{12}C_1 h^3 - \frac{1}{2}\frac{q_u}{b}x^2 + \frac{1}{2}\frac{q_u}{b}xL.$$

Therefore, from conditions (3) we obtain:

$$C_2 = 0\ ,$$

$$C_1 = -\frac{3}{5h}\frac{q_u}{b}\ ,$$

and expression (14) for σ_{xx} takes the form:

$$\sigma_{xx} = -\frac{6}{h^3}\frac{q_u}{b}x\,(x-L)\,z + \frac{4}{h^3}\frac{q_u}{b}z^3 - \frac{3}{5h}\frac{q_u}{b}z. \tag{2-A.15}$$

So, we found that expressions (8), (10) and (15) satisfy the boundary conditions (1)-(3), the equilibrium equations (4), (5) and the equation of compatibility in terms of stress (6). Therefore, expressions (8), (10) and (15) are the solution of the problem. Stress σ_{yy} can be found from the following plane-strain relation:

$$\sigma_{yy} = \nu\,(\sigma_{xx} + \sigma_{zz})\,. \tag{2-A.16}$$

So, the exact solution for stresses in a plate, in cylindrical bending, is

$$\sigma_{xz} = \frac{6}{h^3}\frac{q_u}{b}\left(x - \frac{L}{2}\right)\left(z^2 - \frac{h^2}{4}\right),$$

$$\sigma_{zz} = -\frac{1}{2h^3}\frac{q_u}{b}\,(2z+h)^2\,(z-h)\,,$$

$$\sigma_{xx} = -\frac{6}{h^3}\frac{q_u}{b}x\,(x-L)\,z + \frac{4}{h^3}\frac{q_u}{b}z^3 - \frac{3}{5h}\frac{q_u}{b}z,$$

$$\sigma_{yy} = \nu\,(\sigma_{xx} + \sigma_{zz})\,.$$

2.6 Appendix 2-B

Some components of an element stiffness matrix for an isotropic sandwich plate, for an element with 24 degrees of freedom

The components of the stiffness matrix were derived by exact integration with the use of symbolic computation capabilities of the program "Scientific Workplace". In this Appendix only few components of the stiffness matrix are shown, because of limitations on the size of the dissertation.

$$k_{11} = E_1 \frac{z_1 - z_2 - \nu z_1 + \nu z_2}{l\,(1+\nu)\,(2\nu-1)} - E_2 \frac{-z_2 + z_3 + \nu z_2 - \nu z_3}{l\,(1+\nu)\,(2\nu-1)} - E_3 \frac{-z_3 + z_4 + \nu z_3 - \nu z_4}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{12} = -E_1 \frac{z_1 - z_2 - \nu z_1 + \nu z_2}{l\,(1+\nu)\,(2\nu-1)} + E_2 \frac{-z_2 + z_3 + \nu z_2 - \nu z_3}{l\,(1+\nu)\,(2\nu-1)} + E_3 \frac{-z_3 + z_4 + \nu z_3 - \nu z_4}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{13} = 0,$$

$$k_{14} = -\frac{1}{2} E_1 \frac{z_1^2 - z_2^2 - \nu z_1^2 + \nu z_2^2}{l\,(1+\nu)\,(2\nu-1)} + \frac{1}{2} E_2 \frac{-z_2^2 + z_3^2 + \nu z_2^2 - \nu z_3^2}{l\,(1+\nu)\,(2\nu-1)} + \frac{1}{2} E_3 \frac{-z_3^2 + z_4^2 + \nu z_3^2 - \nu z_4^2}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{15} = 0,$$

$$k_{16} = \frac{1}{2} E_1 \frac{z_1^2 - z_2^2 - \nu z_1^2 + \nu z_2^2}{l\,(1+\nu)\,(2\nu-1)} - \frac{1}{2} E_2 \frac{-z_2^2 + z_3^2 + \nu z_2^2 - \nu z_3^2}{l\,(1+\nu)\,(2\nu-1)} - \frac{1}{2} E_3 \frac{-z_3^2 + z_4^2 + \nu z_3^2 - \nu z_4^2}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{17} = -E_1 \frac{2 z_2 z_1 - z_2^2 - 2 z_2 \nu z_1 + \nu z_2^2 - z_1^2 + \nu z_1^2}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{18} = E_1 \frac{2 z_2 z_1 - z_2^2 - 2 z_2 \nu z_1 + \nu z_2^2 - z_1^2 + \nu z_1^2}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{22} = E_1 \frac{z_1 - z_2 - \nu z_1 + \nu z_2}{l\,(1+\nu)\,(2\nu-1)} - E_2 \frac{-z_2 + z_3 + \nu z_2 - \nu z_3}{l\,(1+\nu)\,(2\nu-1)} - E_3 \frac{-z_3 + z_4 + \nu z_3 - \nu z_4}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{23} = 0,$$

$$k_{24} = \frac{1}{2}E_1\frac{z_1^2 - z_2^2 - \nu z_1^2 + \nu z_2^2}{l\,(1+\nu)\,(2\nu-1)} - \frac{1}{2}E_2\frac{-z_2^2 + z_3^2 + \nu z_2^2 - \nu z_3^2}{l\,(1+\nu)\,(2\nu-1)} - \frac{1}{2}E_3\frac{-z_3^2 + z_4^2 + \nu z_3^2 - \nu z_4^2}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{25} = 0,$$

$$k_{26} = -\frac{1}{2}E_1\frac{z_1^2 - z_2^2 - \nu z_1^2 + \nu z_2^2}{l\,(1+\nu)\,(2\nu-1)} + \frac{1}{2}E_2\frac{-z_2^2 + z_3^2 + \nu z_2^2 - \nu z_3^2}{l\,(1+\nu)\,(2\nu-1)} + \frac{1}{2}E_3\frac{-z_3^2 + z_4^2 + \nu z_3^2 - \nu z_4^2}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{27} = E_1\frac{2z_2z_1 - z_2^2 - 2z_2\nu z_1 + \nu z_2^2 - z_1^2 + \nu z_1^2}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{28} = -E_1\frac{2z_2z_1 - z_2^2 - 2z_2\nu z_1 + \nu z_2^2 - z_1^2 + \nu z_1^2}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{33} = 4E_1\frac{z_1^3 - z_2^3 - \nu z_1^3 + \nu z_2^3}{l^3\,(1+\nu)\,(2\nu-1)} - 4E_2\frac{-z_2^3 + z_3^3 + \nu z_2^3 - \nu z_3^3}{l^3\,(1+\nu)\,(2\nu-1)} - 4E_3\frac{-z_3^3 + z_4^3 + \nu z_3^3 - \nu z_4^3}{l^3\,(1+\nu)\,(2\nu-1)},$$

$$k_{34} = 2E_1\frac{z_1^3 - z_2^3 - \nu z_1^3 + \nu z_2^3}{l^2\,(1+\nu)\,(2\nu-1)} - 2E_2\frac{-z_2^3 + z_3^3 + \nu z_2^3 - \nu z_3^3}{l^2\,(1+\nu)\,(2\nu-1)} - 2E_3\frac{-z_3^3 + z_4^3 + \nu z_3^3 - \nu z_4^3}{l^2\,(1+\nu)\,(2\nu-1)},$$

$$k_{35} = -4E_1\frac{z_1^3 - z_2^3 - \nu z_1^3 + \nu z_2^3}{l^3\,(1+\nu)\,(2\nu-1)} + 4E_2\frac{-z_2^3 + z_3^3 + \nu z_2^3 - \nu z_3^3}{l^3\,(1+\nu)\,(2\nu-1)} + 4E_3\frac{-z_3^3 + z_4^3 + \nu z_3^3 - \nu z_4^3}{l^3\,(1+\nu)\,(2\nu-1)},$$

$$k_{36} = 2E_1\frac{z_1^3 - z_2^3 - \nu z_1^3 + \nu z_2^3}{l^2\,(1+\nu)\,(2\nu-1)} - 2E_2\frac{-z_2^3 + z_3^3 + \nu z_2^3 - \nu z_3^3}{l^2\,(1+\nu)\,(2\nu-1)} - 2E_3\frac{-z_3^3 + z_4^3 + \nu z_3^3 - \nu z_4^3}{l^2\,(1+\nu)\,(2\nu-1)},$$

$$k_{37} = 0,$$

$$k_{38} = 0,$$

$$k_{44} = \frac{4}{3}E_1\frac{z_1^3 - z_2^3 - \nu z_1^3 + \nu z_2^3}{l\,(1+\nu)\,(2\nu-1)} - \frac{4}{3}E_2\frac{-z_2^3 + z_3^3 + \nu z_2^3 - \nu z_3^3}{l\,(1+\nu)\,(2\nu-1)} - \frac{4}{3}E_3\frac{-z_3^3 + z_4^3 + \nu z_3^3 - \nu z_4^3}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{45} = -2E_1\frac{z_1^3 - z_2^3 - \nu z_1^3 + \nu z_2^3}{l^2\,(1+\nu)\,(2\nu-1)} + 2E_2\frac{-z_2^3 + z_3^3 + \nu z_2^3 - \nu z_3^3}{l^2\,(1+\nu)\,(2\nu-1)} + 2E_3\frac{-z_3^3 + z_4^3 + \nu z_3^3 - \nu z_4^3}{l^2\,(1+\nu)\,(2\nu-1)},$$

$$k_{46} = \frac{2}{3}E_1\frac{z_1^3 - z_2^3 - \nu z_1^3 + \nu z_2^3}{l\,(1+\nu)\,(2\nu-1)} - \frac{2}{3}E_2\frac{-z_2^3 + z_3^3 + \nu z_2^3 - \nu z_3^3}{l\,(1+\nu)\,(2\nu-1)} - \frac{2}{3}E_3\frac{-z_3^3 + z_4^3 + \nu z_3^3 - \nu z_4^3}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{47} = \frac{1}{3}E_1\frac{3z_2z_1^2 - z_2^3 - 3z_2\nu z_1^2 + \nu z_2^3 - 2z_1^3 + 2\nu z_1^3}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{48} = -\frac{1}{3}E_1\frac{3z_2z_1^2 - z_2^3 - 3z_2\nu z_1^2 + \nu z_2^3 - 2z_1^3 + 2\nu z_1^3}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{55} = 4E_1\frac{z_1^3 - z_2^3 - \nu z_1^3 + \nu z_2^3}{l^3\,(1+\nu)\,(2\nu-1)} - 4E_2\frac{-z_2^3 + z_3^3 + \nu z_2^3 - \nu z_3^3}{l^3\,(1+\nu)\,(2\nu-1)} - 4E_3\frac{-z_3^3 + z_4^3 + \nu z_3^3 - \nu z_4^3}{l^3\,(1+\nu)\,(2\nu-1)},$$

$$k_{56} = -2E_1\frac{z_1^3 - z_2^3 - \nu z_1^3 + \nu z_2^3}{l^2\,(1+\nu)\,(2\nu-1)} + 2E_2\frac{-z_2^3 + z_3^3 + \nu z_2^3 - \nu z_3^3}{l^2\,(1+\nu)\,(2\nu-1)} + 2E_3\frac{-z_3^3 + z_4^3 + \nu z_3^3 - \nu z_4^3}{l^2\,(1+\nu)\,(2\nu-1)},$$

$$k_{57} = 0,$$

$$k_{58} = 0,$$

$$k_{66} = \frac{4}{3}E_1\frac{z_1^3 - z_2^3 - \nu z_1^3 + \nu z_2^3}{l\,(1+\nu)\,(2\nu-1)} - \frac{4}{3}E_2\frac{-z_2^3 + z_3^3 + \nu z_2^3 - \nu z_3^3}{l\,(1+\nu)\,(2\nu-1)} - \frac{4}{3}E_3\frac{-z_3^3 + z_4^3 + \nu z_3^3 - \nu z_4^3}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{67} = -\frac{1}{3}E_1\frac{3z_2z_1^2 - z_2^3 - 3z_2\nu z_1^2 + \nu z_2^3 - 2z_1^3 + 2\nu z_1^3}{l\,(1+\nu)\,(2\nu-1)},$$

$$k_{68} = \frac{1}{3}E_1\frac{3z_2z_1^2 - z_2^3 - 3z_2\nu z_1^2 + \nu z_2^3 - 2z_1^3 + 2\nu z_1^3}{l\,(1+\nu)\,(2\nu-1)},$$

2.7 Appendix 2-C

Remarks on variational principles and equilibrium equations for a plate in cylindrical bending in terms of force and moment resultants

In chapter 2 we wrote the stress boundary conditions on the upper surface of the sandwich plate $\sigma_{xz}^{(3)} = 0$ at $z = \frac{h}{2} = z_4$. (eqn 2.2.141), $\sigma_{zz}^{(3)} = \frac{q_u}{b}$ at $z = \frac{h}{2} = z_4$ (eqn 2.2.142),[4] in terms of the force resultants[5] :

$$\frac{dN_{xx}}{dx} = 0, \quad \text{(equation 2.2.144)},$$

$$\frac{dQ_{xz}}{dx} + \frac{q_u + q_l}{b} = 0 \quad \text{(equation 2.2.149)}.$$

It was stated in chapter 2 that equations (2.2.144) and (2.2.149) follow also from the virtual work principle, and the reader was referenced to this Appendix. From this we will be able to make a conclusion that the virtual work principle contains information that the second forms of the transverse stresses satisfy the boundary conditions on the upper surface of the plate[6]. Therefore, the finite element formulation, based on the virtual work principle, guarantees that the second forms of the transverse stresses satisfy approximately the boundary conditions on the upper surface of the plate.

Our finite element formulation of the problem of cylindrical bending of the sandwich plate is based on the virtual work principle:

$$\sum_{k=1}^{3} \iiint\limits_{\left(V^{(k)}\right)} \left({}^{H}\sigma_{xx}^{(k)}\, \delta\varepsilon_{xx}^{(k)} + {}^{H}\sigma_{xz}^{(k)}\, \delta\varepsilon_{xz}^{(k)} + {}^{H}\sigma_{zz}^{(k)}\, \delta\varepsilon_{zz}^{(k)} \right) dV$$

[4] where $\sigma_{xz}^{(3)}$ and $\sigma_{zz}^{(3)}$ are second forms of transverse stresses, obtained by integration of the pointwise equilibrium equations $\sigma_{xx,x} + \sigma_{xz,z} = 0$, $\sigma_{xz,x} + \sigma_{zz,z} = 0$

[5] defined by formulas $Q_{xz} \equiv \int\limits_{z_1}^{z_4} \sigma_{xz}\, dz = \sum\limits_{k=1}^{3} \int\limits_{z_k}^{z_{k+1}} \sigma_{xz}^{(k)}\, dz$ and $N_{xx} \equiv \int\limits_{0}^{l} {}^{H}\sigma_{xx}\, dz = \sum\limits_{k=1}^{3} \int\limits_{z_k}^{z_{k+1}} {}^{H}\sigma_{xx}^{(k)}\, dz$

[6] in addition to satisfaction of the boundary conditions on the lower surface of the plate and conditions of continuity of the transverse stresses at the the interfaces between the layers of the layered plate, that is guaranteed by the fact that these boundary and continuity conditions were used in the process of integration of the pointwise equilibrium equations $\sigma_{xx,x} + \sigma_{xz,z} = 0$ and $\sigma_{zx,x} + \sigma_{zz,z} = 0$ in order to obtain stresses σ_{xz} and σ_{zz}.

$$-\int_0^L q_u \left(\delta\, w^{(3)}\Big|_{z=z_4}\right) dx - \int_0^L q_l \left(\delta\, w^{(1)}\Big|_{z=z_1}\right) dx = 0\ , \tag{2-C.1}$$

where the superscript k denotes a number of a layer. The transverse stresses, that enter into equation (2-C.1), are the first forms of the transverse stresses, i.e. they are expressed in terms of the unknown functions with the help of the Hooke's law, equations (2.2.63). If in equation (2-C.1) instead of the first forms of the transverse stresses ${}^H\sigma_{xz}^{(k)}$, ${}^H\sigma_{xz}^{(k)}$ we put the second forms of the transverse stresses $\sigma_{xz}^{(k)}$, $\sigma_{zz}^{(k)}$(equations (2.2.127)-(2.2.129) and (2.2.133)-(2.2.135)), i.e. transverse stresses obtained from the pointwise equilibrium equations, we obtain the virtual work principle, written in the form

$$\sum_{k=1}^{3} \iiint\limits_{\left(V^{(k)}\right)} \left({}^H\sigma_{xx}^{(k)}\ \delta\varepsilon_{xx}^{(k)} + \sigma_{xz}^{(k)}\ \delta\varepsilon_{xz}^{(k)} + \ \sigma_{zz}^{(k)}\ \delta\varepsilon_{zz}^{(k)}\right)\ dV$$

$$-\int_0^L q_u \left(\delta\, w^{(3)}\Big|_{z=z_4}\right) dx - \int_0^L q_l \left(\delta\, w^{(1)}\Big|_{z=z_1}\right) dx = 0\ , \tag{2-C.2}$$

which is equivalent to the virtual work principle, expressed by equation (2-C.1). The equivalency of variational equations (2-C.1) and (2-C.2) is in the sense that both of them produce the same differential equations for the unknown functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ and boundary conditions. This idea is discussed at greater length in Appendix D.

Now, from the virtual work principle, written in the form of equation (2-C.2), let us obtain the equilibrium equations for a sandwich plate in cylindrical bending in terms of force and moment resultants. For this we need to substitute in equation (2-C.2) expressions (2.2.47)-(2.2.49) for $\varepsilon_{xx}^{(k)}$ in terms of the unknown functions u_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{zz}^{(k)}$, perform integration by parts in order to relieve the variations of the unknown functions, collect the coefficients of variations of the unknown functions and set them equal to zero separately. As a result of this, we receive the following equilibrium equations of a sandwich plate in cylindrical bending in terms of force and moment resultants:

$$\delta u_0 : \qquad \frac{dN_{xx}}{dx} = 0, \tag{2-C.3}$$

$$\delta w_0 : \qquad \frac{d^2 M_{xx}}{dx^2} + \frac{q_u + q_l}{b} = 0, \tag{2-C.4}$$

$$\delta\varepsilon_{xz}^{(k)} : \qquad \frac{dM_{xx}^{(k)}}{dx} - Q_{xz}^{(k)} = 0 \quad (k = 1, 2, 3)\,, \tag{2-C.5}$$

$$\delta\varepsilon_{zz}^{(k)} : \qquad \frac{1}{2}\frac{d^2 R_{xx}^{(2)}}{dx^2} - N_{zz}^{(k)} = 0 \quad (k = 1, 2, 3)\,. \tag{2-C.6}$$

where the force and moment resultants are defined as follows:

$$N_{xx} = \int_{z_1}^{z_4} {}^H\sigma_{xx}\ dz = \sum_{k=1}^{k=3} \int_{z_k}^{z_{k+1}} {}^H\sigma_{xx}^{(k)}\ dz, \tag{2-C.7}$$

$$M_{xx}^{(k)} = \int_{z_k}^{z_{k+1}} {}^H\sigma_{xx}^{(k)}\ z\ dz, \tag{2-C.8}$$

$$M_{xx} = \int_{z_1}^{z_4} {}^H\sigma_{xx}\ z\ dx = \sum_{k=1}^{k=3} \int_{z_k}^{z_{k+1}} {}^H\sigma_{xx}^{(k)}\ z\ dz = \sum_{k=1}^{k=3} M_{xx}^{(k)}, \tag{2-C.9}$$

$$Q_{xz}^{(k)} = \int_{z_k}^{z_{k+1}} \sigma_{xz}^{(k)}\ dz, \tag{2-C.10}$$

$$Q_{xz} = \int_{z_1}^{z_4} \sigma_{xz}\ dz = \sum_{k=1}^{3} \int_{z_k}^{z_{k+1}} \sigma_{xz}^{(k)}\ dz = \sum_{k=1}^{3} Q_{xz}^{(k)}, \tag{2-C.11}$$

$$R_{xx}^{(k)} = \int_{z_k}^{z_{k+1}} \sigma_{xx}^{(k)}\ z^2\ dz, \tag{2-C.12}$$

$$N_{zz}^{(k)} = \int_{z_k}^{z_{k+1}} \sigma_{zz}^{(k)}\ dz. \tag{2-C.13}$$

If we sum up equations (2-C.5), we receive

$$\sum_{k=1}^{3} \frac{dM_{xx}^{(k)}}{dx} - \sum_{k=1}^{3} Q_{xz}^{(k)} = 0, \tag{2-C.14}$$

or

$$\frac{dM_{xx}}{dx} - Q_{xz} = 0. \tag{2-C.15}$$

From equations (2-C.4) and (2-C.15) it follows:

$$\frac{dQ_{xz}}{dx} + \frac{q_u + q_l}{b} = 0. \tag{2-C.16}$$

Equilibrium equations (2-C.13) and (2-C.16), obtained from the virtual work principle, are the same as equations (2.2.144) and (2.2.149), which express the statement that the second forms of the transverse stresses satisfy the boundary conditions on the upper surface of the sandwich plate[7]. Therefore, the virtual work principle contains information that the second forms of the transverse stresses satisfy the boundary conditions on the upper surface of the plate

[7]Note that the transverse force resultants $Q_{xz}^{(k)}$, which enter into the equation (2-C.16), are defined the same way as $Q_{xz}^{(k)}$, which enter into the equation (2.2.149): they are defined in terms of $\sigma_{xz}^{(k)}$, the second forms of the transverse shear stresses, not in terms of ${}^{H}\sigma_{xz}^{(k)}$

2.8 Appendix 2-D

Equivalence of the virtual work principle for a plate with transverse stresses obtained from the pointwise equilibrium equations, to the virtual work principle for a plate with transverse stresses obtained from the constitutive equations

In Appendix 2-C a statement was made (with a reference to the Appendix 2-D) that the virtual work principle for a plate can be written in two equivalent forms:

$$\iiint\limits_{(V)} \left({}^{H}\sigma_{xx}\, \delta\varepsilon_{xx} + 2\, {}^{H}\sigma_{xz}\, \delta\varepsilon_{xz} + {}^{H}\sigma_{zz}\, \delta\varepsilon_{zz} \right) dV$$

$$-\int_0^L q_u \left(\delta\, w\Big|_{z=z_4} \right) dx - \int_0^L q_l \left(\delta\, w\Big|_{z=z_1} \right) dx = 0 \tag{2-D.1}$$

and

$$\iiint\limits_{(V)} \left({}^{H}\sigma_{xx}\, \delta\varepsilon_{xx} + 2\sigma_{xz}\, \delta\varepsilon_{xz} + \sigma_{zz}\, \delta\varepsilon_{zz} \right) dV$$

$$-\int_0^L q_u \left(\delta\, w\Big|_{z=z_4} \right) dx - \int_0^L q_l \left(\delta\, w\Big|_{z=z_1} \right) dx = 0\ , \tag{2-D.2}$$

where in the first equation the transverse stresses ${}^{H}\sigma_{xz}$, ${}^{H}\sigma_{zz}$ are obtained from the constitutive equations, and in the second equation the transverse stresses σ_{xz} and σ_{zz} are expressed in terms of the unknown functions by integration of the pointwise equilibrium equations $\sigma_{ij,j} = 0$. The equivalence of variational principles (2-D.1) and (2-D.2) is in the sense that both of these variational principles produce the same differential equations and boundary conditions.

This statement was a necessary logical link in the proof that the finite element formulation, based on the virtual work principle (2-D.1) guarantees that the second forms of the transverse stresses [8] satisfy the boundary conditions on the upper surface of the plate (pages 79 – 82).

[8]obtained by integration of the equilibrium equations $\sigma_{xx,x} + \sigma_{xz,z} = 0$, $\sigma_{zx,x} + \sigma_{zz,z} = 0$.

In this Appendix we will show that for homogeneous isotropic plates the virtual work principles (2-D.1) and (2-D.2) produce the same differential equations and boundary conditions. For a sandwich plate this can be shown in a similar fashion, but the derivation is much more voluminous.

The differential equations and boundary conditions for a homogeneous isotropic plate were derived from the virtual work principle (2-D.1) in chapter 2 (equations (2.1.47)-(2.1.56)), and these equations, written here again, are

$$\delta u_0: \quad (1-\nu)\left(u_0'' - \frac{h^2}{24}\varepsilon_{zz}'''\right) + \nu\varepsilon_{zz}' = 0 \quad (0 \le x \le L) \quad \text{(eqn 2.1.47)},$$

$$\delta w_0: \quad \frac{1-\nu}{(1+\nu)(1-2\nu)}\frac{bEh^3}{12}\left(w_0^{IV} - 2\varepsilon_{xz}'''\right) = q_u + q_l \quad (0 \le x \le L) \quad \text{(eqn 2.1.48)},$$

$$\delta\varepsilon_{xz}: \quad \varepsilon_{xz} + \frac{h^2(1-\nu)}{12(1-2\nu)}\left(w_0''' - 2\varepsilon_{xz}''\right) = 0 \quad (0 \le x \le L) \quad \text{(eqn 2.1.49)},$$

$$\delta\varepsilon_{zz}: \quad \nu\left(u_0' - \frac{h^2}{24}\varepsilon_{zz}''\right) + (1+\nu)\left[\varepsilon_{zz} + \frac{h^2}{8}\left(\frac{h^2}{40}\varepsilon_{zz}^{IV} - \frac{1}{3}u_0'''\right)\right] =$$
$$= \frac{(1+\nu)(1-2\nu)}{bE}(q_u - q_l) \quad (0 \le x \le L) \quad \text{(eqn 2.1.50)}.$$

$$\text{Either } (1-\nu)\left(u_0' - \frac{h^2}{24}\varepsilon_{zz}''\right) + \nu\varepsilon_{zz} = 0 \text{ or } u_0 \text{ specified at } x = 0, L \quad \text{(eqn 2.1.51)};$$

$$\text{either } 2\varepsilon_{xz}' - w_0'' = 0 \text{ or } \varepsilon_{xz} \text{ specified at } x = 0, L \quad \text{(eqn 2.1.52)};$$

$$\text{either } 2\varepsilon_{xz}' - w_0'' = 0 \text{ or } w_0' \text{ specified at } x = 0, L \quad \text{(eqn 2.1.53)};$$

$$\text{either } 2\varepsilon_{xz}'' - w_0''' = 0 \text{ or } w_0 \text{ specified at } x = 0, L \quad \text{(eqn 2.1.54)};$$

$$\text{either } (1-\nu)\left(u_0' - \frac{3}{40}h^2\varepsilon_{zz}''\right) + \nu\varepsilon_{zz} = 0 \text{ or } \varepsilon_{zz}' \text{ specified at } x = 0, L \quad \text{(eqn 2.1.55)};$$

$$\text{either } (1-\nu)\left(u_0'' - \frac{3}{40}h^2\varepsilon_{zz}'''\right) + \nu\varepsilon_{zz}' = 0 \text{ or } \varepsilon_{zz} \text{ specified at } x = 0, L \quad \text{(eqn 2.1.56)}.$$

Now, let us derive differential equations and boundary conditions from the virtual work principle (2-D.2). The expressions for the strain and the stresses in terms of the unknown functions $u_0(x)$,

$w_0(x)$, $\varepsilon_{xz}(x)$, $\varepsilon_{zz}(x)$, that enter into the virtual work principle (2-D.2), were found in chapter 2. These expressions are

$$w = w_0(x) + \varepsilon_{zz}(x)z \quad \text{(eqn 2.1.24)},$$

$$\varepsilon_{xx} = u_0' + (2\varepsilon_{xz}' - w_0'')\,z - \frac{1}{2}\varepsilon_{zz}''z^2 \quad \text{(eqn 2.1.27)},$$

$$\sigma_{xx}^H = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\left[u_0' + (2\varepsilon_{xz}' - w_0'')\,z - \frac{1}{2}\varepsilon_{zz}''z^2\right] + \nu\varepsilon_{zz}\right\} \quad \text{(eqn 2.1.28)},$$

$$\sigma_{xz} = -\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[u_0''\left(z+\frac{h}{2}\right) + \frac{1}{2}\left(2\varepsilon_{xz}'' - w_0'''\right)\left(z^2 - \frac{h^2}{4}\right) - \frac{1}{6}\varepsilon_{zz}'''\left(z^3 + \frac{h^3}{8}\right)\right] -$$
$$\frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{zz}'\left(z+\frac{h}{2}\right) \quad \text{(eqn 2.1.29)},$$

$$\sigma_{zz} = -\frac{q_l}{b} + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\frac{1}{8}(2z+h)^2u_0''' + \frac{1}{24}(z-h)(2z+h)^2\left(2\varepsilon_{xz}''' - w_0^{IV}\right)\right.$$
$$\left. -\frac{1}{384}\left(4z^2 - 4hz + 3h^2\right)(2z+h)^2\varepsilon_{zz}^{IV}\right] + \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{1}{8}(2z+h)^2\varepsilon_{zz}'' \quad (2.1.30).$$

Substitution of equations (2.1.24), (2.1.27) – (2.1.30) into equation (2-D.2) yields:

$$0 = b\int_0^L\int_{-h/2}^{h/2}\frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\left[u_0' + (2\varepsilon_{xz}' - w_0'')\,z - \frac{1}{2}\varepsilon_{zz}''z^2\right] + \nu\varepsilon_{zz}\right\}\times$$

$$\times\left[\delta u_0' + (2\delta\varepsilon_{xz}' - \delta w_0'')\,z - \frac{1}{2}\delta\varepsilon_{zz}''z^2\right]\,dz\,dx +$$

$$+b\int_0^L\int_{-h/2}^{h/2}\left\{-\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[u_0''\left(z+\frac{h}{2}\right) + \frac{1}{2}\left(2\varepsilon_{xz}'' - w_0'''\right)\left(z^2 - \frac{h^2}{4}\right) - \frac{1}{6}\varepsilon_{zz}'''\left(z^3 + \frac{h^3}{8}\right)\right] -\right.$$

$$\left.\frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{zz}'\left(z+\frac{h}{2}\right)\right\}2\,\delta\varepsilon_{xz}\,dz\,dx +$$

$$+b\int_0^L\int_{-h/2}^{h/2}\left\{-\frac{q_l}{b} + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\frac{1}{8}(2z+h)^2u_0''' + \frac{1}{24}(z-h)(2z+h)^2\left(2\varepsilon_{xz}''' - w_0^{IV}\right)\right.\right.$$

$$-\frac{1}{384}\left(4z^2-4hz+3h^2\right)(2z+h)^2\varepsilon_{zz}^{IV}\Big]+\frac{E\nu}{(1+\nu)(1-2\nu)}\frac{1}{8}(2z+h)^2\varepsilon_{zz}^{''}\Big\}\ \delta\varepsilon_{zz}\ dz\ dx+$$

$$+\int_0^L\frac{h}{2}(q_l-q_u)(\delta\varepsilon_{zz})\ dx-\int_0^L(q_l+q_u)(\delta w_0)\ dx. \tag{2-D.3}$$

Performing integration with respect to z in equation (2-D.3), we obtain:

$$\frac{E}{(1+\nu)(1-2\nu)}hb\int_0^L\left[(1-\nu)\left(u_0'-\frac{1}{24}h^2\varepsilon_{zz}''\right)+\nu\varepsilon_{zz}\right]\delta u_0'\ dx+$$

$$+\frac{E}{(1+\nu)(1-2\nu)}(1-\nu)\frac{1}{6}h^3b\int_0^L\left(2\varepsilon_{xz}'-w_0''\right)\ \delta\varepsilon_{xz}'\ dx$$

$$-\frac{E}{(1+\nu)(1-2\nu)}(1-\nu)\frac{1}{12}h^3b\int_0^L\left(2\varepsilon_{xz}'-w_0''\right)\ \delta w_0''\ dx+$$

$$+\frac{E}{(1+\nu)(1-2\nu)}h^3b\int_0^L\left[(1-\nu)\left(-\frac{1}{24}u_0'+\frac{1}{320}h^2\varepsilon_{zz}''\right)-\frac{1}{24}\nu\varepsilon_{zz}\right]\delta\varepsilon_{zz}''\ dx+$$

$$+\frac{E}{(1+\nu)(1-2\nu)}b\int_0^L\left\{-(1-\nu)\left[\frac{1}{2}h^2u_0''-\frac{1}{12}h^3\left(2\varepsilon_{xz}''-w_0'''\right)-\frac{1}{48}h^4\varepsilon_{zz}'''\right]-\nu\frac{1}{2}h^2\varepsilon_{zz}'\right\}2\ \delta\varepsilon_{xz}\ dx+$$

$$+b\int_0^L\Big\{-\frac{q_l}{b}h+\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\Big[\frac{1}{6}h^3u_0'''-\frac{1}{24}h^4\left(2\varepsilon_{xz}'''-w_0^{IV}\right)$$

$$-\frac{1}{120}h^5\varepsilon_{zz}^{IV}\Big]+\frac{E\nu}{(1+\nu)(1-2\nu)}\frac{1}{6}h^3\varepsilon_{zz}''\Big\}\ \delta\varepsilon_{zz}\ dx+$$

$$+\int_0^L\frac{h}{2}(q_l-q_u)(\delta\varepsilon_{zz})\ dx-\int_0^L(q_l+q_u)(\delta w_0)\ dx. \tag{2-D.4}$$

Integration by parts in equation (2-D.4) yields:

$$\frac{Ehb}{(1+\nu)(1-2\nu)}\left[(1-\nu)\left(u_0'-\frac{1}{24}h^2\varepsilon_{zz}''\right)+\nu\varepsilon_{zz}\right](\delta u_0)\Big|_0^L$$

$$-\frac{Ehb}{(1+\nu)(1-2\nu)}\int_0^L \left[(1-\nu)\left(u_0'' - \frac{1}{24}\ h^2\varepsilon_{zz}'''\right) \ +\ \nu\varepsilon_{zz}'\right](\delta u_0)\ dx+$$

$$+\frac{1}{6}h^3b\frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}\left(2\varepsilon_{xz}' - w_0''\right)\ (\delta\varepsilon_{xz})\Big|_0^L$$

$$-\frac{1}{6}h^3b\frac{E\,(1-\nu)}{(1+\nu)(1-2\nu)}\int_0^L \left(2\varepsilon_{xz}'' - w_0'''\right)\ (\delta\varepsilon_{xz})\ dx$$

$$-\frac{1}{12}h^3b\frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)}\left(2\varepsilon_{xz}' - w_0''\right)(\delta w_0')\Big|_0^L +$$

$$+\frac{1}{12}h^3b\frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)}\left(2\varepsilon_{xz}'' - w_0'''\right)(\delta w_0)\Big|_0^L$$

$$-\frac{1}{12}h^3b\frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)}\int_0^L \left(2\varepsilon_{xz}''' - w_0^{IV}\right)\ (\delta w_0)\ dx$$

$$+\frac{E}{(1+\nu)\,(1-2\nu)}h^3b\left[(1-\nu)\left(-\frac{1}{24}u_0' + \frac{1}{320}h^2\varepsilon_{zz}''\right) - \frac{1}{24}\nu\varepsilon_{zz}\right](\delta\varepsilon_{zz}')\Big|_0^L$$

$$-\frac{E}{(1+\nu)\,(1-2\nu)}h^3b\left[(1-\nu)\left(-\frac{1}{24}u_0'' + \frac{1}{320}h^2\varepsilon_{zz}'''\right) - \frac{1}{24}\nu\varepsilon_{zz}'\right](\delta\varepsilon_{zz})\Big|_0^L +$$

$$+\frac{E}{(1+\nu)\,(1-2\nu)}h^3b\int_0^L \left[(1-\nu)\left(-\frac{1}{24}u_0''' + \frac{1}{320}h^2\varepsilon_{zz}^{IV}\right) - \frac{1}{24}\nu\varepsilon_{zz}''\right](\delta\varepsilon_{zz})\ dx+$$

$$+\frac{E}{(1+\nu)(1-2\nu)}b\int_0^L \left\{-(1-\nu)\left[\frac{1}{2}h^2u_0'' - \frac{1}{12}h^3\left(2\varepsilon_{xz}'' - w_0'''\right) - \frac{1}{48}h^4\varepsilon_{zz}'''\right] - \nu\frac{1}{2}h^2\varepsilon_{zz}'\right\} 2\ \delta\varepsilon_{xz}\ dx+$$

$$+b\int_0^L \left\{-\frac{q_l}{b}h + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\frac{1}{6}h^3u_0''' - \frac{1}{24}h^4\left(2\varepsilon_{xz}''' - w_0^{IV}\right)\right.\right.$$

$$-\frac{1}{120}h^5\varepsilon_{zz}^{IV}\bigg] + \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{1}{6}h^3\varepsilon_{zz}''\bigg\}\ \delta\varepsilon_{zz}\ dx +$$

$$+\int_0^L \frac{h}{2}(q_l - q_u)(\delta\varepsilon_{zz})\ dx - \int_0^L (q_l + q_u)(\delta w_0)\ dx. \tag{2-D.5}$$

From the last equation we obtain the following differential equations:

$$\delta u_0: \qquad (1-\nu)\left(u_0'' - \frac{1}{24}\,h^2\varepsilon_{zz}'''\right) + \nu\varepsilon_{zz}' = 0, \tag{2-D.6}$$

$$\delta w_0: \qquad E\frac{h^3 b}{12}\frac{(1-\nu)}{(1+\nu)(1-2\nu)}\left(w_0^{IV} - 2\varepsilon_{xz}'''\right) - (q_l + q_u) = 0, \tag{2-D.7}$$

$$\delta\varepsilon_{xz}: \ -\frac{1}{6}h^3 b\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(2\varepsilon_{xz}'' - w_0'''\right) + \frac{E}{(1+\nu)(1-2\nu)}2b\ \times$$
$$\times\left\{-(1-\nu)\left[\frac{1}{2}h^2u_0'' - \frac{1}{12}h^3\left(2\varepsilon_{xz}'' - w_0'''\right) - \frac{1}{48}h^4\varepsilon_{zz}'''\right] - \nu\frac{1}{2}h^2\varepsilon_{zz}'\right\} = 0, \tag{2-D.8}$$

$$\delta\varepsilon_{zz}: \ \frac{h}{2}(q_l - q_u) + b\left\{-\frac{q_l}{b}h + \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\frac{1}{6}h^3u_0''' - \frac{1}{24}h^4\left(2\varepsilon_{xz}''' - w_0^{IV}\right)\right.\right.$$

$$\left.\left.-\frac{1}{120}h^5\varepsilon_{zz}^{IV}\right] + \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{1}{6}h^3\varepsilon_{zz}''\right\} +$$

$$+\frac{E}{(1+\nu)(1-2\nu)}h^3 b\left[(1-\nu)\left(-\frac{1}{24}u_0''' + \frac{1}{320}h^2\varepsilon_{zz}^{IV}\right) - \frac{1}{24}\nu\varepsilon_{zz}''\right] = 0. \tag{2-D.9}$$

After simple transformations, equation (2-D.8) can be written in the form

$$\delta\varepsilon_{xz}: (1-\nu)\left(u_0'' - \frac{1}{24}h^2\varepsilon_{zz}'''\right) + \nu\varepsilon_{zz}' = 0\ , \tag{2-D.8'}$$

and equation (2-D.9) can be written in the form:

$$\delta\varepsilon_{zz}:\ \ E\frac{h^3 b}{12}\frac{(1-\nu)}{(1+\nu)(1-2\nu)}\left(w_0^{IV} - 2\varepsilon_{xz}'''\right) - (q_l + q_u) +$$

$$+\frac{Eh^2 b}{4(1+\nu)(1-2\nu)}\underbrace{\left[(1-\nu)\left(u_0''' - \frac{1}{24}h^2\varepsilon_{zz}^{IV}\right) + \nu\varepsilon_{zz}''\right]}_{0\ \text{ because of eqn (2-E.6)}} = 0. \tag{2-D.9'}$$

In summary, the differential equations that follow from the virtual work principle (2-D.2), are the following:

$$\delta u_0 : \quad (1-\nu)\left(u_0'' - \frac{1}{24}\, h^2 \varepsilon_{zz}''' \right) \; + \; \nu\varepsilon_{zz}' = 0, \tag{2-D.6}$$

$$\delta w_0 : \quad E\frac{h^3 b}{12}\frac{(1-\nu)}{(1+\nu)(1-2\nu)}\left(w_0^{IV} - 2\varepsilon_{xz}'''\right) - (q_l + q_u) = 0 \tag{2-D.7}$$

$$\delta\varepsilon_{xz} : (1-\nu)\left(u_0'' - \frac{1}{24}h^2\varepsilon_{zz}'''\right) + \nu\varepsilon_{zz}' = 0 \; , \tag{2-D.8'}$$

$$\delta\varepsilon_{zz}\!: \quad E\frac{h^3 b}{12}\frac{(1-\nu)}{(1+\nu)(1-2\nu)}\left(w_0^{IV} - 2\varepsilon_{xz}'''\right) - (q_l + q_u) = 0 \; . \tag{2-D.9'}$$

We see that only two out of these four equations are independent, but these two equations are the same equations that follow from the virtual work principle (2-D.1). As can be seen from equation (2-D.5), the boundary conditions, that follow from the virtual work principle (2-D.2), are the same as the boundary conditions that follow from the virtual work principle (2-D.1) (equations 2.1.51 – 2.1.56).

In a similar fashion it can be shown that the same conclusions can be made for the layerwise model of the sandwich plate. But for the layerwise model of the sandwich plate the proof is much more voluminous.

2.9 Appendix 2-E

Exact Elasticity Solution for a Simply Supported Isotropic Sandwich Plate in Static Cylindrical Bending under a Uniform Load on the Upper Surface

Let us consider cylindrical bending of a wide symmetric sandwich plate with isotropic face sheets and the core (Figure 2.3). The upper surface of the plate is under a uniform load with intensity (force per unit length) q. By q we denoted not an absolute value of the load intensity, but a projection of the load intensity on the z-axis, i.e. q can be positive or negative, depending on the direction of the load. Along the edges $x = 0, L$ the plate is simply supported. We will denote a number of layer of the plate by a superscript k $(k = 1, 2, 3)$. The Young's moduli of the face sheets are different from that of the core $(E^{(1)} = E^{(3)} \neq E^{(2)})$, but the Poisson ratio is the same for all layers $(\nu^{(1)} = \nu^{(2)} = \nu^{(3)})$.

The equations of linear elasticity, as applied to this problem, have the form:

equilibrium equations:

$$\sigma_{xx,x}^{(k)} + \sigma_{xz,z}^{(k)} = 0, \tag{2-E.1}$$

$$\sigma_{xz,x}^{(k)} + \sigma_{zz,z}^{(k)} = 0; \tag{2-E.2}$$

strain-displacement relations for plane strain:

$$\varepsilon_{xx}^{(k)} = u_{,x}^{(k)}, \tag{2-E.3}$$

$$\varepsilon_{zz}^{(k)} = w_{,z}^{(k)}, \tag{2-E.4}$$

$$2\varepsilon_{xz}^{(k)} = u_{,z}^{(k)} + w_{,x}^{(k)}, \tag{2-E.5}$$

$$\varepsilon_{yy}^{(k)} = \varepsilon_{yz}^{(k)} = \varepsilon_{xy}^{(k)} = 0; \tag{2-E.6}$$

constitutive relations for plane strain:

$$\sigma_{xx}^{(k)} = \frac{E^{(k)}}{(1+\nu)(1-2\nu)} \left[(1-\nu)\,\varepsilon_{xx}^{(k)} + \nu \varepsilon_{zz}^{(k)} \right]; \tag{2-E.7}$$

$$\sigma_{zz}^{(k)} = \frac{E^{(k)}}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\varepsilon_{zz}^{(k)} + \nu\varepsilon_{xx}^{(k)}\right]; \tag{2-E.8}$$

$$\sigma_{yy}^{(k)} = \frac{E^{(k)}}{(1+\nu)(1-2\nu)}\left(\varepsilon_{xx}^{(k)} + \varepsilon_{zz}^{(k)}\right) = \nu\left(\sigma_{xx}^{(k)} + \sigma_{zz}^{(k)}\right); \tag{2-E.9}$$

$$\sigma_{xz}^{(k)} = \frac{E^{(k)}}{(1+\nu)}\varepsilon_{xz}^{(k)}; \tag{2-E.10}$$

$$\sigma_{xy}^{(k)} = \sigma_{yz}^{(k)} = 0; \tag{2-E.11}$$

or, in the inverse form

$$\varepsilon_{xx}^{(k)} = \frac{1-\nu^2}{E^{(k)}}\left(\sigma_{xx}^{(k)} - \frac{\nu}{1-\nu}\sigma_{zz}^{(k)}\right); \tag{2-E.12}$$

$$\varepsilon_{zz}^{(k)} = \frac{1-\nu^2}{E^{(k)}}\left(\sigma_{zz}^{(k)} - \frac{\nu}{1-\nu}\sigma_{xx}^{(k)}\right); \tag{2-E.13}$$

$$\varepsilon_{xz}^{(k)} = \frac{1+\nu}{E^{(k)}}\sigma_{xz}^{(k)}; \tag{2-E.14}$$

$$\varepsilon_{yy}^{(k)} = \varepsilon_{xy}^{(k)} = \varepsilon_{yz}^{(k)} = 0; \tag{2-E.15}$$

boundary conditions:

$$w = 0 \text{ at } x = 0, L \text{ and } z = 0; \tag{2-E.16}$$

$$\left.\begin{aligned}
&\int_{-h/2}^{-t/2} \sigma_{xx}^{(1)}\, dz = 0 \text{ at } x = 0, L \\
&\int_{-t/2}^{t/2} \sigma_{xx}^{(2)}\, dz = 0 \text{ at } x = 0, L \\
&\int_{t/2}^{h/2} \sigma_{xx}^{(3)}\, dz = 0 \text{ at } x = 0, L
\end{aligned}\right\} \tag{2-E.17}$$

$$\int_{-h/2}^{h/2} \sigma_{xx}\, z\, dz = 0 \text{ at } x = 0, L$$

or

$$\int_{-h/2}^{-t/2} \sigma_{xx}^{(1)}\ z\ dz + \int_{-t/2}^{t/2} \sigma_{xx}^{(2)}\ z\ dz + \int_{t/2}^{h/2} \sigma_{xx}^{(3)}\ z\ dz = 0 \text{ at } x = 0, L; \tag{2-E.18}$$

$$\int_{-h/2}^{h/2} \sigma_{xz}\ dz = \frac{q_u}{b}\frac{L}{2} \text{ at } x = 0,$$

$$\int_{-h/2}^{h/2} \sigma_{xz}\ dz = -\frac{q_u}{b}\frac{L}{2} \text{ at } x = L$$

or

$$\int_{-h/2}^{-t/2} \sigma_{xz}^{(1)}\ dz + \int_{-t/2}^{t/2} \sigma_{xz}^{(2)} dz + \int_{t/2}^{h/2} \sigma_{xz}^{(3)} dz = \frac{q_u}{b}\frac{L}{2} \text{ at } x = 0; \tag{2-E.19}$$

$$\int_{-h/2}^{-t/2} \sigma_{xz}^{(1)}\ dz + \int_{-t/2}^{t/2} \sigma_{xz}^{(2)} dz + \int_{t/2}^{h/2} \sigma_{xz}^{(3)} dz = -\frac{q_u}{b}\frac{L}{2} \text{ at } x = L; \tag{2-E.20}$$

$$\sigma_{xz}^{(1)} = 0, \sigma_{zz}^{(1)} = 0 \text{ at } z = -\frac{h}{2}; \tag{2-E.21}$$

$$\sigma_{xz}^{(3)} = 0, \sigma_{zz}^{(3)} = \frac{q_u}{b} \text{ at } z = \frac{h}{2}; \tag{2-E.22}$$

symmetry condition:

$$u\left(\frac{L}{2}\right) = 0; \tag{2-E.23}$$

continuity of displacements and stresses at the interfaces between the core and the face sheets:

$$u^{(1)} = u^{(2)}, w^{(1)} = w^{(2)}, \sigma_{xz}^{(1)} = \sigma_{xz}^{(2)}, \sigma_{zz}^{(1)} = \sigma_{zz}^{(2)} \text{ at } z = -\frac{t}{2}; \tag{2-E.24}$$

$$u^{(2)} = u^{(3)}, w^{(2)} = w^{(3)}, \sigma_{xz}^{(2)} = \sigma_{xz}^{(3)}, \sigma_{zz}^{(2)} = \sigma_{zz}^{(3)} \text{ at } z = \frac{t}{2}. \tag{2-E.25}$$

We will find exact elasticity solution of this problem, following a procedure, suggested by Pikul (1977) for a problem with different boundary conditions.

Let us take shear strains of the layers in the form

$$\varepsilon_{xz}^{(k)} = R\left(z^2 - c^{(k)}\right)x, \tag{2-E.26}$$

where R and $c^{(k)}$ are the unknown constants, which are to be defined. Upon substitution of (2-E.26) into the constitutive relations (2-E.10), we receive

$$\sigma_{xz}^{(k)} = \frac{E^{(k)}}{1+\nu}R\left(z^2 - c^{(k)}\right)x. \tag{2-E.27}$$

Let us substitute expression (2-E.27) into the equilibrium equations (2-E.1) and (2-E.2), and integrate them with respect to x and z correspondingly:

$$\sigma_{xx}^{(k)} = -\frac{E^{(k)}}{1+\nu}R\left[x^2 z + \varphi^{(k)}(z)\right], \tag{2-E.28}$$

$$\sigma_{zz}^{(k)} = -\frac{E^{(k)}}{1+\nu}R\left[\frac{z^3}{3} - c^{(k)}z + \psi^{(k)}(x)\right], \tag{2-E.29}$$

where $\varphi^{(k)}(z)$ and $\psi^{(k)}(x)$ are the arbitrary functions of integration. Substitution of expressions (2-E.28) and (2-E.29) into the constitutive relations (2-E.12) and (2-E.13) yields:

$$\varepsilon_{xx}^{(k)} = -(1-\nu)R\left[x^2 z + \varphi^{(k)}(z) - \frac{\nu}{1-\nu}\left(\frac{z^3}{3} - c^{(k)}z + \varphi^{(k)}(z)\right)\right], \tag{2-E.30}$$

$$\varepsilon_{zz}^{(k)} = -(1-\nu)R\left[\frac{z^3}{3} - c^{(k)}z + \psi^{(k)}(x) - \frac{\nu}{1-\nu}\left(x^2 z + \varphi^{(k)}(z)\right)\right]. \tag{2-E.31}$$

Substitution of (2-E.30) into (2-E.3) and integration of the resulting equation with respect to x yields:

$$u^{(k)} = -(1-\nu)R\left[\frac{x^3}{3}z + x\varphi^{(k)}(z) - \frac{\nu}{1-\nu}\left(\frac{z^3}{3} - c^{(k)}z\right)x - \frac{\nu}{1-\nu}\int \psi^{(k)}(x)\ dx + \chi^{(k)}(z)\right]. \tag{2-E.32}$$

where $\chi^{(k)}(z)$ is an arbitrary function of integration. Substitution of (2-E.31) into (2-E.4) and integration of the resulting equation with respect to z yields

$$w^{(k)} = -R(1-\nu)\left[\frac{z^4}{12} - c^{(k)}\frac{z^2}{2} + z\psi^{(k)}(x) - \frac{\nu}{1-\nu}x^2\frac{z^2}{2} - \frac{\nu}{1-\nu}\int \varphi^{(k)}(z)\ dz + \lambda^{(k)}(x)\right]. \tag{2-E.33}$$

Upon substitution of expressions (2-E.32)and (2-E.33) for displacements into the strain-displacement relation (2-E.5), we receive the second form of expression for $\varepsilon_{xz}^{(k)}$:

$$\varepsilon_{xz}^{(k)} = -R(1-\nu)\left[\frac{x^3}{3} + x\frac{d\varphi^k(z)}{dz} - \frac{\nu}{1-\nu}\left(z^2 - c^{(k)}\right)x + \frac{d\chi^{(k)}(z)}{dz} + z\frac{d\psi^{(k)}(x)}{dx} - \frac{\nu}{1-\nu}z^2 x + \frac{d\lambda^{(k)}(x)}{dx}\right]. \quad (2\text{-E.34})$$

Exact elasticity solution is possible if both expressions for $\varepsilon_{xz}^{(k)}$, (2-E.26) and (2-E.34), are identically equal:

$$R\left(z^2 - c^{(k)}\right)x \equiv -R(1-\nu)\left[\frac{x^3}{3} + x\frac{d\varphi^k(z)}{dz} - \frac{\nu}{1-\nu}\left(z^2 - c^{(k)}\right)x + \frac{d\chi^{(k)}(z)}{dz} + z\frac{d\psi^{(k)}(x)}{dx} - \frac{\nu}{1-\nu}z^2 x + \frac{d\lambda^{(k)}(x)}{dx}\right]. \quad (2\text{-E.35})$$

In order to find the functions $\varphi^{(k)}(z)$, $\psi^{(k)}(x)$, $\lambda^{(k)}(x)$ and $\chi^{(k)}(z)$, which make the identity (2-E.35) possible, let us represent the functions $\varphi^{(k)}(z)$, $\psi^{(k)}(x)$ and $\lambda^{(k)}(x)$ in the form:

$$\begin{aligned}
\varphi^{(k)}(z) &= \varphi_1^{(k)}(z) + \varphi_2^{(k)}(z) + \varphi_3^{(k)}(z) + \varphi_4^{(k)}(z), \\
\psi^{(k)}(x) &= \psi_1^{(k)}(x) + \psi_2^{(k)}(x), \\
\lambda^{(k)}(x) &= \lambda_1^{(k)}(x) + \lambda_2^{(k)}(x).
\end{aligned} \quad (2\text{-E.36})$$

Substitution of (2-E.36) into (2-E.35) yields

$$\left(\frac{x^3}{3} + \frac{d\lambda_2^{(k)}(x)}{dx}\right) + \left[x\frac{d\varphi_1^{(k)}(z)}{dz} - \frac{\nu}{1-\nu}\left(z^2 - c^{(k)}\right)x + \frac{\left(z^2 - c^{(k)}\right)x}{1+\nu}\right] + \left(x\frac{d\varphi_2^{(k)}(z)}{dz} - \frac{\nu}{\nu-1}xz^2\right) + \left(x\frac{d\varphi_3^{(k)}(z)}{dz} + z\frac{d\psi_1^{(k)}(x)}{dx}\right) + \left(x\frac{d\varphi_4^{(k)}(z)}{dz} + \frac{d\lambda_1^{(k)}}{dx}\right) + \left(\frac{d\chi^{(k)}(z)}{dz} + z\frac{d\psi_2^{(k)}(x)}{dx}\right) \equiv 0. \quad (2\text{-E.37})$$

The identity (2-E.37) will take place, if each term in brackets in (2-E.37) is equal to zero. This leads us to differential equations for the functions $\varphi_i^{(k)}(z)$ $(i = 1,2,3,4)$, $\psi_1^{(k)}(x)$, $\psi_2^{(k)}(x)$, $\lambda_1^{(k)}(x)$, $\lambda_2^{(k)}(x)$. When we solve these differential equations and substitute the found functions into expressions (2-

E.36), we find

$$
\begin{aligned}
\varphi^{(k)}(z) &= \frac{\nu(3-\nu)-1}{1-\nu}\left(\frac{z^3}{3} - c^{(k)}z\right) + \frac{\nu}{1-\nu}\frac{z^3}{3} + \beta^{(k)}\frac{z^2}{2} + \kappa^{(k)}z + a^{(k)}, \\
\psi^{(k)}(x) &= -\beta^{(k)}\frac{x^2}{2} + e^{(k)}x + b^{(k)}, \\
\lambda^{(k)}(x) &= -\frac{x^4}{12} - -^{(k)}\frac{x^2}{2} + d^{(k)}, \\
\chi^{(k)}(z) &= -e^{(k)}\frac{z^2}{2} + \kappa^{(k)},
\end{aligned}
\tag{2-E.38}
$$

where $a^{(k)}, b^{(k)}, d^{(k)}, e^{(k)}, \beta^{(k)}, -^{(k)}$ and $\kappa^{(k)}$ are constants of integration. Substitution of (2-E.38) into (2-E.28), (2-E.29), (2-E.32) and (2-E.33) yields

$$
\begin{aligned}
\sigma_{xx}^{(k)} = -\frac{E^{(k)}}{1+\nu}R\Bigg[x^2z + \frac{\nu(3-\nu)-1}{1-\nu}\left(\frac{z^3}{3} - c^{(k)}z\right) + \\
\frac{\nu}{1-\nu}\frac{z^3}{3} + \beta^{(k)}\frac{z^2}{2} + -^{(k)}z + a^{(k)}\Bigg],
\end{aligned}
\tag{2-E.39}
$$

$$
\sigma_{zz}^{(k)} = -\frac{E^{(k)}}{1+\nu}R\left[\frac{z^3}{3} - c^{(k)}z - \beta^{(k)}\frac{x^2}{2} + e^{(k)}x + b^{(k)}\right],
\tag{2-E.40}
$$

$$
\begin{aligned}
u^{(k)} = -(1-\nu)R\Bigg[\frac{x^3}{3}z + \frac{1}{3}(\nu-1)\left(z^2 - 3c^{(k)}\right)zx + \\
\frac{\nu}{1-\nu}\frac{z^3}{3}x + \beta^{(k)}\frac{z^2}{2}x + -^{(k)}zx + a^{(k)}x + \\
\frac{\nu}{1-\nu}\left(\beta^{(k)}\frac{x^3}{6} - e^{(k)}\frac{x^2}{2} - b^{(k)}x\right) - e^{(k)}\frac{z^2}{2} + \kappa^{(k)}\Bigg],
\end{aligned}
\tag{2-E.41}
$$

$$
\begin{aligned}
w^{(k)} = -R(1-\nu)\Bigg[\frac{z^4}{12} - c^{(k)}\frac{z^2}{2} + z\left(-\beta^{(k)}\frac{x^2}{2} + e^{(k)}x + b^{(k)}\right) - \\
\frac{\nu}{1-\nu}\frac{x^2z^2}{2} - \frac{\nu^2(3-\nu)-\nu}{(1-\nu)^2}\left(\frac{z^4}{12} - c^{(k)}\frac{z^2}{2}\right) - \frac{\nu^2}{(1-\nu)^2}\frac{z^4}{12} - \\
\frac{\nu}{1-\nu}\beta^{(k)}\frac{z^3}{6} - \frac{\nu}{1-\nu}-^{(k)}\frac{z^2}{2} - \frac{\nu}{1-\nu}a^{(k)}z - \frac{x^4}{12} - -^{(k)}\frac{x^2}{2} + d^{(k)}\Bigg].
\end{aligned}
\tag{2-E.42}
$$

Substitution of expressions (2-E.39)-(2-E.42) into the boundary conditions, symmetry conditions and continuity conditions (2-E.16)-(2-E.25) yields equations for the constants of integration. Solving these equations and substituting expressions for the constants of integration into expressions (2-E.27), (2-E.39) and (2-E.40) for stresses, we receive

$$
\sigma_{xz}^{(1)} = \frac{q_u}{b}\frac{6E^{(1)}}{\left(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)}\right)}\left(z^2 - \frac{1}{4}h^2\right)\left(x - \frac{1}{2}L\right),
\tag{2-E.43}
$$

$$\sigma_{xz}^{(2)} = \frac{q_u}{b}\frac{6E^{(2)}}{\left(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)}\right)}\left(z^2 - \frac{1}{4}\frac{h^2E^{(1)} - t^2E^{(1)} + t^2E^{(2)}}{E^{(2)}}\right) \times$$
$$\times \left(x - \frac{1}{2}L\right), \tag{2-E.44}$$

$$\sigma_{xz}^{(3)} = \frac{q_u}{b}\frac{6E^{(1)}}{\left(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)}\right)}\left(z^2 - \frac{1}{4}h^2\right)\left(x - \frac{1}{2}L\right), \tag{2-E.45}$$

$$\sigma_{xx}^{(1)} = \frac{q_u}{b}\frac{6E^{(1)}}{\left(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)}\right)}\left\{(L - x)\, xz - \frac{1}{5}\left[\frac{3}{4}\left(h^2 + t^2\right) + ht\right] z + \right.$$
$$\left. \frac{2}{3}z^3 - \frac{1}{15}ht\,(h + t) - \frac{1}{60}\left(t^3 + h^3\right)\right\}, \tag{2-E.46}$$

$$\sigma_{xx}^{(2)} = \frac{q_u}{b}\frac{E^{(2)}}{\left(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)}\right)}\left[6\,(L - x)\, xz + 4z^3 - \frac{3}{5}t^2 z\right], \tag{2-E.47}$$

$$\sigma_{xx}^{(3)} = \frac{q_u}{b}\frac{6E^{(1)}}{\left(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)}\right)}\left\{(L - x)\, xz - \frac{1}{5}\left[\frac{3}{4}\left(h^2 + t^2\right) + ht\right] z \right.$$
$$\left. + \frac{2}{3}z^3 + \frac{1}{15}ht\,(t + h) + \frac{1}{60}\left(t^3 + h^3\right)\right\}, \tag{2-E.48}$$

$$\sigma_{zz}^{(1)} = -\frac{q_u}{b}\frac{6E^{(1)}}{\left(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)}\right)}\left(\frac{1}{3}z^3 - \frac{1}{4}h^2 z - \frac{1}{12}h^3\right), \tag{2-E.49}$$

$$\sigma_{zz}^{(2)} = -\frac{q_u}{b}\frac{6E^{(2)}}{\left(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)}\right)}\left[\frac{1}{3}z^3 - \frac{1}{4}\frac{h^2E^{(1)} - t^2E^{(1)} + t^2E^{(2)}}{E^{(2)}} z \right.$$
$$\left. - \frac{1}{12}\frac{h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)}}{E^{(2)}}\right], \tag{2-E.50}$$

$$\sigma_{zz}^{(3)} = -\frac{q_u}{b}\frac{6E^{(1)}}{\left(h^3E^{(1)} - t^3E^{(1)} + t^3E^{(2)}\right)}\left[\frac{1}{3}z^3 - \frac{1}{4}h^2 z - \frac{1}{12}h^3 \right.$$
$$\left. + \frac{1}{6}\left(1 - \frac{E^{(2)}}{E^{(1)}}\right) t^3\right]. \tag{2-E.51}$$

$$w_0 \equiv w^{(2)}\Big|_{z=0} = -\frac{q}{b}\frac{3\left(1 - \nu^2\right)}{E_1h^3 - E_1t^3 + E_2t^3} x\,(L - x) \times$$
$$\times \left[\frac{1}{6}\left(x - \frac{L}{2}\right)^2 - \frac{5}{24}L^2 - \left(\frac{2}{5} + \frac{1}{4}\frac{\nu}{1 - \nu}\right)\frac{E_1h^5 - E_1t^5 + E_2t^5}{E_1h^3 - E_1t^3 + E_2t^3} \right.$$
$$\left. - \frac{3}{4}\frac{E_1\left(h^2 - t^2\right)}{E_1h^3 - E_1t^3 + E_2t^3}\frac{E_1 - E_2}{1 - \nu}\left(\frac{t\left(h^2 - t^2\right)}{E_2} - \frac{\nu t^3}{3E_1}\right)\right]. \tag{2-E.52}$$

If $E^{(1)} = E^{(2)}$, equations (2-E.43)-(2-E.51) give stresses in a homogeneous simply supported plate under a uniform load, derived in Appendix 2-A.

$$w_1 = -\frac{6\frac{q_u}{b}(1-\nu^2)}{E^{(1)}\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}\left\{\frac{L^2-4\left(x-\frac{L}{2}\right)^2}{8}\left[\frac{L^2+4\left(x-\frac{L}{2}\right)^2}{24}-\frac{L^2}{4}-\left(8+\frac{5\nu}{1-\nu}\right)\right.\right.$$

$$\times\frac{h^5-(1-\frac{E^{(2)}}{E^{(1)}})t^5}{20\left[h^3-(1-\frac{E^{(2)}}{E^{(1)}})t^3\right]}-\left(\frac{E^{(1)}-E^{(2)}}{E^{(2)}}(1+\frac{\nu}{1-\nu})\frac{t(h^2-t^2)}{2}+\frac{\nu}{1-\nu}\frac{h^3}{12}\right.$$

$$-\left.\left.\frac{\nu}{1-\nu}\frac{h^3-2(1-\frac{E^{(2)}}{E^{(1)}})t^3}{12}\right)\frac{3(h^2-t^2)}{2\left[h^3-(1-\frac{E^{(2)}}{E^{(1)}})t^3\right]}\right]+\frac{1+\nu}{12(1-\nu)}z^4-\left[\frac{\nu}{1-\nu}\left(\left(x-\frac{L}{2}\right)^2-\frac{L^2}{4}\right.\right.$$

$$-(8+\frac{5\nu}{1-\nu})\frac{h^5-(1-\frac{E^{(2)}}{E^{(1)}})t^5}{20\left[h^3-(1-\frac{E^{(2)}}{E^{(1)}})t^3\right]}-\left(\frac{E^{(1)}-E^{(2)}}{E^{(2)}}(1+\frac{\nu}{1-\nu})\frac{t(h^2-t^2)}{2}+\frac{\nu}{1-\nu}\frac{h^3}{12}\right.$$

$$-\left.\left.\frac{\nu}{1-\nu}\frac{h^3-2(1-\frac{E^{(2)}}{E^{(1)}})t^3}{12}\right)\frac{3\left(h^2-t^2\right)}{2\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}\right)+\left(1+\frac{\nu}{1-\nu}\right)^2\frac{h^2}{4}\right]\frac{z^2}{2}$$

$$+\left[\frac{h^3-2\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3}{12}-\frac{\nu}{1-\nu}\left(-\frac{E^{(1)}-E^{(2)}}{E^{(2)}}\left(1+\frac{\nu}{1-\nu}\right)\frac{t\left(h^2-t^2\right)}{4}\right.\right.$$

$$-\left.\left.\frac{\nu}{1-\nu}\frac{E^{(1)}h}{tE^{(2)}+(h-t)E^{(1)}}\frac{h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3}{12}+\frac{\nu}{1-\nu}\frac{h^3-2\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3}{12}\right)\right]z$$

$$-\frac{E^{(1)}-E^{(2)}}{E^{(2)}}\left(1-\frac{3\nu}{1-\nu}\right)\left(1+\frac{\nu}{1-\nu}\right)\frac{t^2}{32}\left(h^2-t^2\right)$$

$$\left.+\left(1-\frac{\nu^2}{(1-\nu)^2}\right)\frac{t}{24}\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]\left(2-\frac{E^{(1)}}{E^{(2)}}\right)-h^3\right\}; \qquad (2\text{-E.53})$$

$$w_2 = -\frac{6\frac{q_u}{b}(1-\nu^2)}{E^{(1)}\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}\Bigg\{\frac{L^2-4\left(x-\frac{L}{2}\right)^2}{8}\Bigg[\frac{L^2+4\left(x-\frac{L}{2}\right)^2}{24}-\frac{L^2}{4}-\left(8+\frac{5\nu}{1-\nu}\right)$$

$$\times\frac{h^5-(1-\frac{E^{(2)}}{E^{(1)}})t^5}{20\left[h^3-(1-\frac{E^{(2)}}{E^{(1)}})t^3\right]}-\Bigg(\frac{E^{(1)}-E^{(2)}}{E^{(2)}}(1-\frac{\nu}{1-\nu})\frac{t(h^2-t^2)}{2}+\frac{\nu}{1-\nu}\frac{h^3}{12}$$

$$-\frac{\nu}{1-\nu}\,\frac{h^3-2(1-\frac{E^{(2)}}{E^{(1)}})t^3}{12}\Bigg)\frac{3(h^2-t^2)}{2\left[h^3-(1-\frac{E^{(2)}}{E^{(1)}})t^3\right]}\Bigg]+\frac{1+\nu}{12(1-\nu)}z^4-\Bigg[\frac{\nu}{1-\nu}\Bigg(\left(x-\frac{L}{2}\right)^2-\frac{L^2}{4}$$

$$-(8+\frac{5\nu}{1-\nu})\frac{h^5-(1-\frac{E^{(2)}}{E^{(1)}})t^5}{20\left[h^3-(1-\frac{E^{(2)}}{E^{(1)}})t^3\right]}-\Bigg(\frac{E^{(1)}-E^{(2)}}{E^{(2)}}(1+\frac{\nu}{1-\nu})\frac{t(h^2-t^2)}{2}+\frac{\nu}{1-\nu}\frac{h^3}{12}$$

$$-\frac{\nu}{1-\nu}\,\frac{h^3-2(1-\frac{E^{(2)}}{E^{(1)}})t^3}{12}\Bigg)\frac{3\left(h^2-t^2\right)}{2\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}\Bigg)$$

$$+\left(1+\frac{\nu}{1-\nu}\right)^2\frac{\left[h^2-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^2\right]E^{(1)}}{4E^{(2)}}\Bigg]\frac{z^2}{2}+\Bigg[\Bigg[\frac{\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]E^{(1)}}{12E^{(2)}}$$

$$-\frac{\nu}{1-\nu}\Bigg(-\frac{\nu}{1-\nu}\frac{E^{(1)}h}{tE^{(2)}+(h-t)E^{(1)}}\frac{h^3-\left(1-\frac{E^{(2)}}{E_1^{(1)}}\right)t^3}{12}+\frac{\nu}{1-\nu}\frac{E^{(1)}\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}{12E^{(2)}}\Bigg)\Bigg]z\Bigg\};$$

(2-E.54)

$$w_3 = -\frac{6\frac{q_u}{b}(1-\nu^2)}{E^{(1)}\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}\Bigg\{\frac{L^2-4\left(x-\frac{L}{2}\right)^2}{8}\Bigg(\frac{L^2+4\left(x-\frac{L}{2}\right)^2}{24}-\frac{L^2}{4}-\left(8+\frac{5\nu}{1-\nu}\right)$$

$$\times\frac{h^5-(1-\frac{E^{(2)}}{E^{(1)}})t^5}{20\left[h^3-(1-\frac{E^{(2)}}{E^{(1)}})t^3\right]}-\Bigg(\frac{E^{(1)}-E^{(2)}}{E^{(2)}}(1+\frac{\nu}{1-\nu})\frac{t(h^2-t^2)}{2}+\frac{\nu}{1-\nu}\frac{h^3}{12}$$

$$-\frac{\nu}{1-\nu}\frac{h^3-2(1-\frac{E^{(2)}}{E^{(1)}})t^3}{12}\bigg)\frac{3(h^2-t^2)}{2\left[h^3-(1-\frac{E^{(2)}}{E^{(1)}})t^3\right]})+\frac{1+\nu}{12(1-\nu)}z^4-\left[\frac{\nu}{1-\nu}\left(\left(x-\frac{L}{2}\right)^2-\frac{L^2}{4}\right.\right.$$

$$-(8+\frac{5\nu}{1-\nu})\frac{h^5-(1-\frac{E^{(2)}}{E^{(1)}})t^5}{20\left[h^3-(1-\frac{E^{(2)}}{E^{(1)}})t^3\right]}-\left(\frac{E^{(1)}-E^{(2)}}{E^{(2)}}(1+\frac{\nu}{1-\nu})\frac{t(h^2-t^2)}{2}+\frac{\nu}{1-\nu}\frac{h^3}{12}\right.$$

$$\left.\left.-\frac{\nu}{1-\nu}\frac{h^3-2(1-\frac{E^{(2)}}{E^{(1)}})t^3}{12}\right)\frac{3\left(h^2-t^2\right)}{2\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}\right)$$

$$+\left(1+\frac{\nu}{1-\nu}\right)^2\frac{h^2}{4}]\frac{z^2}{2}+[\frac{h^3}{12}-\frac{\nu}{1-\nu}(\frac{E^{(1)}-E^{(2)}}{E^{(2)}}\left(1+\frac{\nu}{1-\nu}\right)\frac{t\left(h^2-t^2\right)}{4}$$

$$-\frac{\nu}{1-\nu}\frac{E^{(1)}h}{tE^{(2)}+(h-t)E^{(1)}}\frac{h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3}{12}+\frac{\nu}{1-\nu}\frac{h^3}{12})]z-\frac{E^{(1)}-E^{(2)}}{E^{(2)}}\left(1-\frac{3\nu}{1-\nu}\right)$$

$$\left.\times\left(1+\frac{\nu}{1-\nu}\right)\frac{t^{2(h^2-t^2)}}{32}-\left(1-\frac{\nu^2}{(1-\nu)^2}\right)\frac{t}{2}\frac{E^{(2)}h^3-E^{(1)}\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}{12E^{(2)}}\right\};\qquad\text{(2-E.55)}$$

$$u_1=-\frac{6\frac{q_u}{b}\left(x-\frac{L}{2}\right)\left(1-\nu^2\right)}{E^{(1)}\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}\left\{\left[\left(\frac{\left(x-\frac{L}{2}\right)^2}{3}-\frac{L^2}{4}-\left(8+\frac{5\nu}{1-\nu}\right)\right.\right.\right.$$

$$\times\frac{h^5-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^5}{20\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}-\left[\frac{E^{(1)}-E^{(2)}}{E^{(2)}}\left(1+\frac{\nu}{1-\nu}\right)\frac{t\left(h^2-t^2\right)}{4}-\frac{\nu}{1-\nu}\right.$$

$$\times\frac{E^{(1)}h}{tE^{(2)}+(h-t)E^{(1)}}\frac{h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3}{12}+\frac{\nu h^3}{12(1-\nu)}+\frac{E^{(1)}-E^{(2)}}{E^{(2)}}\left(1+\frac{\nu}{1-\nu}\right)\frac{t\left(h^2-t^2\right)}{4}$$

$$\left.\left.+\frac{\nu}{1-\nu}\frac{E^{(1)}h}{tE^{(2)}+(h-t)E^{(1)}}\frac{h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3}{12}+\frac{h^3-2\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3}{12}\right]\frac{3\left(h^2-t^2\right)}{2\left[h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}\right)$$

$$- \frac{2 + \frac{\nu}{1-\nu}}{3} z^2 + 2\left(1 + \frac{\nu}{1-\nu}\right)\frac{h^2}{4}\Bigg] z + \frac{E^{(1)} - E^{(2)}}{E^{(2)}}\left(1 + \frac{\nu}{1-\nu}\right)\frac{t\left(h^2 - t^2\right)}{4}$$

$$- \frac{\nu}{1-\nu}\, \frac{E^{(1)}h}{tE^{(2)} + (h-t)E^{(1)}} \frac{h^3 - \left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^3}{12} \Bigg\}; \qquad \text{(2-E.56)}$$

$$u_2 = -\frac{6\frac{q_u}{b}\left(x - \frac{L}{2}\right)\left(1-\nu^2\right)}{E^{(1)}\left[h^3 - \left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^3\right]} \Bigg\{\Bigg[\Bigg(\frac{\left(x - \frac{L}{2}\right)^2}{3} - \frac{L^2}{4} - \left(8 + \frac{5\nu}{1-\nu}\right)$$

$$\times \frac{h^5 - \left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^5}{20\left[h^3 - \left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^3\right]} - \Bigg[\frac{E^{(1)} - E^{(2)}}{E^{(2)}}\left(1 + \frac{\nu}{1-\nu}\right)\frac{t\left(h^2 - t^2\right)}{2}$$

$$+ \frac{\nu}{1-\nu}\frac{\left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^3}{6}\Bigg] \frac{3\left(h^2 - t^2\right)}{2\left[h^3 - \left(1 - \frac{E_2}{E_1}\right)t^3\right]}\Bigg) - \frac{2 + \frac{\mu}{1-\mu}}{3} z^2 + \left(1 + \frac{\nu}{1-\nu}\right)$$

$$\times \frac{h^2 - \left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^2}{2\frac{E^{(2)}}{E^{(1)}}}\Bigg] z - \frac{\nu}{1-\nu}\frac{E^{(1)}h}{tE^{(2)} + (h-t)\,E^{(1)}} \frac{h^3 - \left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^3}{12} \Bigg\}; \qquad \text{(2-E.57)}$$

$$u_3 = -\frac{6\frac{q_u}{b}\left(x - \frac{L}{2}\right)\left(1-\nu^2\right)}{E_1^{(1)}\left[h^3 - \left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^3\right]} \Bigg\{\Bigg[\Bigg(\frac{\left(x - \frac{L}{2}\right)^2}{3} - \frac{L^2}{4} - \left(8 + \frac{5\nu}{1-\mu}\right)$$

$$\times \frac{h^5 - \left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^5}{20\left[h^3 - \left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^3\right]} - \Bigg[\frac{E^{(1)} - E^{(2)}}{E^{(2)}}\left(1 + \frac{\nu}{1-\nu}\right)\frac{t\left(h^2 - t^2\right)}{2}$$

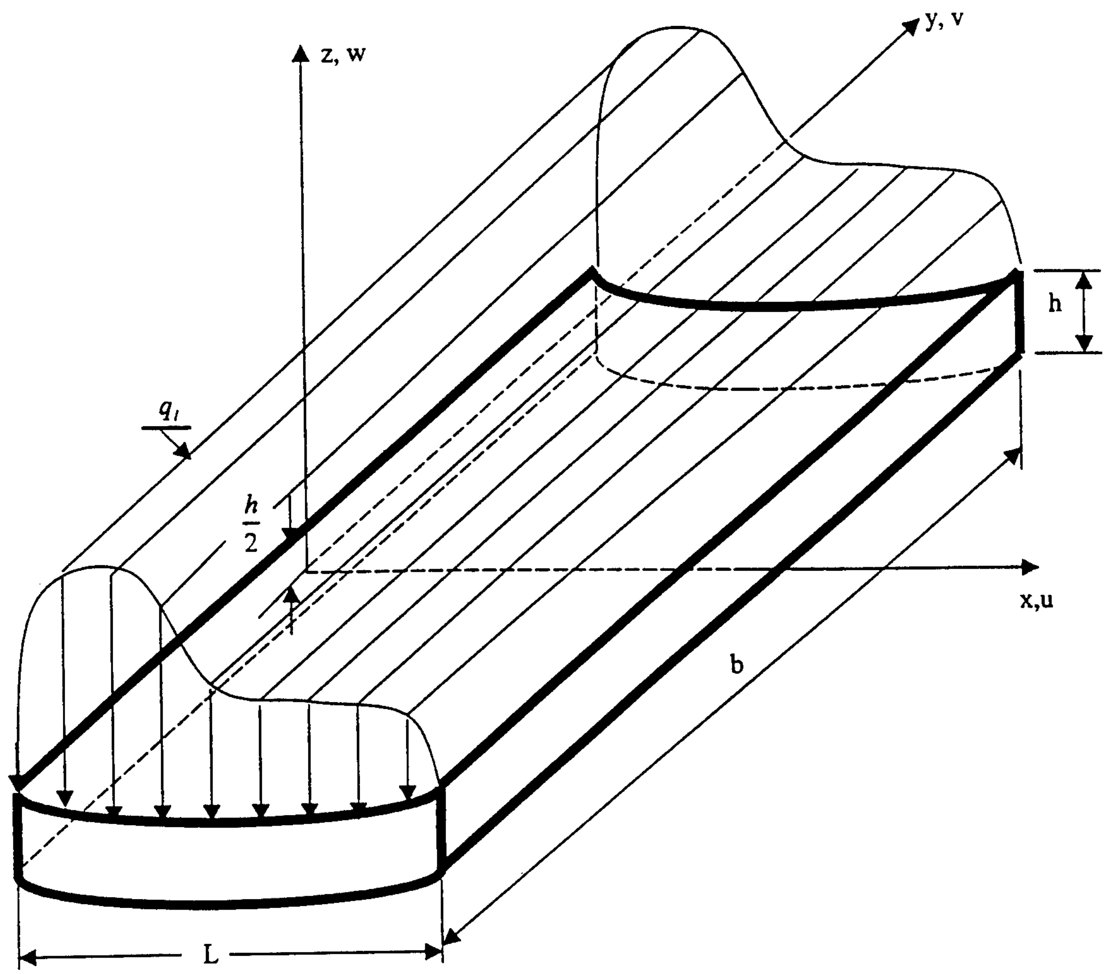
$$+ \frac{\nu}{1-\nu}\frac{h^3}{12} - \frac{\nu}{1-\nu}\frac{h^3 - 2\left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^3}{12}\Bigg] \frac{3\left(h^2 - t^2\right)}{2\left[h^3 - \left(1 - \frac{E^{(2)}}{E^{(1)}}\right)t^3\right]}\Bigg) - \frac{2\left(1 + \frac{\nu}{1-\nu}\right)}{3} z^2$$

$$+\frac{\left(1+\frac{\nu}{1-\nu}\right)h^2}{2}\Bigg]z+\frac{E^{(1)}-E^{(2)}}{E^{(2)}}\left(1+\frac{\nu}{1-\nu}\right)\frac{t\left(h^2-t^2\right)}{4}$$

$$-\frac{\nu}{1-\nu}\frac{E^{(1)}h}{tE^{(2)}+(h-t)E^{(1)}}\frac{h^3-\left(1-\frac{E^{(2)}}{E^{(1)}}\right)t^3}{12}\Bigg\}. \tag{2-E.58}$$

Figure 2.1

Wide plate under a load that does not vary in the width direction



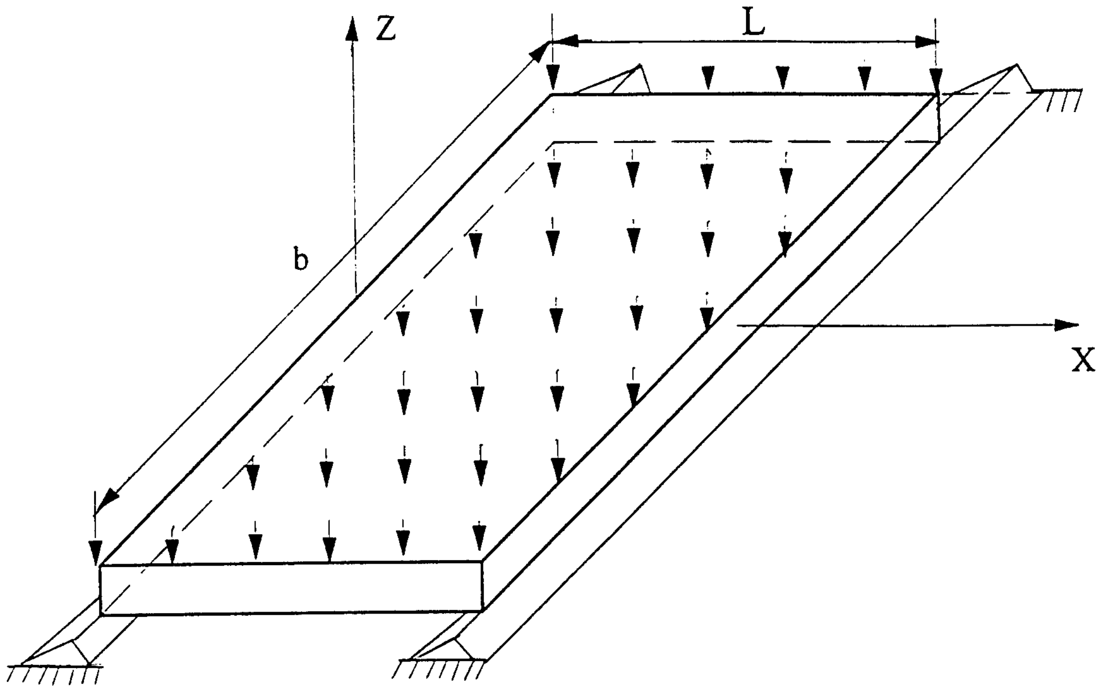


Figure 2.2
Wide simply supported plate under a uniform load on the upper surface

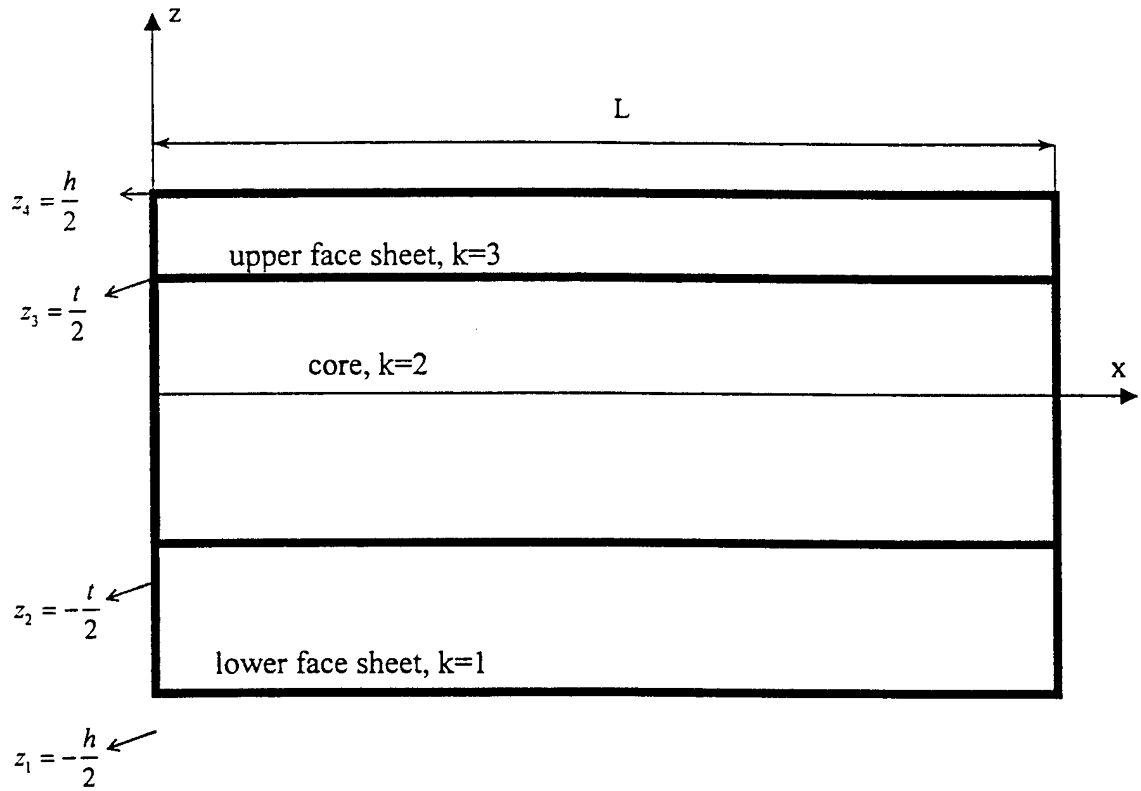


Figure 2.3. The coordinate system and notations for the sandwich plate, h is thickness of the whole plate, t is thickness of the core

Figure 2.4

The element coordinate system and nodal variables associated with one node of a finite element

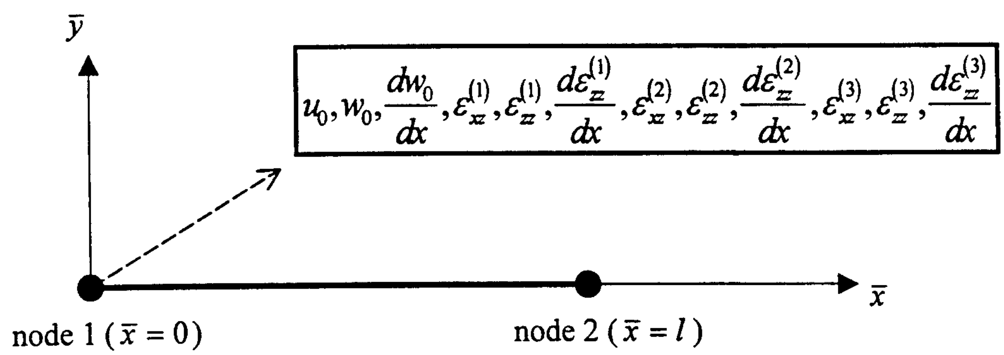
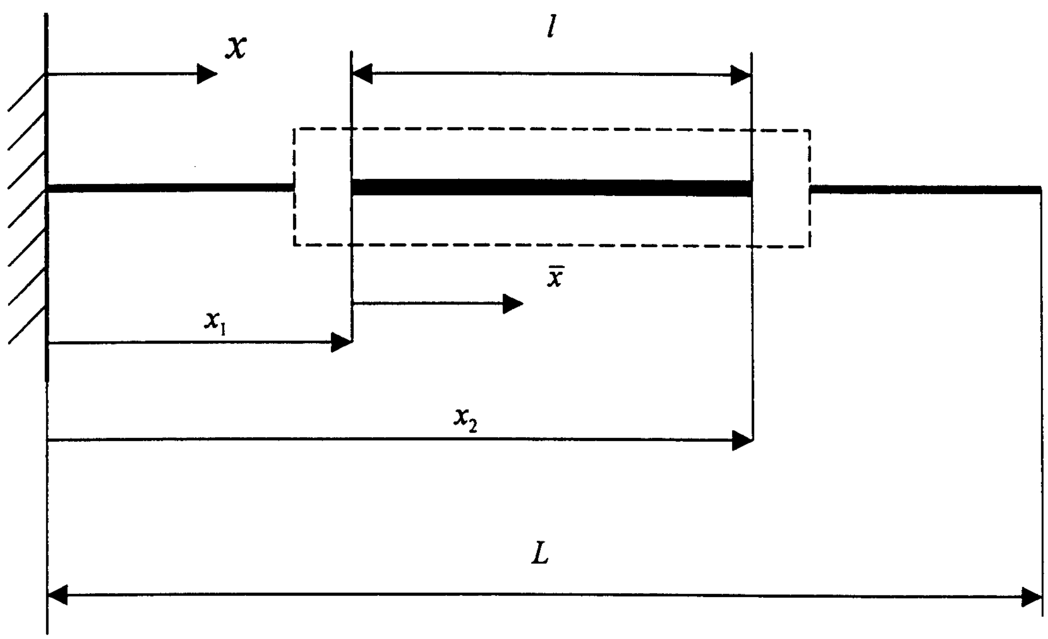


Figure 2.5

The problem and element coordinate systems



Chapter 3

Two-Dimensional Model of a Composite Cargo Platform, Dropped on Elastic Foundation

In this chapter we consider the problem of computation of stresses, strains and displacements in a sandwich composite platform, loaded by a cargo on its upper face sheet, dropped from the aircraft on the ground , which is modelled as an elastic Winkler foundation. The sandwich plate is analyzed with a layer-wise theory with three conventional layers representing the core and the upper and lower face sheets (Figure 2.3).

3.1 Three-dimensional formulation of the problem

As work-conjugate measures of strain and stress, we use the Green-Lagrange strain tensor and the second Piola-Kirchhoff stress tensor. We limit our research to a practically important case of small strains, moderate displacements (of the order of thickness of the plate) and moderate rotations ($10^o - 15^o$). This means that of all the higher order terms in strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{s,i}u_{s,j}\right) \qquad (3.1.1)$$

only $u_{3,\alpha}u_{3,\beta}$ (α, $\beta = 1, 2$) are not negligible compared to $u_{\alpha,i}$ ($\alpha = 1, 2$; i =1, 2, 3) (von Karman, 1910). Therefore, the strain-displacement relations for the k-th conventional layer (sublaminate)

become

$$\varepsilon_{\alpha\beta}^{(k)} = \frac{1}{2}\left(u_{\alpha,\beta}^{(k)} + u_{\beta,\alpha}^{(k)} + u_{3,\alpha}^{(k)}u_{3,\beta}^{(k)}\right) \quad (\alpha,\beta = 1,2) \tag{3.1.2}$$

(no summation with respect to k),

$$\varepsilon_{i3}^{(k)} = \frac{1}{2}\left(u_{i,3}^{(k)} + u_{3,i}^{(k)}\right) \quad (i = 1,2,3), \tag{3.1.3}$$

or, in unabridged form,

$$\varepsilon_{xx}^{(k)} = u_{,x}^{(k)} + \frac{1}{2}\left(w_{,x}^{(k)}\right)^2, \tag{3.1.2-a}$$

$$\varepsilon_{yy}^{(k)} = v_{,y}^{(k)} + \frac{1}{2}\left(w_{,y}^{(k)}\right)^2, \tag{3.1.2-b}$$

$$\varepsilon_{xy}^{(k)} = \frac{1}{2}\left(u_{,y}^{(k)} + v_{,x}^{(k)} + w_{,x}^{(k)}w_{,y}^{(k)}\right) \quad \text{(no summation with respect to k)}, \tag{3.1.2-c}$$

$$\varepsilon_{xz}^{(k)} = \frac{1}{2}\left(u_{,z}^{(k)} + w_{,x}^{(k)}\right), \tag{3.1.3-a}$$

$$\varepsilon_{yz}^{(k)} = \frac{1}{2}\left(v_{,z}^{(k)} + w_{,y}^{(k)}\right), \tag{3.1.3-b}$$

$$\varepsilon_{zz}^{(k)} = w_{,z}^{(k)}. \tag{3.1.3-c}$$

Now we need to find the simplified equations of motion and boundary conditions, such that their accuracy corresponds to the accuracy of the adopted von-Karman strain-displacement relations. These equations of motion will be used for computation of the transverse stresses in the post-processing stage of the finite element analysis.

One way to do this is to simplify the general non-linear equations of motion. Such an approach is adopted in books by Novozhilov (1961), Stoker (1968), Ambartsumyan (1969) and other. Thus, to find the equations of motion, corresponding to the von Karman strain-displacement relations, Stoker (1968) retains in the general non-linear equations of motion those non-linear terms which involve products of stresses and plate slopes $w_{,x}$ and $w_{,y}$, and neglects all other non-linear terms. The resulting equations of motion are

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \overline{F}_x = \rho\ddot{u}, \tag{3.1.4}$$

$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \overline{F}_y = \rho \ddot{v}, \tag{3.1.5}$$

$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \frac{\partial}{\partial x}(\sigma_{xx} w_{,x} + \sigma_{yx} w_{,y}) +$$

$$+ \frac{\partial}{\partial y}(\sigma_{xy} w_{,x} + \sigma_{yy} w_{,y}) + \frac{\partial}{\partial z}(\sigma_{xz} w_{,x} + \sigma_{yz} w_{,y}) + \overline{F}_z = \rho \ddot{w}. \tag{3.1.6}$$

Another known method of deriving the simplified non-linear equations of motion is the variational method, based on substituting strain-displacement relations into the virtual work principle. Such a method is adopted by Reddy (1984, 1996), Lu and Liu (1992), Yu (1997) and other authors for deriving the two-dimensional equations of motion of plates, i.e. equations of motion averaged over thickness of plates. Pikul (1985) used this method to derive nonlinear three-dimensional pointwise equilibrium equations for shells, under assumed strain-displacement relations different from those, which are used in the present work. Equations of motion and boundary conditions, obtained by substituting strain-displacement relations into the virtual work principle, are called "variationally consistent" with the strain-displacement relations (terminology of Reddy, 1984, 1996). Following this idea, let us receive equations of motion and boundary conditions, variationally consistent with the von-Karman strain-displacement relations (3.1.2) and (3.1.3).

Let us substitute variations of strains, defined by equations (3.1.2) and (3.1.3),

$$\delta\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\delta u_{\alpha,\beta} + \delta u_{\beta,\alpha} + u_{3,\alpha}\ \delta u_{3,\beta} + u_{3,\beta}\ \delta u_{3,\alpha}\right) \qquad (\alpha = 1,2;\ \beta = 1,2)\,, \tag{3.1.7}$$

$$\delta\varepsilon_{i3} = \frac{1}{2}\left(\delta u_{i,3} + \delta u_{3,i}\right) \qquad (i = 1,2,3) \tag{3.1.8}$$

into the virtual work principle

$$\iiint\limits_{(V)} \sigma_{ij}\ \delta\varepsilon_{ij}\ dV = \iiint\limits_{(V)} \left(\overline{F}_i - \rho\ddot{u}_i\right)\ \delta u_i\ dV + \iint\limits_{(S)} \bar{t}_i\ \delta u_i\ dS\ , \tag{3.1.9}$$

where $\overline{F}_i$ is a known body force per unit volume, $\bar{t}_i$ is a known surface traction. Expression $\sigma_{ij}\ \delta\varepsilon_{ij}$ can be presented in the form

$$\sigma_{ij}\ \delta\varepsilon_{ij} = \sigma_{\alpha\beta}\ \delta\varepsilon_{\alpha\beta} + 2\sigma_{\alpha 3}\ \delta\varepsilon_{\alpha 3} + \sigma_{33}\ \delta\varepsilon_{33} \tag{3.1.10}$$

$$(\alpha = 1,2;\ \beta = 1,2;\ i = 1,2,3;\ j = 1,2,3)\,.$$

When we substitute equations (3.1.7) and (3.1.8) into equation (3.1.10), we receive

$$
\begin{aligned}
\sigma_{ij}\ \delta\varepsilon_{ij} = {} & \sigma_{\alpha\beta}\frac{1}{2}\left(\delta u_{\alpha,\beta} + \delta u_{\beta,\alpha} + u_{3,\alpha}\ \delta u_{3,\beta} + u_{3,\beta}\ \delta u_{3,\alpha}\right) + \\
& 2\sigma_{\alpha 3}\frac{1}{2}\left(\delta u_{\alpha,3} + \delta u_{3,\alpha}\right) + \sigma_{33}\ \delta u_{3,3} = \\
& = \frac{1}{2}\left[\sigma_{\alpha\beta}\ \delta u_{\alpha,\beta} + \sigma_{\beta\alpha}\ \delta u_{\beta,\alpha} + \sigma_{\alpha\beta} u_{3,\alpha}\ \delta u_{3,\beta} + \sigma_{\beta\alpha} u_{3,\beta}\ \delta u_{3,\alpha}\right] \\
& + \sigma_{\alpha 3}\left(\delta u_{\alpha,3} + \delta u_{3,\alpha}\right) + \sigma_{33}\ \delta u_{3,3} \\
& = \frac{1}{2}\left[\sigma_{\alpha\beta}\ \delta u_{\alpha,\beta} + \sigma_{\alpha\beta}\ \delta u_{\alpha,\beta} + \sigma_{\alpha\beta} u_{3,\alpha}\ \delta u_{3,\beta} + \sigma_{\alpha\beta} u_{3,\alpha}\ \delta u_{3,\beta}\right] \\
& + \sigma_{\alpha 3}\left(\delta u_{\alpha,3} + \delta u_{3,\alpha}\right) + \sigma_{33}\ \delta u_{3,3} \\
& = \frac{1}{2}\left[2\sigma_{\alpha\beta}\ \delta u_{\alpha,\beta} + 2\sigma_{\alpha\beta} u_{3,\alpha}\ \delta u_{3,\beta}\right] + \sigma_{\alpha 3}\left(\delta u_{\alpha,3} + \delta u_{3,\alpha}\right) + \sigma_{33}\ \delta u_{3,3} \\
& = \sigma_{\alpha\beta}\ \delta u_{\alpha,\beta} + \sigma_{\alpha\beta}\ u_{3,\alpha}\ \delta u_{3,\beta} + \sigma_{\alpha 3}\left(\delta u_{\alpha,3} + \delta u_{3,\alpha}\right) + \sigma_{33}\ \delta u_{3,3} \\
& = \left(\sigma_{\alpha\beta}\ \delta u_{\alpha,\beta} + \sigma_{\alpha 3}\ \delta u_{\alpha,3} + \sigma_{3\alpha}\ \delta u_{3,\alpha} + \sigma_{33}\ \delta u_{3,3}\right) + \sigma_{\alpha\beta}\ u_{3,\alpha}\ \delta u_{3,\beta} \\
& = \sigma_{ij}\ \delta u_{i,j} + \sigma_{\alpha\beta}\ u_{3,\alpha}\ \delta u_{3,\beta} \qquad (3.1.11) \\
& (\alpha = 1,2;\ \beta = 1,2;\ i = 1,2,3;\ j = 1,2,3)\,.
\end{aligned}
$$

Substituting expression (3.1.11) into the left-hand side of equation (3.1.9), we receive

$$
\begin{aligned}
\iiint\limits_{(V)} \sigma_{ij}\ \delta\varepsilon_{ij}\ dV = {} & \iiint\limits_{(V)} \left(\sigma_{ij}\ \delta u_{i,j} + \sigma_{\alpha\beta}\ u_{3,\alpha}\ \delta u_{3,\beta}\right)\ dV \\
& = \iiint\limits_{(V)} \left[\left(\sigma_{ij}\ \delta u_i\right)_{,j} - \sigma_{ij,j}\ \delta u_i + \left(\sigma_{\alpha\beta}\ u_{3,\alpha}\ \delta u_3\right)_{,\beta} - \left(\sigma_{\alpha\beta}\ u_{3,\alpha}\right)_{,\beta}\ \delta u_3\right]\ dV \\
& = \iint\limits_{(S)} \left(\sigma_{ij}\ n_j\ \delta u_i + \sigma_{\alpha\beta}\ u_{3,\alpha}\ n_\beta\ \delta u_3\right)\ dS - \\
& \iiint\limits_{(V)} \left[\sigma_{ij,j}\ \delta u_i + \left(\sigma_{\alpha\beta}\ u_{3,\beta}\right)_{,\beta}\ \delta u_3\right]\ dV \\
& = \iint\limits_{(S)} \left[\sigma_{\alpha j}\ n_j\ \delta u_\alpha + \left(\sigma_{3j} n_j + \sigma_{\alpha\beta}\ u_{3,\alpha}\ n_\beta\ \right)\ \delta u_3\right]\ dS \\
& - \iiint\limits_{(V)} \left\{\sigma_{\alpha j,j}\ \delta u_\alpha + \left[\sigma_{3j,j} + \left(\sigma_{\alpha\beta}\ u_{3,\alpha}\right)_{,\beta}\right]\ \delta u_3\right\}\ dV \qquad (3.1.12) \\
& (\alpha = 1,2;\ \beta = 1,2;\ i = 1,2,3;\ j = 1,2,3)\,,
\end{aligned}
$$

where n_1, n_2 and n_3 are components of the outward unit normal vector to the surface. The substi-

tution of expression (3.1.12) into the virtual work principle (3.1.9) yields

$$
\begin{aligned}
0 &= \iiint\limits_{(V)} \sigma_{ij}\ \delta\varepsilon_{ij}\ dV - \iiint\limits_{(V)} \left(\overline{F}_i - \rho\ddot{u}_i\right)\ \delta u_i\ dV + \iint\limits_{(S)} \bar{t}_i\ \delta u_i\ dS \\
&= \iint\limits_{(S)} \left[\sigma_{\alpha j}\ n_j\ \delta u_\alpha + \left(\sigma_{3j} n_j + \sigma_{\alpha\beta}\ u_{3,\alpha}\ n_\beta\ \right)\ \delta u_3\right]\ dS \\
&- \iiint\limits_{(V)} \left\{\sigma_{\alpha j,j}\ \delta u_\alpha + \left[\sigma_{3j,j} + \left(\sigma_{\alpha\beta}\ u_{3,\alpha}\right)_{,\beta}\right]\ \delta u_3\right\}\ dV \\
&- \iiint\limits_{(V)} \left(\overline{F}_i - \rho\ddot{u}_i\right)\ \delta u_i\ dV + \iint\limits_{(S)} \bar{t}_i\ \delta u_i\ dS \\
&= \iint\limits_{(S)} \left[\left(\sigma_{\alpha j}\ n_j - \bar{t}_\alpha\right)\ \delta u_\alpha\ dS + \left(\sigma_{3j}\ n_j + \sigma_{\alpha\beta}\ u_{3,\alpha} n_\beta - \bar{t}_3\right)\ \delta u_3\right]\ dS \\
&- \iiint\limits_{(V)} \left\{\left(\sigma_{\alpha j,j} + \overline{F}_\alpha - \rho\ddot{u}_\alpha\right)\ \delta u + \left[\sigma_{3j,j} + \left(\sigma_{\alpha\beta}\ u_{3,\alpha}\right)_{,\beta} + \overline{F}_3 - \rho\ddot{u}_3\right]\ \delta u_3\right\}\ dV \\
&(\alpha = 1,2;\ \beta = 1,2;\ j = 1,2,3)\,. \qquad (3.1.13)
\end{aligned}
$$

If one equates to zero the coefficients of variations of displacements, one obtains the equations of motion

$$
\sigma_{\alpha j,j} + F_\alpha = \rho\ddot{u}_\alpha; \quad \sigma_{3j,j} + \left(\sigma_{\alpha\beta}\ u_{3,\alpha}\right)_{,\beta} + \overline{F}_3 = \rho\ddot{u}_3 \quad (\alpha = 1,2;\ \beta = 1,2;\ j = 1,2,3) \qquad (3.1.14)
$$

and natural boundary conditions

$$
\sigma_{\alpha j}\ n_j = t_\alpha; \quad \sigma_{3j}\ n_j + \sigma_{\alpha\beta}\ u_{3,\alpha}\ n_\beta = t_3 \ \text{ at } S_\sigma \ \ (\alpha = 1,2;\ \beta = 1,2;\ j = 1,2,3)\,, \qquad (3.1.15)
$$

where S_σ is part of the surface on which displacements are not specified. Equations of motion (3.1.14) in unabridged form are

$$
\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \overline{F}_x = \rho\ddot{u}, \qquad (3.1.14\text{-a})
$$

$$
\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \overline{F}_y = \rho\ddot{v}, \qquad (3.1.14\text{-b})
$$

$$
\begin{aligned}
&\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \frac{\partial}{\partial x}\left(\sigma_{xx} w_{,x} + \sigma_{yx} w_{,y}\right) + \\
&\frac{\partial}{\partial y}\left(\sigma_{xy}\ w_{,x} + \sigma_{yy}\ w_{,y}\right) + \overline{F}_z = \rho\ddot{w}. \qquad (3.1.14\text{-c})
\end{aligned}
$$

The boundary conditions (3.1.15) in unabridged form are

$$\sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z = \bar{t}_x \ , \tag{3.1.15-a}$$

$$\sigma_{yx}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z = \bar{t}_y \ , \tag{3.1.15-b}$$

$$\sigma_{zx}n_x + \sigma_{zy}n_y + \sigma_{zz}n_z + \sigma_{xx}w_{,x}n_x + \sigma_{yy}w_{,y}n_y + \sigma_{xy}\left(w_{,x}n_y + w_{,y}n_x\right) = \bar{t}_z \ . \tag{3.1.15-c}$$

We see that the third equation of motion, derived from the virtual work principle (equation 3.1.14-c), is different from the corresponding equation (3.1.6), obtained by simplification of the general non-linear equations of motion, namely, in equation (3.1.14-c) the term $\frac{\partial}{\partial z}\left(\sigma_{xz}\ w_{,x} + \sigma_{yz}\ w_{,y}\right)$ is not present. In single-layer theories of plates, if tangential components of surface tractions are equal to zero, this term does not influence the two-dimensional (averaged over the thickness) plate equations of motion (Whitney, 1987). But in the layer-wise theories, these terms influence the two-dimensional equations of motion for individual layers, because stresses σ_{xz} and σ_{yz} do not vanish at the interfaces between the layers. Therefore, a question arises: what simplified non-linear equations of motion are to be used in our analysis.

To make such a decision one needs to keep in mind that the simplified non-linear equations of motion must be consistent with a finite element formulation, that will be based on the virtual work principle (3.1.9)[1]. In case of fully nonlinear Green-Lagrange strain-displacement relations, the virtual work principle (3.1.9) is derived (Washizu, 1982) from the equilibrium equations[2] in terms of the second Piola-Kirchhoff stress tensor

$$\left[\left(\delta_{\lambda\mu} + u_{\lambda,\mu}\right)\sigma_{\kappa\mu}\right]_{,\kappa} + \overline{F}_\lambda - \rho\ \ddot{u}_\lambda = 0 \qquad (\lambda = 1,2,3) \tag{3.1.16}$$

and stress boundary conditions

$$\sigma_{ij}(\delta_{\lambda i} + u_{\lambda,i})n_j - \bar{t}_\lambda = 0 \tag{3.1.17}$$

[1]In case of elastic material and conservative external forces, that is the case in the problem of the dissertation, the virtual work principle takes the form of the Hamilton's principle: $\delta\left(K - U - V\right) = 0$, where K is kinetic energy of the system, U is strain energy of the system and V is potential energy of the system in an external force field (potential energy due to the load).

[2]In the dynamic problems, the term equilibrium equations implies dynamic equilibrium equations (or equations of motion), i.e. implies that inertia forces are part of the body forces.

as follows: first, the left-hand parts of the equilibrium equations (3.1.16) and stress boundary conditions (3.1.17) are multiplied by variations of displacements and integrated:

$$-\iiint\limits_{(V)} \left\{ \left[(\delta_{\lambda\mu} + u_{\lambda,\mu})\, \sigma_{\kappa\mu} \right]_{,\kappa} + \overline{F}_\lambda - \rho\, \ddot{u}_\lambda \right\} \delta u_\lambda \, dV + \iint\limits_{(S_1)} \left[\sigma_{ij}(\delta_{\lambda i} + u_{\lambda,i}) n_j - \bar{t}_\lambda = 0 \right] \delta u_\lambda dS = 0, \tag{3.1.18}$$

where S_1 is part of the surface where the stresses are specified. Integration by parts in the equation (3.1.18) yields

$$\iiint\limits_{(V)} \sigma_{\lambda\mu}\, \delta \left[\frac{1}{2} \left(u_{\lambda,\mu} + u_{\mu,\lambda} + u_{\kappa,\lambda}\, u_{\kappa,\mu} \right) \right] dV - \iiint\limits_{(V)} \left(\overline{F}_\lambda - \rho\, \ddot{u}_\lambda \right) \delta u_\lambda \, dV - \iint\limits_{(S_1)} \bar{t}_\lambda \, \delta u_\lambda \, dS = 0. \tag{3.1.19}$$

In equation (3.1.19), the expression under the variation sign in the first term is recognizable as the Green-Lagrange strain tensor. In a similar fashion, in order to derive the virtual work principle with the von-Karman strains[3] , i.e. equation

$$\iiint\limits_{(V)} \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \sigma_{\alpha\beta}\, \delta \left[\frac{1}{2} \left(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{3,\alpha}\, u_{3,\beta} \right) \right] dV + \iiint\limits_{(V)} \sum_{i=1}^{3} \sigma_{i3}\, \delta \left[\frac{1}{2} \left(u_{i,3} + u_{3,i} \right) \right] dV$$

$$- \iiint\limits_{(V)} \left(\overline{F}_\lambda - \rho\, \ddot{u}_\lambda \right) \delta u_\lambda \, dV - \iint\limits_{(S_1)} \bar{t}_\lambda \, \delta u_\lambda \, dS = 0, \tag{3.1.20}$$

it is necessary to use in the derivation such equilibrium equations, that they lead to the virtual work principle (3.1.20) with von-Karman strains. Such equilibrium equations can be found by starting from the virtual work principle (3.1.20) and performing the same derivations as those that led to virtual work principle (3.1.19), but in the reverse order. This has been done already in this chapter, with the result being equilibrium equations (3.1.14) and natural boundary conditions (3.1.15). If in conjunction with the von-Karman strains some other equilibrium equations are used (for example, equations (3.1.4)–(3.1.6)), then the virtual work principle (3.1.20) is non-existent. Then, the finite element formulation on the basis of the virtual work principle (3.1.20) (i.e. with the von-Karman strains) can not be made.

Therefore, in the post-processing stage of the finite element analysis, the computation of the transverse stresses needs to be done with the use of the equations of motion (3.1.14-a), (3.1.14-b),

[3] given by equations (3.1.2) and (3.1.3)

(3.1.14-c), variationally consistent with the von-Karman strains. This opinion is shared by other authors. For example, according to Reddy (1984), "the correct forms of differential equations and boundary conditions for any theory, based on assumed displacement field, are not known without using the virtual work principle".

The equations of motion (3.1.14) will be written for each of the three conventional layers: upper and lower face sheets and the core. Besides, we will take into account that in our problem the body force is the gravity force, therefore $F_x = F_y = 0$ and $F_z = -\rho g$, where ρ is mass density and $g = 9.81\frac{m}{s^2}$.

$$\sigma_{xx,x}^{(k)} + \sigma_{xy,y}^{(k)} + \sigma_{xz,z}^{(k)} = \rho^{(k)}\ddot{u}^{(k)} \ , \tag{3.1.21}$$

$$\sigma_{yx,x}^{(k)} + \sigma_{yy,y}^{(k)} + \sigma_{yz,z}^{k} = \rho^{(k)}\ddot{v}^{(k)} \ , \tag{3.1.22}$$

$$\sigma_{zx,x}^{(k)} + \sigma_{zy,y}^{(k)} + \sigma_{zz,z}^{(k)} + \frac{\partial}{\partial x}\left(\sigma_{xx}^{(k)} w_{,x}^{(k)} + \sigma_{yx}^{(k)} w_{,y}^{(k)}\right) +$$

$$+\frac{\partial}{\partial y}\left(\sigma_{xy}^{(k)}\ w_{,x}^{(k)} + \sigma_{yy}^{(k)}\ w_{,y}^{(k)}\right) - \rho^{(k)}g = \rho^{(k)}\ddot{w}^{(k)} \tag{3.1.23}$$

$$(k = 1, 2, 3)\,.$$

The boundary conditions (3.1.15) will be written for the upper and lower surfaces of the plate and for the interfaces between the face sheets and the core, i.e. for the surfaces $z = z_1$, $z = z_2^-$, $z = z_2^+$, $z = z_3^-$, $z = z_3^+$ and $z = z_4$ (Figure 3.1). At these surfaces $n_x = n_y = 0$, $n_z = \pm 1$. Therefore, in our problem the boundary conditions (3.1.15) take the form:

$$\text{at } z = z_1 \ \ \sigma_{xz}^{(1)}(z_1) = t_x(z_1) = 0, \ \ \sigma_{yz}^{(1)}(z_1) = t_y(z_1) = 0, \ \ \sigma_{zz}^{(1)}(z_1)\underbrace{n_z(z_1)}_{-1} = t_z(z_1)\,.\ ; \tag{3.1.24}$$

$$\text{at } z = z_2^- \ \ \ \sigma_{xz}^{(1)}\left(z_2^-\right) = t_x\left(z_2^-\right), \ \sigma_{yz}^{(1)}\left(z_2^-\right) = t_y\left(z_2^-\right), \ \ \sigma_{zz}^{(1)}(z_2) \ \underbrace{n_z\left(z_2^-\right)}_{1} = t_z\left(z_2^-\right) \ ; \tag{3.1.25}$$

$$\text{at } z = z_2^+ \ \ \ \sigma_{xz}^{(2)}\left(z_2^+\right) = t_x\left(z_2^-\right), \ \sigma_{yz}^{(2)}\left(z_2^+\right) = t_y\left(z_2^+\right), \ \ \sigma_{zz}^{(2)}(z_2)\underbrace{n_z\left(z_2^+\right)}_{-1} = t_z\left(z_2^+\right) \ ; \tag{3.1.26}$$

$$\text{at } z = z_3^- \quad \sigma_{xz}^{(2)}\left(z_3^-\right) = t_x\left(z_3^-\right), \; \sigma_{yz}^{(2)}\left(z_3^-\right) = t_y\left(z_3^-\right), \; \sigma_{zz}^{(2)}\left(z_3\right)\underbrace{n_z\left(z_3^-\right)}_{1} = t_z\left(z_3^-\right) \; ; \quad (3.1.27)$$

$$\text{at } z = z_3^+ \quad \sigma_{xz}^{(3)}\left(z_3^+\right) = t_x\left(z_3^+\right), \; \sigma_{yz}^{(3)}\left(z_3^+\right) = t_y\left(z_3^+\right), \; \sigma_{zz}^{(3)}\left(z_3\right)\underbrace{n_z\left(z_3^+\right)}_{-1} = t_z\left(z_3^+\right) \; ; \quad (3.1.28)$$

$$\text{at } z = z_4 \;\; \sigma_{xz}^{(3)}\left(z_4\right) = t_x\left(z_4\right) = 0, \;\; \sigma_{yz}^{(3)}\left(z_4\right) = t_y\left(z_4\right) = 0, \;\; \sigma_{zz}^{(3)}\left(z_4\right)\underbrace{n_z\left(z_4\right)}_{1} = t_z\left(z_4\right) \; . \quad (3.1.29)$$

At each of the interfaces the absolute values of forces, acting at the adjacent layers, are equal:

$$\vec{t}\left(z_2^-\right) = -\vec{t}\left(z_2^+\right), \; \vec{t}\left(z_3^-\right) = -\vec{t}\left(z_3^+\right) \; , \quad (3.1.30)$$

or

$$t_x\left(z_2^-\right) = -t_x\left(z_2^+\right), \; t_y\left(z_2^-\right) = -t_y\left(z_2^+\right), \; t_z\left(z_2^-\right) = -t_z\left(z_2^+\right) \; , \quad (3.1.31)$$

$$t_x\left(z_3^-\right) = -t_x\left(z_3^+\right), \; t_y\left(z_3^-\right) = -t_y\left(z_3^+\right), \; t_z\left(z_3^-\right) = -t_z\left(z_3^+\right) \; . \quad (3.1.32)$$

Therefore, from equations (3.1.25) and (3.1.26) it follows that:

$$\sigma_{xz}^{(1)}\left(z_2\right) = \sigma_{xz}^{(2)}\left(z_2\right), \; \sigma_{yz}^{(1)}\left(z_2\right) = \sigma_{yz}^{(2)}\left(z_2\right), \; \sigma_{zz}^{(1)}\left(z_2\right) = \sigma_{zz}^{(2)}\left(z_2\right) \; , \quad (3.1.33)$$

and from equations (3.1.27) and (3.1.28) it follows that:

$$\sigma_{xz}^{(2)}\left(z_3\right) = \sigma_{xz}^{(3)}\left(z_3\right), \; \sigma_{yz}^{(2)}\left(z_3\right) = \sigma_{yz}^{(3)}\left(z_3\right), \; \sigma_{zz}^{(2)}\left(z_3\right) = \sigma_{zz}^{(3)}\left(z_3\right). \quad (3.1.34)$$

Equations (3.1.33) and (3.1.34) are conditions of continuity of the transverse stresses at the interfaces between the face sheets and the core.

At the edges of the plate $x = 0, L$, where $n_x = \mp 1$, $n_y = n_z = 0$, the boundary conditions (3.1.15) take the form:

$$\mp\sigma_{xx} = t_x \; , \; \mp\sigma_{yx} = t_y \; , \; \mp\sigma_{zx} \mp \sigma_{xx}\, w_{,x} \mp \sigma_{yx}\, w_{,y} = t_z \;\; \text{at } x = 0, L \; . \quad (3.1.35)$$

At the edges of the plate $y = 0, B$ (Figure 3.1.2), where $n_y = \mp 1$, $n_x = n_z = 0$, the boundary conditions (3.1.15) take the form

$$\mp\sigma_{xy} = t_x, \;\; \mp\sigma_{yy} = t_y, \;\; \mp\sigma_{zy} \mp \sigma_{xy}\, w_{,x} \mp \sigma_{yy}\, w_{,y} = t_z \;\; \text{at } \; y = 0, B \; . \quad (3.1.36)$$

For a plate with the edges free from loads, that is the case for a cargo platform dropped on the ground,

$$t_x = t_y = t_z = 0 \text{ at } x = 0, L \text{ and } y = 0, B \; . \tag{3.1.37}$$

Therefore, the boundary conditions (3.1.35) and (3.1.36) in this case take the form:

$$\sigma_{xx} = 0, \;\; \sigma_{yx} = 0, \quad \mp \sigma_{zx} \mp \underbrace{\sigma_{xx}}_{0} w_{,x} \mp \underbrace{\sigma_{yx}}_{0} w_{,y} = 0 \text{ at } x = 0, L \; , \tag{3.1.38}$$

$$\sigma_{xy} = 0, \;\; \sigma_{yy} = 0, \quad \mp \sigma_{zy} \mp \underbrace{\sigma_{xy}}_{0} w_{,x} \mp \underbrace{\sigma_{yy}}_{0} \, w_{,y} = 0 \;\text{ at }\; y = 0, B \; . \tag{3.1.39}$$

The stress boundary conditions at the edges $x = 0, L$ and $y = 0, B$, namely the boundary conditions expressed by equations (3.1.35) and (3.1.36), or (3.1.38) and (3.1.39), can not be satisfied exactly within the framework of a plate theory, in which some simplifying assumptions are introduced in addition to the 3-D formulation. In any plate theory the stress boundary conditions at the edges $x = 0, L$ and $y = 0, B$ are satisfied approximately, in the integral sense. The approximate, integral stress boundary conditions at the edges of a plate can be derived from the Hamilton's principle (or virtual work principle), as natural boundary conditions, the same way as it was done in Chapter 2 for a homogeneous plate in cylindrical bending. In a problem of a cargo platform, dropped on elastic foundation, the boundary conditions at the edges are the stress boundary conditions (3.1.38) and (3.1.39). Therefore, the corresponding approximate boundary conditions, which follow from the Hamilton's principle, are the natural boundary conditions. When we solve the problem by a finite element method, based on the Hamilton's principle, the natural boundary conditions will be automatically satisfied approximately, with no need to impose any constraints on the degrees of freedom of nodes at the boundaries. Therefore, if we solve the problem by the finite element method, based on the Hamilton's principle, we do not have to derive the approximate stress boundary conditions, as it was done for a problem of a homogeneous plate in Chapter 2, that was solved analytically by solving differential equations with boundary conditions, equations (2.1.47)–(2.1.56).

In conclusion, let us write again those equations, which will be used in subsequent derivations:

strain-displacement relations

$$\varepsilon_{xx}^{(k)} = u_{,x}^{(k)} + \frac{1}{2}\left(w_{,x}^{(k)}\right)^2 \quad \text{(eqn 3.1.2-a)},$$

$$\varepsilon_{yy}^{(k)} = v_{,y}^{(k)} + \frac{1}{2}\left(w_{,y}^{(k)}\right)^2 \quad \text{(eqn 3.1.2-b)},$$

$$\varepsilon_{xy}^{(k)} = \frac{1}{2}\left(u_{,y}^{(k)} + v_{,x}^{(k)} + w_{,x}^{(k)} w_{,y}^{(k)}\right) \quad \text{(no summation with respect to k)} \quad \text{(eqn 3.1.2-c)},$$

$$\varepsilon_{xz}^{(k)} = \frac{1}{2}\left(u_{,z}^{(k)} + w_{,x}^{(k)}\right) \quad \text{(eqn 3.1.3-a)},$$

$$\varepsilon_{yz}^{(k)} = \frac{1}{2}\left(v_{,z}^{(k)} + w_{,y}^{(k)}\right) \quad \text{(eqn 3.1.3-b)},$$

$$\varepsilon_{zz}^{(k)} = w_{,z}^{(k)} \quad \text{(eqn 3.1.3-c)},$$

equations of motion

$$\sigma_{xx,x}^{(k)} + \sigma_{xy,y}^{(k)} + \sigma_{xz,z}^{(k)} = \rho^{(k)} \ddot{u}^{(k)} \quad \text{(eqn 3.1.21)},$$

$$\sigma_{yx,x}^{(k)} + \sigma_{yy,y}^{(k)} + \sigma_{yz,z}^{k} = \rho^{(k)} \ddot{v}^{(k)} \quad \text{(eqn 3.1.22)},$$

$$\sigma_{zx,x}^{(k)} + \sigma_{zy,y}^{(k)} + \sigma_{zz,z}^{(k)} + \frac{\partial}{\partial x}\left(\sigma_{xx}^{(k)} w_{,x}^{(k)} + \sigma_{yx}^{(k)} w_{,y}^{(k)}\right) +$$

$$\frac{\partial}{\partial y}\left(\sigma_{xy}^{(k)}\ w_{,x}^{(k)} + \sigma_{yy}^{(k)}\ w_{,y}^{(k)}\right) - \rho^{(k)} g = \rho^{(k)} \ddot{w}^{(k)} \quad (k = 1, 2, 3) \quad \text{(eqn 3.1.23)},$$

stress boundary conditions on the lower and upper surfaces

$$\text{at } z = z_1 \ \ \sigma_{xz}^{(1)} = 0, \ \ \sigma_{yz}^{(1)} = 0, \ \ \sigma_{zz}^{(1)} = -t_z(z_1) \quad \text{(eqn 3.1.24)},$$

$$\text{at } z = z_4 \ \ \sigma_{xz}^{(3)} = 0, \ \ \sigma_{yz}^{(3)} = 0, \ \ \sigma_{zz}^{(3)} = t_z(z_4) \quad \text{(eqn 3.1.29)}.$$

continuity of the transverse stresses at the interfaces between the face sheets and the core

$$\sigma_{xz}^{(1)}(z_2) = \sigma_{xz}^{(2)}(z_2), \ \ \sigma_{yz}^{(1)}(z_2) = \sigma_{yz}^{(2)}(z_2), \ \ \sigma_{zz}^{(1)}(z_2) = \sigma_{zz}^{(2)}(z_2) \quad \text{(eqn 3.1.33)},$$

$$\sigma_{xz}^{(2)}(z_3) = \sigma_{xz}^{(3)}(z_3), \ \ \sigma_{yz}^{(2)}(z_3) = \sigma_{yz}^{(3)}(z_3), \ \ \sigma_{zz}^{(2)}(z_3) = \sigma_{zz}^{(3)}(z_3) \quad \text{(eqn 3.1.34)}.$$

In addition, continuity of displacements at the interfaces between the face sheets and the core is required:

$$u^{(1)} = u^{(2)},\ v^{(1)} = v^{(2)},\ w^{(1)} = w^{(2)} \ \text{ at } z = z_2\ , \tag{3.1.40}$$

$$u^{(2)} = u^{(3)},\ v^{(2)} = v^{(3)},\ w^{(2)} = w^{(3)} \ \text{ at } z = z_3. \tag{3.1.41}$$

The formulation of the problem includes also the constitutive relations, that are demonstrated in section 3.6 of this chapter.

3.2 Simplifying Assumptions of the Plate Theory

In order to apply the failure criteria to sandwich composite structures, the full three-dimensional state of stress must be known. A finite element analysis using three-dimensional elements could provide this, however the effort is enormous and often not acceptable for real structures. The computational cost can be cut down by reducing the problem to a two-dimensional one, i.e. by using a plate formulation. The improved values of transverse stress components σ_{xz}, σ_{yz}, σ_{zz} can then be computed in a postprocessing procedure, utilizing equations of motion of a three-dimensional continuum. To construct a plate theory, in addition to the three-dimensional formulation of the problem we will make simplifying assumptions regarding distribution of the transverse strain components in the thickness direction. In chapter 2 we considered the construction of a plate theory of a sandwich plate in cylindrical bending, based on the assumption that the transverse strain components do not vary in the thickness direction within a conventional layer of a sandwich plate (a face sheet or the core), but can be different in different layers. This theory was based on linear elasticity and its results were compared with the exact solution of linear elasticity. The comparison showed the validity of these assumptions. Therefore, considering nonlinear dynamics of a sandwich composite plate, we will make the similar simplifying assumptions, leading to a plate theory, i.e. we will assume that within the face sheets and the core the transverse strains do not depend on the z-coordinate, but they can be different functions of coordinates x, y and time t in different face sheets and the core:

$$\left.\begin{array}{c}\varepsilon_{xz}^{(k)} = \varepsilon_{xz}^{(k)}\left(x,y,t\right) \ , \\ \varepsilon_{yz}^{(k)} = \varepsilon_{yz}^{(k)}\left(x,y,t\right) \ , \\ \varepsilon_{zz}^{(k)} = \varepsilon_{zz}^{(k)}\left(x,y,t\right) \\ (k=1,2,3) \ , \end{array}\right\} \tag{3.2.1}$$

where the superscript k denotes the number of a sublaminate: $k = 1$ denotes the lower face sheet, $k = 2$ denotes the core and $k = 3$ denotes the upper face sheet. As in chapter 2, the assumed transverse strains will be called the first form of the transverse strains, and they will be denoted also as

$$\left.\begin{array}{c}\varepsilon_{xz}^{(k)} \equiv \left(\varepsilon_{xz}^{(k)}\right)^{(I)} \ , \\ \varepsilon_{yz}^{(k)} \equiv \left(\varepsilon_{yz}^{(k)}\right)^{(I)} \ , \\ \varepsilon_{zz}^{(k)} \equiv \left(\varepsilon_{zz}^{(k)}\right)^{(I)} \ . \end{array}\right\} \tag{3.2.2}$$

The assumed transverse strains (3.2.1), together with displacements of the middle surface of the plate

$$\left.\begin{aligned} u_0(x,y,t) &\equiv u^{(2)}\big|_{z=0} \ , \\ v_0(x,y,t) &\equiv v^{(2)}\big|_{z=0} \ , \\ w_0(x,y,t) &\equiv w^{(2)}\big|_{z=0} \end{aligned}\right\} \tag{3.2.3}$$

are the **unknown functions of the problem**, which will be computed by the finite element method. Therefore, all displacements, strains and stresses must be expressed in terms of these functions.

3.3 Displacements in Terms of the Unknown Functions

In this section we will integrate strain-displacement relations for the transverse strains in order to obtain expressions for displacements in terms of the unknown functions $\varepsilon_{xz}^{(k)}$, $\varepsilon_{yz}^{(k)}$, $\varepsilon_{zz}^{(k)}$, u_0, v_0, w_0. The von-Karman strain-displacement relations (3.1.2) and (3.1.3), written here again, are

$$\varepsilon_{xx}^{(k)} = u_{,x}^{(k)} + \frac{1}{2}\left(w_{,x}^{(k)}\right)^2, \tag{3.3.1}$$

$$\varepsilon_{yy}^{(k)} = v_{,y}^{(k)} + \frac{1}{2}\left(w_{,y}^{(k)}\right)^2, \tag{3.3.2}$$

$$\varepsilon_{xy}^{(k)} = \frac{1}{2}\left(u_{,y}^{(k)} + v_{,x}^{(k)} + w_{,x}^{(k)} w_{,y}^{(k)}\right) \quad \text{(no summation with respect to k)}, \tag{3.3.3}$$

$$\varepsilon_{xz}^{(k)} = \frac{1}{2}\left(u_{,z}^{(k)} + w_{,x}^{(k)}\right), \tag{3.3.4}$$

$$\varepsilon_{yz}^{(k)} = \frac{1}{2}\left(v_{,z}^{(k)} + w_{,y}^{(k)}\right), \tag{3.3.5}$$

$$\varepsilon_{zz}^{(k)} = w_{,z}^{(k)}, \tag{3.3.6}$$

where the superscript k is the number of a sublaminate (a face sheet or the core). Let us integrate strain-displacement relation (3.3.6). For the core ($k = 2$), which contains the plane $z = 0$, we receive

$$w^{(2)}(x,y,z,t) - \underbrace{w^{(2)}\Big|_{z=0}}_{w_0} = \int_0^z \frac{\partial w^{(2)}}{\partial z} dz = \int_0^z \varepsilon_{zz}^{(2)}(x,y,t)\ dz = \varepsilon_{zz}^{(2)}(x,y,t)\ z \quad (z_2 \le z \le z_3) \tag{3.3.7}$$

or

$$w^{(2)}(x,y,z,t) = w_0(x,y,t) + \varepsilon_{zz}^{(2)}(x,y,t)\ z, \tag{3.3.8}$$

where

$$w_0 \equiv w^{(2)}\Big|_{z=0}\ . \tag{3.3.10}$$

Integration of equation $\varepsilon_{zz}^{(1)} = \frac{\partial w^{(1)}}{\partial z}$ from z_2 to z, where z belongs to the lower face sheet ($z_1 \le z \le z_2$), yields:

$$w^{(1)} - \underbrace{w^{(1)}(z_2)}_{w^{(2)}(z_2)} = \int_{z_2}^z \frac{\partial w^{(1)}}{\partial z} dz = \int_{z_2}^z \varepsilon_{zz}^{(1)}\ dz \quad (z_1 \le z \le z_2), \tag{3.3.11}$$

or, due to the continuity condition, $w^{(1)}(z_2) = w^{(2)}(z_2)$,

$$w^{(1)} = w^{(2)}(z_2) + \int_{z_2}^{z} \varepsilon_{zz}^{(1)}\, dz\ . \tag{3.3.12}$$

From equation (3.3.7), it follows that

$$w^{(2)}(z_2) = w_0 + \int_{0}^{z_2} \varepsilon_{zz}^{(2)} dz\ . \tag{3.3.13}$$

The substitution of (3.3.13) into (3.3.12) yields:

$$w^{(1)}(x,y,z,t) = w_0(x,y,t) + \int_{0}^{z_2} \varepsilon_{zz}^{(2)}(x,y,t)\, dz + \int_{z_2}^{z} \varepsilon_{zz}^{(1)}(x,y,t)\ dz \qquad (z_1 \le z \le z_2)\ ,$$

or

$$w^{(1)}(x,y,z,t) = w_0(x,y,t) + \varepsilon_{zz}^{(2)}(x,y,t)\ z_2 + \varepsilon_{zz}^{(1)}(x,y,t)(z - z_2) \qquad (z_1 \le z \le z_2)\ . \tag{3.3.14}$$

Analogously, integrating equation $\varepsilon_{zz}^{(3)} = \frac{\partial w^{(3)}}{\partial z}$ and satisfying the continuity condition at the interface between the core and the upper face sheet, $w^{(3)}(z_3) = w^{(2)}(z_3)$, we receive

$$w^{(3)}(x,y,z,t) = w_0(x,y,t) + \int_{0}^{z_3} \varepsilon_{zz}^{(2)}(x,y,t)\, dz + \int_{z_3}^{z} \varepsilon_{zz}^{(3)}(x,y,t)\ dz \qquad (z_3 \le z \le z_4)\ ,$$

or

$$w^{(3)}(x,y,z,t) = w_0(x,y,t) + \varepsilon_{zz}^{(2)}(x,y,t)\ z_3 + \varepsilon_{zz}^{(3)}(x,y,t)(z - z_3) \qquad (z_3 \le z \le z_4)\ . \tag{3.3.15}$$

Now, let us integrate strain-displacement relations (3.3.4) and (3.3.5) in order to obtain expressions for displacements $u^{(k)}(x,y,z,t)$ and $v^{(k)}(x,y,z,t)$ in terms of the unknown functions. In tensorial notations relations (3.3.4) and (3.3.5) can be written as

$$\varepsilon_{\alpha 3}^{(k)} = \frac{1}{2}\left(u_{\alpha,3}^{(k)} + u_{3,\alpha}^{(k)}\right) \qquad (\alpha = 1,2;\ k = 1,2,3)\ . \tag{3.3.16}$$

Integrating equations (3.3.16) with respect to z, we receive

$$u_\alpha^{(2)} - u_\alpha^{(2)}\Big|_{z=0} = \int_0^z \frac{\partial u_\alpha^{(2)}}{\partial z}\, dz = \int_0^z \left(2\varepsilon_{\alpha 3}^{(2)} - u_{3,\alpha}^{(2)}\right) dz \quad (\alpha = 1,2;\ \ z_2 \le z \le z_3)\ . \tag{3.3.17}$$

$$u_\alpha^{(1)} - u_\alpha^{(1)}(z_2) = \int_{z_2}^z \frac{\partial u_\alpha^{(1)}}{\partial z}\, dz = \int_{z_2}^z \left(2\varepsilon_{\alpha 3}^{(1)} - u_{3,\alpha}^{(1)}\right) dz \quad (\alpha = 1,2;\ \ z_1 \le z \le z_2)\ . \tag{3.3.18}$$

$$u_{\alpha}^{(3)} - u_{\alpha}^{(3)}(z_3) = \int_{z_3}^{z} \frac{\partial u_{\alpha}^{(3)}}{\partial z}\, dz = \int_{z_3}^{z} \left(2\varepsilon_{\alpha 3}^{(3)} - u_{3,\alpha}^{(3)}\right) dz \quad (\alpha = 1,2;\ \ z_1 \le z \le z_2)\ . \tag{3.3.19}$$

The substitution of expressions (3.3.13)-(3.3.15) for $w^{(k)} \equiv u_3^{(k)}$ into equations (3.3.17)-(3.3.19), performing integration in these equations and finding the constants of integration from the conditions of continuity of displacements u and v at the interfaces between the face sheets and the core

$$u^{(1)}(z_2) = u^{(2)}(z_2)\,,\ \ u^{(2)}(z_3) = u^{(3)}(z_3)\,,\ \ v^{(1)}(z_2) = v^{(2)}(z_2)\,,\ \ v^{(2)}(z_3) = v^{(3)}(z_3)\,, \tag{3.3.20}$$

yields expressions for $u^{(k)}$ and $v^{(k)}$ in terms of u_0, v_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{yz}^{(k)}$, $\varepsilon_{zz}^{(k)}$, where $u_0 \equiv u|_{z=0}$, $v_0 \equiv v|_{z=0}$:

$$\begin{aligned} u^{(1)} = u_0 &+ \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right) z_2 - \frac{1}{2}\varepsilon_{zz,x}^{(2)}\, z_2^2 + \left(2\varepsilon_{xz}^{(1)} - w_{0,x} - \varepsilon_{zz,x}^{(2)}\, z_2\right)(z - z_2) \\ &- \frac{1}{2}\varepsilon_{zz,x}^{(1)}(z - z_2)^2 \qquad\qquad (z_1 \le z \le z_2)\ , \end{aligned} \tag{3.3.21}$$

$$u^{(2)} = u_0 + \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right) z - \frac{1}{2}\varepsilon_{zz,x}^{(2)}\, z^2 \qquad\qquad (z_2 \le z \le z_3)\ , \tag{3.3.22}$$

$$\begin{aligned} u^{(3)} = u_0 &+ \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right) z_3 - \frac{1}{2}\varepsilon_{zz,x}^{(2)}\, z_3^2 + \left(2\varepsilon_{xz}^{(3)} - w_{0,x} - \varepsilon_{zz,x}^{(2)} z_3\right)(z - z_3) \\ &- \frac{1}{2}\varepsilon_{zz,x}^{(3)}(z - z_3)^2 \qquad\qquad (z_3 \le z \le z_4)\ , \end{aligned} \tag{3.3.23}$$

$$\begin{aligned} v^{(1)} = v_0 &+ \left(2\varepsilon_{yz}^{(2)} - w_{0,y}\right) z_2 - \frac{1}{2}\varepsilon_{zz,y}^{(2)}\, z_2^2 + \left(2\varepsilon_{yz}^{(1)} - w_{0,y} - \varepsilon_{zz,y}^{(2)}\, z_2\right)(z - z_2) \\ &- \frac{1}{2}\varepsilon_{zz,y}^{(1)}(z - z_2)^2 \qquad\qquad (z_1 \le z \le z_2)\ , \end{aligned} \tag{3.3.24}$$

$$v^{(2)} = v_0 + \left(2\varepsilon_{yz}^{(2)} - w_{0,y}\right) z - \frac{1}{2}\varepsilon_{zz,y}^{(2)}\, z^2 \qquad (z_2 \le z \le z_3)\ , \tag{3.3.25}$$

$$\begin{aligned} v^{(3)} = v_0 &+ \left(2\varepsilon_{yz}^{(2)} - w_{0,y}\right) z_3 - \frac{1}{2}\varepsilon_{zz,y}^{(2)}\, z_3^2 + \left(2\varepsilon_{yz}^{(3)} - w_{0,y} - \varepsilon_{zz,y}^{(2)} z_3\right)(z - z_3) \\ &- \frac{1}{2}\varepsilon_{zz,y}^{(3)}(z - z_3)^2 \qquad\qquad (z_3 \le z \le z_4)\ , \end{aligned} \tag{3.3.26}$$

Expressions (3.3.8), (3.3.14), (3.3.15) and (3.3.21)-(3.3.26) for displacements in terms of the unknown functions u_0, v_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{yz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ can be written in a more convenient form:

$$\left\{\begin{matrix} u \\ v \\ w \end{matrix}\right\}^{(k)} = \left\{\begin{matrix} \psi_{u0} \\ \psi_{v0} \\ \psi_{w0} \end{matrix}\right\}^{(k)} + \left\{\begin{matrix} \psi_{u1} \\ \psi_{v1} \\ \psi_{w1} \end{matrix}\right\}^{(k)} z + \left\{\begin{matrix} \psi_{u2} \\ \psi_{v2} \\ 0 \end{matrix}\right\}^{(k)} z^2\ , \tag{3.3.27}$$

where

$$\psi_{u0}^{(1)} = u_0 + 2z_2\left(\varepsilon_{xz}^{(2)} - \varepsilon_{xz}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(1)}\right) \ , \tag{3.3.28}$$

$$\psi_{u1}^{(1)} = 2\varepsilon_{xz}^{(1)} - w_{0,x} + z_2\left(\varepsilon_{zz,x}^{(1)} - \varepsilon_{zz,x}^{(2)}\right) \ , \tag{3.3.29}$$

$$\psi_{u2}^{(1)} = -\frac{1}{2}\varepsilon_{zz,x}^{(1)} \ , \tag{3.3.30}$$

$$\psi_{u0}^{(2)} = u_0 \ , \tag{3.3.31}$$

$$\psi_{u1}^{(2)} = 2\varepsilon_{xz}^{(2)} - w_{0,x} \ , \tag{3.3.32}$$

$$\psi_{u2}^{(2)} = -\frac{1}{2}\varepsilon_{zz,x}^{(2)} \ , \tag{3.3.33}$$

$$\psi_{u0}^{(3)} = u_0 + 2z_3\left(\varepsilon_{xz}^{(2)} - \varepsilon_{xz}^{(3)}\right) + \frac{1}{2}z_3^2\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(3)}\right) \ , \tag{3.3.34}$$

$$\psi_{u1}^{(3)} = 2\varepsilon_{xz}^{(3)} - w_{0,x} + z_3\left(\varepsilon_{zz,x}^{(3)} - \varepsilon_{zz,x}^{(2)}\right) \ , \tag{3.3.35}$$

$$\psi_{u2}^{(3)} = -\frac{1}{2}\varepsilon_{zz,x}^{(3)} \ , \tag{3.3.36}$$

$$\psi_{v0}^{(1)} = v_0 + 2z_2\left(\varepsilon_{yz}^{(2)} - \varepsilon_{yz}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,y}^{(2)} - \varepsilon_{zz,y}^{(1)}\right) \ , \tag{3.3.37}$$

$$\psi_{v1}^{(1)} = 2\varepsilon_{yz}^{(1)} - w_{0,y} + z_2\left(\varepsilon_{zz,y}^{(1)} - \varepsilon_{zz,y}^{(2)}\right) \ , \tag{3.3.38}$$

$$\psi_{v2}^{(1)} = -\frac{1}{2}\varepsilon_{zz,y}^{(1)} \ , \tag{3.3.39}$$

$$\psi_{v0}^{(2)} = v_0 \ , \tag{3.3.40}$$

$$\psi_{v1}^{(2)} = 2\varepsilon_{yz}^{(2)} - w_{0,y} \ , \tag{3.3.41}$$

$$\psi_{v2}^{(2)} = -\frac{1}{2}\varepsilon_{zz,y}^{(2)} \ , \tag{3.3.42}$$

$$\psi_{v0}^{(3)} = v_0 + 2z_3\left(\varepsilon_{yz}^{(2)} - \varepsilon_{yz}^{(3)}\right) + \frac{1}{2}z_3^2\left(\varepsilon_{zz,y}^{(2)} - \varepsilon_{zz,y}^{(3)}\right) \ , \tag{3.3.43}$$

$$\psi_{v1}^{(3)} = 2\varepsilon_{yz}^{(3)} - w_{0,y} + z_3\left(\varepsilon_{zz,y}^{(3)} - \varepsilon_{zz,y}^{(2)}\right) \ , \tag{3.3.44}$$

$$\psi_{v2}^{(3)} = -\frac{1}{2}\varepsilon_{zz,y}^{(3)} \ , \tag{3.3.45}$$

$$\psi_{w0}^{(1)} = w_0 + z_2\left(\varepsilon_{zz}^{(2)} - \varepsilon_{zz}^{(1)}\right) \ , \tag{3.3.46}$$

$$\psi_{w1}^{(1)} = \varepsilon_{zz}^{(1)} \ , \tag{3.3.47}$$

$$\psi_{w0}^{(2)} = w_0 \ , \tag{3.3.48}$$

$$\psi_{w1}^{(2)} = \varepsilon_{zz}^{(2)} \ , \tag{3.3.49}$$

$$\psi_{w0}^{(3)} = w_0 + z_3\left(\varepsilon_{zz}^{(2)} - \varepsilon_{zz}^{(3)}\right) \ , \tag{3.3.50}$$

$$\psi_{w1}^{(3)} = \varepsilon_{zz}^{(3)} \ . \tag{3.3.51}$$

Matrix equation (3.3.27) can be written in the form

$$\begin{Bmatrix} u^{(k)} \\ v^{(k)} \\ w^{(k)} \end{Bmatrix} = \underset{(3\times 8)}{\left[\ \widetilde{Z}\ \right]}\left\{\psi^{(k)}\right\} \ , \tag{3.3.52}$$

where

$$\underset{(3\times 8)}{\left[\ \widetilde{Z}\ \right]} = \begin{bmatrix} 1 & z & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z \end{bmatrix} \tag{3.3.53}$$

and

$$\left\{\psi^{(k)}\right\} \equiv \left\lfloor\ \psi_{u0}^{(k)} \quad \psi_{u1}^{(k)} \quad \psi_{u2}^{(k)} \quad \psi_{v0}^{(k)} \quad \psi_{v1}^{(k)} \quad \psi_{v2}^{(k)} \quad \psi_{w0}^{(k)} \quad \psi_{w1}^{(k)}\ \right\rfloor^T \ . \tag{3.3.54}$$

Then, displacements of the lower face sheet (k=1) can be written in the form

$$
\left\{\begin{array}{c} u^{(1)} \\ v^{(1)} \\ w^{(1)} \end{array}\right\} = \underset{(3\times 8)}{\left[\,\widetilde{Z}\,\right]}\underset{(8\times 1)}{\left\{\psi^{(1)}\right\}} = \left[\,\widetilde{Z}\,\right]\left\{\begin{array}{c} \psi_{u0}^{(1)} \\ \psi_{u1}^{(1)} \\ \psi_{u2}^{(1)} \\ \psi_{v0}^{(1)} \\ \psi_{v1}^{(1)} \\ \psi_{v2}^{(1)} \\ \psi_{w0}^{(1)} \\ \psi_{w1}^{(1)} \end{array}\right\} =
$$

$$
= \underset{(3\times 8)}{\left[\,\widetilde{Z}\,\right]}\left\{\begin{array}{c}
u_0 + 2z_2\left(\varepsilon_{xz}^{(2)} - \varepsilon_{xz}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(1)}\right) \\
2\varepsilon_{xz}^{(1)} - w_{0,x} + z_2\left(\varepsilon_{zz,x}^{(1)} - \varepsilon_{zz,x}^{(2)}\right) \\
-\frac{1}{2}\varepsilon_{zz,x}^{(1)} \\
v_0 + 2z_2\left(\varepsilon_{yz}^{(2)} - \varepsilon_{yz}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,y}^{(2)} - \varepsilon_{zz,y}^{(1)}\right) \\
2\varepsilon_{yz}^{(1)} - w_{0,y} + z_2\left(\varepsilon_{zz,y}^{(1)} - \varepsilon_{zz,y}^{(2)}\right) \\
-\frac{1}{2}\varepsilon_{zz,y}^{(1)} \\
w_0 + z_2\left(\varepsilon_{zz}^{(2)} - \varepsilon_{zz}^{(1)}\right) \\
\varepsilon_{zz}^{(1)}
\end{array}\right\} =
$$

$$
= \underset{(3\times 8)}{\left[\,\widetilde{Z}\,\right]}\underset{(8\times 12)}{\left[\begin{array}{cccccccccccc}
1 & 0 & 0 & -2z_2 & 0 & -\frac{1}{2}z_2^2\frac{\partial}{\partial x} & 2z_2 & 0 & \frac{1}{2}z_2^2\frac{\partial}{\partial x} & 0 & 0 & 0 \\
0 & 0 & -\frac{\partial}{\partial x} & 2 & 0 & z_2\frac{\partial}{\partial x} & 0 & 0 & -z_2\frac{\partial}{\partial x} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & w_0 & 0 & -2z_2 & -\frac{1}{2}z_2^2\frac{\partial}{\partial y} & 0 & 2z_2 & \frac{1}{2}z_2^2\frac{\partial}{\partial y} & 0 & 0 & 0 \\
0 & 0 & -\frac{\partial}{\partial y} & 0 & 2 & z_2\frac{\partial}{\partial y} & 0 & 0 & -z_2\frac{\partial}{\partial y} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -z_2 & 0 & 0 & z_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]}
\underset{(12\times 1)}{\left\{\begin{array}{c} u_0 \\ v_0 \\ w_0 \\ \varepsilon_{xz}^{(1)} \\ \varepsilon_{yz}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{yz}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ \varepsilon_{xz}^{(3)} \\ \varepsilon_{yz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{array}\right\}}
$$

or

$$\left\{\begin{array}{c} u^{(1)} \\ v^{(1)} \\ w^{(1)} \end{array}\right\} = \underset{(3\times 8)}{\left[\ \widetilde{Z}\ \right]} \underbrace{\underset{(8\times 12)}{\left[\ \widetilde{\partial}^{(1)}\right]} \underset{(12\times 1)}{\{f\}}}_{\{\psi^{(1)}\}} , \tag{3.3.55}$$

where matrix $\left[\ \widetilde{Z}\ \right]$ is defined by formula (3.3.53); $\left[\ \widetilde{\partial}^{(1)}\right]$ is a matrix of differential operators, defined as

$$\left[\ \widetilde{\partial}^{(1)}\right] \equiv \begin{bmatrix} 1 & 0 & 0 & -2z_2 & 0 & -\frac{1}{2}z_2^2\frac{\partial}{\partial x} & 2z_2 & 0 & \frac{1}{2}z_2^2\frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial x} & 2 & 0 & z_2\frac{\partial}{\partial x} & 0 & 0 & -z_2\frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & w_0 & 0 & -2z_2 & -\frac{1}{2}z_2^2\frac{\partial}{\partial y} & 0 & 2z_2 & \frac{1}{2}z_2^2\frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial y} & 0 & 2 & z_2\frac{\partial}{\partial y} & 0 & 0 & -z_2\frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -z_2 & 0 & 0 & z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} , \tag{3.3.56}$$

and $\{f\}$ is a **column-matrix of the unknown functions of the problem**, defined as

$$\{f\} \equiv \left\lfloor\ u_0 \quad v_0 \quad w_0 \quad \varepsilon_{xz}^{(1)} \quad \varepsilon_{yz}^{(1)} \quad \varepsilon_{zz}^{(1)} \quad \varepsilon_{xz}^{(2)} \quad \varepsilon_{yz}^{(2)} \quad \varepsilon_{zz}^{(2)} \quad \varepsilon_{xz}^{(3)} \quad \varepsilon_{yz}^{(3)} \quad \varepsilon_{zz}^{(3)}\ \right\rfloor^T . \tag{3.3.57}$$

Displacements of the core ($k = 2$) can be written in the form

$$\left\{\begin{array}{c} u^{(2)} \\ v^{(2)} \\ w^{(2)} \end{array}\right\} = \underset{(3\times 8)}{\left[\ \widetilde{Z}\ \right]} \underset{(8\times 1)}{\left\{\psi^{(2)}\right\}} = \left[\ \widetilde{Z}\ \right] \left\{\begin{array}{c} \psi_{u0}^{(2)} \\ \psi_{u1}^{(2)} \\ \psi_{u2}^{(2)} \\ \psi_{v0}^{(2)} \\ \psi_{v1}^{(2)} \\ \psi_{v2}^{(2)} \\ \psi_{w0}^{(2)} \\ \psi_{w1}^{(2)} \end{array}\right\} =$$

$$= \left[\,\widetilde{Z}\,\right] \left\{ \begin{array}{c} u_0 \\ 2\varepsilon_{xz}^{(2)} - w_{0,x} \\ -\frac{1}{2}\varepsilon_{zz,x}^{(2)} \\ v_0 \\ 2\varepsilon_{yz}^{(2)} - w_{0,y} \\ -\frac{1}{2}\varepsilon_{zz,y}^{(2)} \\ w_0 \\ \varepsilon_{zz}^{(2)} \end{array} \right\} = \underset{(3\times 8)}{\left[\,\widetilde{Z}\,\right]} \underset{(8\times 12)}{\left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial x} & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]} \underset{(12\times 1)}{\left\{ \begin{array}{c} u_0 \\ v_0 \\ w_0 \\ \varepsilon_{xz}^{(1)} \\ \varepsilon_{yz}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{yz}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ \varepsilon_{xz}^{(3)} \\ \varepsilon_{yz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{array} \right\}},$$

or

$$\left\{ \begin{array}{c} u^{(2)} \\ v^{(2)} \\ w^{(2)} \end{array} \right\} = \underset{(3\times 8)}{\left[\,\widetilde{Z}\,\right]} \underbrace{\underset{(8\times 12)}{\left[\,\widetilde{\partial}^{(2)}\,\right]} \underset{(12\times 1)}{\{f\}}}_{\{\psi^{(2)}\}}, \tag{3.3.58}$$

where

$$\left[\,\widetilde{\partial}^{(2)}\,\right] = \underset{(8\times 12)}{\left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial x} & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]}, \tag{3.3.59}$$

$\left[\,\widetilde{Z}\,\right]$ is a matrix, defined by equation (3.3.53), $\{f\}$ is a column-matrix of the unknown functions of the problem, defined by equation (3.3.57).

The displacements of the upper face sheet ($k = 3$) can be written in the form

$$\left\{\begin{array}{c} u^{(3)} \\ v^{(3)} \\ w^{(3)} \end{array}\right\} = \underset{(3\times 8)}{\left[\ \widetilde{Z}\ \right]} \underset{(8\times 1)}{\left\{\psi^{(3)}\right\}} = \left[\ \widetilde{Z}\ \right] \left\{\begin{array}{c} \psi_{u0}^{(3)} \\ \psi_{u1}^{(3)} \\ \psi_{u2}^{(3)} \\ \psi_{v0}^{(3)} \\ \psi_{v1}^{(3)} \\ \psi_{v2}^{(3)} \\ \psi_{w0}^{(3)} \\ \psi_{w1}^{(3)} \end{array}\right\} =$$

$$= \left[\ \widetilde{Z}\ \right] \left\{\begin{array}{c} u_0 + 2z_3\left(\varepsilon_{xz}^{(2)} - \varepsilon_{xz}^{(3)}\right) + \frac{1}{2}z_3^2\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(3)}\right) \\ 2\varepsilon_{xz}^{(3)} - w_{0,x} + z_3\left(\varepsilon_{zz,x}^{(3)} - \varepsilon_{zz,x}^{(2)}\right) \\ -\frac{1}{2}\varepsilon_{zz,x}^{(3)} \\ v_0 + 2z_3\left(\varepsilon_{yz}^{(2)} - \varepsilon_{yz}^{(3)}\right) + \frac{1}{2}z_3^2\left(\varepsilon_{zz,y}^{(2)} - \varepsilon_{zz,y}^{(3)}\right) \\ 2\varepsilon_{yz}^{(3)} - w_{0,y} + z_3\left(\varepsilon_{zz,y}^{(3)} - \varepsilon_{zz,y}^{(2)}\right) \\ -\frac{1}{2}\varepsilon_{zz,y}^{(3)} \\ w_0 + z_3\left(\varepsilon_{zz}^{(2)} - \varepsilon_{zz}^{(3)}\right) \\ \varepsilon_{zz}^{(3)} \end{array}\right\} =$$

$$= \left[\ \widetilde{Z}\ \right] \left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 2z_3 & 0 & \frac{1}{2}z_3^2\frac{\partial}{\partial x} & -2z_3 & 0 & -\frac{1}{2}z_3^2\frac{\partial}{\partial x} \\ 0 & 0 & -\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & -z_3\frac{\partial}{\partial x} & 2 & 0 & z_3\frac{\partial}{\partial x} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial x} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2z_3 & \frac{1}{2}z_3^2\frac{\partial}{\partial y} & 0 & -2z_3 & -\frac{1}{2}z_3^2\frac{\partial}{\partial y} \\ 0 & 0 & -\frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 & -z_3\frac{\partial}{\partial y} & 0 & 2 & z_3\frac{\partial}{\partial y} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial y} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & z_3 & 0 & 0 & -z_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left\{\begin{array}{c} u_0 \\ v_0 \\ w_0 \\ \varepsilon_{xz}^{(1)} \\ \varepsilon_{yz}^{(1)} \\ \varepsilon_{zz}^{(1)} \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{yz}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ \varepsilon_{xz}^{(3)} \\ \varepsilon_{yz}^{(3)} \\ \varepsilon_{zz}^{(3)} \end{array}\right\},$$

or

$$\left\{\begin{array}{c} u^{(3)} \\ v^{(3)} \\ w^{(3)} \end{array}\right\} = \underset{(3\times8)}{\left[\ \widetilde{Z}\ \right]} \underbrace{\underset{(8\times12)}{\left[\ \widetilde{\partial}^{(3)}\right]} \underset{(12\times1)}{\{f\}}}_{\psi^{(3)}} , \qquad (3.3.60)$$

where

$$\underset{(8\times12)}{\left[\widetilde{\partial}^{(3)}\right]} = \left[\begin{array}{cccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 2z_3 & 0 & \frac{1}{2}z_3^2\frac{\partial}{\partial x} & -2z_3 & 0 & -\frac{1}{2}z_3^2\frac{\partial}{\partial x} \\
0 & 0 & -\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & -z_3\frac{\partial}{\partial x} & 2 & 0 & z_3\frac{\partial}{\partial x} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial x} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 2z_3 & \frac{1}{2}z_3^2\frac{\partial}{\partial y} & 0 & -2z_3 & -\frac{1}{2}z_3^2\frac{\partial}{\partial y} \\
0 & 0 & -\frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 & -z_3\frac{\partial}{\partial y} & 0 & 2 & z_3\frac{\partial}{\partial y} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial y} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & z_3 & 0 & 0 & -z_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] , \qquad (3.3.61)$$

$\left[\ \widetilde{Z}\ \right]$ is a matrix, defined by equation (3.3.53), $\{f\}$ is a column-matrix of the unknown functions of the problem, defined by equation (3.3.57).

In summary, the column-matrices of displacements in each of the sublaminates can be written in the form:

$$\left\{\begin{array}{c} u^{(1)} \\ v^{(1)} \\ w^{(1)} \end{array}\right\} = \underset{(3\times8)}{\left[\ \widetilde{Z}\ \right]} \underbrace{\underset{(8\times12)}{\left[\ \widetilde{\partial}^{(1)}\right]} \underset{(12\times1)}{\{f\}}}_{\{\psi^{(1)}\}} , \qquad (3.3.62)$$

$$\left\{\begin{array}{c} u^{(2)} \\ v^{(2)} \\ w^{(2)} \end{array}\right\} = \underset{(3\times8)}{\left[\ \widetilde{Z}\ \right]} \underbrace{\underset{(8\times12)}{\left[\ \widetilde{\partial}^{(2)}\ \right]} \underset{(12\times1)}{\{f\}}}_{\{\psi^{(2)}\}} , \qquad (3.3.63)$$

$$\left\{\begin{array}{c} u^{(3)} \\ v^{(3)} \\ w^{(3)} \end{array}\right\} = \underset{(3\times8)}{\left[\ \widetilde{Z}\ \right]} \underbrace{\underset{(8\times12)}{\left[\ \widetilde{\partial}^{(3)}\right]} \underset{(12\times1)}{\{f\}}}_{\psi^{(3)}} , \qquad (3.3.64)$$

where $\left[\ \widetilde{Z}\ \right]$ is a matrix, that depends only on z-coordinate; $\left[\ \widetilde{\partial}^{(1)}\right]$, $\left[\ \widetilde{\partial}^{(2)}\right]$, $\left[\ \widetilde{\partial}^{(3)}\right]$ are matrices of differential operators; and $\{f\}$ is a column-matrix of the unknown functions of the problem.

3.4 In-Plane Strains in Terms of the Unknown Functions

In order to perform the finite element formulation, it is necessary to have an expression for the strain energy in terms of the unknown functions u_0, v_0, w_0, $\varepsilon_{xz}^{(1)}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{xz}^{(3)}$, $\varepsilon_{yz}^{(1)}$, $\varepsilon_{yz}^{(2)}$, $\varepsilon_{yz}^{(3)}$, $\varepsilon_{zz}^{(1)}$, $\varepsilon_{zz}^{(2)}$, $\varepsilon_{zz}^{(3)}$. This requires expressions for the strains in term of the unknown functions. The transverse strains $\varepsilon_{xz}^{(1)}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{xz}^{(3)}$, $\varepsilon_{yz}^{(1)}$, $\varepsilon_{yz}^{(2)}$, $\varepsilon_{yz}^{(3)}$, $\varepsilon_{zz}^{(1)}$, $\varepsilon_{zz}^{(2)}$, $\varepsilon_{zz}^{(3)}$ are the unknown functions themselves. Therefore, it is necessary to express the in-plane strains $\varepsilon_{xx}^{(1)}$, $\varepsilon_{xx}^{(2)}$, $\varepsilon_{xx}^{(3)}$, $\varepsilon_{xy}^{(1)}$, $\varepsilon_{xy}^{(2)}$, $\varepsilon_{xy}^{(3)}$, $\varepsilon_{yy}^{(1)}$, $\varepsilon_{yy}^{(2)}$, $\varepsilon_{yy}^{(3)}$ in terms of the unknown functions.

In order to find the in-plane strains $\varepsilon_{xx}^{(k)}$, $\varepsilon_{xy}^{(k)}$and $\varepsilon_{yy}^{(k)}$ in terms of the unknown functions we will substitute expressions (3.3.27), written here again,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}^{(k)} = \begin{Bmatrix} \psi_{u0} \\ \psi_{v0} \\ \psi_{w0} \end{Bmatrix}^{(k)} + \begin{Bmatrix} \psi_{u1} \\ \psi_{v1} \\ \psi_{w1} \end{Bmatrix}^{(k)} z + \begin{Bmatrix} \psi_{u2} \\ \psi_{v2} \\ \psi_{w2} \end{Bmatrix}^{(k)} z^2 \qquad \text{(eqn 3.3.27)}$$

into the strain-displacement relations (3.1.2), written here again,

$$\varepsilon_{xx}^{(k)} = u_{,x}^{(k)} + \frac{1}{2}\left(w_{,x}^{(k)}\right)^2 ,$$

$$\varepsilon_{yy}^{(k)} = v_{,y}^{(k)} + \frac{1}{2}\left(w_{,y}^{(k)}\right)^2 , \qquad \text{(eqn 3.1.12)}$$

$$\varepsilon_{xy}^{(k)} = \frac{1}{2}\left(u_{,y}^{(k)} + v_{,x}^{(k)} + w_{,x}^{(k)} w_{,y}^{(k)}\right) \quad \text{(no summation with respect to k).}$$

The result can be written in the form

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(k)} = \begin{Bmatrix} \varphi_{xx0} \\ \varphi_{yy0} \\ \varphi_{xy0} \end{Bmatrix}^{(k)} + \begin{Bmatrix} \varphi_{xx1} \\ \varphi_{yy1} \\ \varphi_{xy1} \end{Bmatrix}^{(k)} z + \begin{Bmatrix} \varphi_{xx2} \\ \varphi_{yy2} \\ \varphi_{xy2} \end{Bmatrix}^{(k)} z^2 , \tag{3.4.1}$$

where expressions for $\varphi_{xxm}^{(k)}$, $\varphi_{yym}^{(k)}$, $\varphi_{xym}^{(k)}$ $(m = 0, 1, 2)$ in terms of the unknown functions are (the non-linear terms are underbraced):

$$\begin{aligned} \varphi_{xx0}^{(1)} = u_{0,x} + 2z_2\left(\varepsilon_{xz,x}^{(2)} - \varepsilon_{xz,x}^{(1)}\right) + \\ \frac{1}{2}z_2^2\left(\varepsilon_{zz,xx}^{(2)} - \varepsilon_{zz,xx}^{(1)}\right) + \\ \underbrace{\frac{1}{2}\left[w_{0,x} + z_2\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(1)}\right)\right]^2} , \end{aligned} \tag{3.4.2}$$

$$\varphi^{(1)}_{xx1} = 2\varepsilon^{(1)}_{xz,x} - w_{0,xx} + z_2\left(\varepsilon^{(1)}_{zz,xx} - \varepsilon^{(2)}_{zz,xx}\right) + \underbrace{\left[w_{0,x} + z_2\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(1)}_{zz,x}\right)\right]\varepsilon^{(1)}_{zz,x}} , \tag{3.4.3}$$

$$\varphi^{(1)}_{xx2} = -\frac{1}{2}\varepsilon^{(1)}_{zz,xx} + \underbrace{\frac{1}{2}\left(\varepsilon^{(1)}_{zz,x}\right)^2} , \tag{3.4.4}$$

$$\varphi^{(1)}_{yy0} = v_{0,y} + 2z_2\left(\varepsilon^{(2)}_{yz,y} - \varepsilon^{(1)}_{yz,y}\right) + \frac{1}{2}z_2^2\left(\varepsilon^{(2)}_{zz,yy} - \varepsilon^{(1)}_{zz,yy}\right) + \underbrace{\frac{1}{2}\left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right]^2} , \tag{3.4.5}$$

$$\varphi^{(1)}_{yy1} = 2\varepsilon^{(1)}_{yz,y} - w_{0,yy} + z_2\left(\varepsilon^{(1)}_{zz,yy} - \varepsilon^{(2)}_{zz,yy}\right) + \underbrace{\left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right]\varepsilon^{(1)}_{zz,y}} , \tag{3.4.6}$$

$$\varphi^{(1)}_{yy2} = -\frac{1}{2}\varepsilon^{(1)}_{zz,yy} + \underbrace{\frac{1}{2}\left(\varepsilon^{(1)}_{zz,y}\right)^2} , \tag{3.4.7}$$

$$\varphi^{(1)}_{xy0} = u_{0,y} + v_{0,x} + 2z_2\left(\varepsilon^{(2)}_{xz,y} - \varepsilon^{(1)}_{xz,y} + \varepsilon^{(2)}_{yz,x} - \varepsilon^{(1)}_{yz,x}\right) + z_2^2\left(\varepsilon^{(2)}_{zz,xy} - \varepsilon^{(1)}_{zz,xy}\right) + \underbrace{\left[w_{0,x} + z_2\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(1)}_{zz,x}\right)\right]\left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right]} , \tag{3.4.8}$$

$$\varphi^{(1)}_{xy1} = 2\left(\varepsilon^{(1)}_{xz,y} + \varepsilon^{(1)}_{yz,x}\right) - 2w_{0,xy} + 2z_2\left(\varepsilon^{(1)}_{zz,xy} - \varepsilon^{(2)}_{zz,xy}\right) + \underbrace{\varepsilon^{(1)}_{zz,y}\left[w_{0,x} + z_2\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(1)}_{zz,x}\right)\right] + \varepsilon^{(1)}_{zz,x}\left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right]} , \tag{3.4.9}$$

$$\varphi^{(1)}_{xy2} = -\varepsilon^{(1)}_{zz,xy} + \underbrace{\varepsilon^{(1)}_{zz,x}\varepsilon^{(1)}_{zz,y}} \tag{3.4.10}$$

$$\varphi_{xx0}^{(2)} = u_{0,x} + \underbrace{\frac{1}{2}(w_{0,x})^2}\ , \tag{3.4.11}$$

$$\varphi_{xx1}^{(2)} = 2\varepsilon_{xz,x}^{(2)} - w_{0,xx} + \underbrace{w_{0,x}\varepsilon_{zz,x}^{(2)}}\ , \tag{3.4.12}$$

$$\varphi_{xx2}^{(2)} = -\frac{1}{2}\varepsilon_{zz,xx}^{(2)} + \underbrace{\frac{1}{2}\left(\varepsilon_{zz,x}^{(2)}\right)^2}\ , \tag{3.4.13}$$

$$\varphi_{yy0}^{(2)} = v_{0,y} + \underbrace{\frac{1}{2}(w_{0,y})^2}\ , \tag{3.4.14}$$

$$\varphi_{yy1}^{(2)} = 2\varepsilon_{yz,y}^{(2)} - w_{0,yy} + \underbrace{w_{0,y}\varepsilon_{zz,y}^{(2)}}\ , \tag{3.4.15}$$

$$\varphi_{yy2}^{(2)} = -\frac{1}{2}\varepsilon_{zz,yy}^{(2)} + \underbrace{\frac{1}{2}\left(\varepsilon_{zz,y}^{(2)}\right)^2}\ , \tag{3.4.16}$$

$$\varphi_{xy0}^{(2)} = u_{0,y} + v_{0,x} + \underbrace{w_{0,x}w_{0,y}}\ , \tag{3.4.17}$$

$$\varphi_{xy1}^{(2)} = 2\left(\varepsilon_{xz,y}^{(2)} + \varepsilon_{yz,x}^{(2)} - w_{0,xy}\right) + \underbrace{w_{0,y}\varepsilon_{zz,x}^{(2)} + w_{0,x}\varepsilon_{zz,y}^{(2)}}\ , \tag{3.4.18}$$

$$\varphi_{xy2}^{(2)} = -\varepsilon_{zz,xy}^{(2)} + \underbrace{\varepsilon_{zz,x}^{(2)}\varepsilon_{zz,y}^{(2)}}\ , \tag{3.4.19}$$

$$\begin{aligned}\varphi_{xx0}^{(3)} = {} & u_{0,x} + 2z_3\left(\varepsilon_{xz,x}^{(2)} - \varepsilon_{xz,x}^{(3)}\right) + \frac{1}{2}z_3^2\left(\varepsilon_{zz,xx}^{(2)} - \varepsilon_{zz,xx}^{(3)}\right) + \\ & \underbrace{\frac{1}{2}\left[w_{0,x} + z_3\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(3)}\right)\right]^2}\ ,\end{aligned} \tag{3.4.20}$$

$$\varphi^{(3)}_{xx1} = 2\varepsilon^{(3)}_{xz,x} - w_{0,xx} + z_3\left(\varepsilon^{(3)}_{zz,xx} - \varepsilon^{(2)}_{zz,xx}\right) + \underbrace{\left[w_{0,x} + z_3\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(3)}_{zz,x}\right)\right]\varepsilon^{(3)}_{zz,x}} \; , \tag{3.4.21}$$

$$\varphi^{(3)}_{xx2} = -\frac{1}{2}\varepsilon^{(3)}_{zz,xx} + \underbrace{\frac{1}{2}\left(\varepsilon^{(3)}_{zz,x}\right)^2} \; , \tag{3.4.22}$$

$$\varphi^{(3)}_{yy0} = v_{0,y} + 2z_3\left(\varepsilon^{(2)}_{yz,y} - \varepsilon^{(3)}_{yz,y}\right) + \frac{1}{2}z_3^2\left(\varepsilon^{(2)}_{zz,yy} - \varepsilon^{(3)}_{zz,yy}\right) + \underbrace{\frac{1}{2}\left[w_{0,y} + z_3\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(3)}_{zz,y}\right)\right]^2} \; , \tag{3.4.23}$$

$$\varphi^{(3)}_{yy1} = 2\varepsilon^{(3)}_{yz,y} - w_{0,yy} + z_3\left(\varepsilon^{(3)}_{zz,yy} - \varepsilon^{(2)}_{zz,yy}\right) + \underbrace{\left[w_{0,y} + z_3\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(3)}_{zz,y}\right)\right]\varepsilon^{(3)}_{zz,y}} \; , \tag{3.4.24}$$

$$\varphi^{(3)}_{yy2} = -\frac{1}{2}\varepsilon^{(3)}_{zz,yy} + \underbrace{\frac{1}{2}\left(\varepsilon^{(3)}_{zz,y}\right)^2} \; , \tag{3.4.25}$$

$$\varphi^{(3)}_{xy0} = u_{0,y} + v_{0,x} + 2z_3\left(\varepsilon^{(2)}_{xz,y} - \varepsilon^{(3)}_{xz,y} + \varepsilon^{(2)}_{yz,x} - \varepsilon^{(3)}_{yz,x}\right) + z_3^2\left(\varepsilon^{(2)}_{zz,xy} - \varepsilon^{(3)}_{zz,xy}\right) + \underbrace{\left[w_{0,x} + z_3\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(3)}_{zz,x}\right)\right]\left[w_{0,y} + z_3\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(3)}_{zz,y}\right)\right]} \; , \tag{3.4.26}$$

$$\varphi^{(3)}_{xy1} = 2\left[\varepsilon^{(3)}_{xz,y} + \varepsilon^{(3)}_{yz,x} - w_{0,xy} + z_3\left(\varepsilon^{(3)}_{zz,xy} - \varepsilon^{(2)}_{zz,xy}\right)\right] + \underbrace{\varepsilon^{(3)}_{zz,x}\left[w_{0,y} + z_3\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(3)}_{zz,y}\right)\right] + \varepsilon^{(3)}_{zz,y}\left[w_{0,x} + z_3\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(3)}_{zz,x}\right)\right]} \; , \tag{3.4.27}$$

$$\varphi^{(3)}_{xy2} = -\varepsilon^{(3)}_{zz,xy} + \underbrace{\varepsilon^{(3)}_{zz,x}\varepsilon^{(3)}_{zz,y}} \; . \tag{3.4.28}$$

3.5 Expressions for All Strains in Terms of the Unknown Functions in Matrix Form

In performing the finite element formulation it is convenient to write the expression for the strain energy in matrix form. Therefore, it is convenient to form column-matrices of strains of each sublaminate as follows

$$\underset{(6\times 1)}{\left\{\varepsilon^{(k)}\right\}} \equiv \left\lfloor \; \varepsilon_{xx}^{(k)} \quad \varepsilon_{yy}^{(k)} \quad \varepsilon_{zz}^{(k)} \quad 2\varepsilon_{yz}^{(k)} \quad 2\varepsilon_{xz}^{(k)} \quad 2\varepsilon_{xy}^{(k)} \; \right\rfloor^T \qquad (k=1,2,3)\,, \tag{3.5.1}$$

(where the superscript k denotes the number of a sublaminate) and to write the expressions for these column-matrices in terms of the unknown functions u_0, v_0, w_0, $\varepsilon_{xz}^{(1)}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{xz}^{(3)}$, $\varepsilon_{yz}^{(1)}$, $\varepsilon_{yz}^{(2)}$, $\varepsilon_{yz}^{(3)}$, $\varepsilon_{zz}^{(1)}$, $\varepsilon_{zz}^{(2)}$, $\varepsilon_{zz}^{(3)}$ in matrix form.

Then, using expressions (3.4.1) for the in-plane strains in terms of the unknown functions, one can write

$$\underset{(6\times 1)}{\left\{\varepsilon^{(k)}\right\}} = \underset{(6\times 12)}{[\,Z\,]}\underset{(12\times 1)}{\left\{\varphi^{(k)}\right\}}, \tag{3.5.2}$$

where

$$\underset{(12\times 1)}{\left\{\varphi^{(k)}\right\}} \equiv \left\lfloor \; \varphi_{xx0}^{(k)} \quad \varphi_{xx1}^{(k)} \quad \varphi_{xx2}^{(k)} \quad \varphi_{yy0}^{(k)} \quad \varphi_{yy1}^{(k)} \quad \varphi_{yy2}^{(k)} \quad \varphi_{xy0}^{(k)} \quad \varphi_{xy1}^{(k)} \quad \varphi_{xy2}^{(k)} \quad 2\varepsilon_{xz}^{(k)} \quad 2\varepsilon_{yz}^{(k)} \quad \varepsilon_{zz}^{(k)} \; \right\rfloor^T, \tag{3.5.3}$$

and

$$\underset{(6\times 12)}{[\,Z\,]} = \begin{bmatrix} 1 & z & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z & z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z & z^2 & 0 & 0 & 0 \end{bmatrix}. \tag{3.5.4}$$

In the column-matrix $\left\{\varphi^{(k)}\right\}$, the functions $\varphi_{xx0}^{(k)}$, $\varphi_{xx1}^{(k)}$, $\varphi_{xx2}^{(k)}$, $\varphi_{yy0}^{(k)}$, $\varphi_{yy1}^{(k)}$, $\varphi_{yy2}^{(k)}$, $\varphi_{xy0}^{(k)}$, $\varphi_{xy1}^{(k)}$, $\varphi_{xy2}^{(k)}$ are expressed in terms of derivatives of the unknown functions by formulas (3.4.2)–(3.4.28). With the help of a matrix of differential operators, let us express the column-matrices $\left\{\varphi^{(k)}\right\}$ in terms of a column-matrix $\{f\}$, which contains only the unknown functions. Let us define the **column-matrix**

of the unknown functions as follows:

$$\{f\} \equiv \left\lfloor\ u_0 \quad v_0 \quad w_0 \quad \varepsilon_{xz}^{(1)} \quad \varepsilon_{yz}^{(1)} \quad \varepsilon_{zz}^{(1)} \quad \varepsilon_{xz}^{(2)} \quad \varepsilon_{yz}^{(2)} \quad \varepsilon_{zz}^{(2)} \quad \varepsilon_{xz}^{(3)} \quad \varepsilon_{yz}^{(3)} \quad \varepsilon_{zz}^{(3)}\ \right\rfloor^T . \tag{3.5.5}$$

Then

$$\underset{(12\times1)}{\left\{\varphi^{(1)}\right\}} \equiv \left\lfloor\ \varphi_{xx0}^{(1)} \quad \varphi_{xx1}^{(1)} \quad \varphi_{xx2}^{(1)} \quad \varphi_{yy0}^{(1)} \quad \varphi_{yy1}^{(1)} \quad \varphi_{yy2}^{(1)} \quad \varphi_{xy0}^{(1)} \quad \varphi_{xy1}^{(1)} \quad \varphi_{xy2}^{(1)} \quad 2\varepsilon_{xz}^{(1)} \quad 2\varepsilon_{yz}^{(1)} \quad \varepsilon_{zz}^{(1)}\ \right\rfloor^T \equiv$$

$$\equiv \left\{ \begin{array}{c} u_{0,x} + 2z_2\left(\varepsilon_{xz,x}^{(2)} - \varepsilon_{xz,x}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,xx}^{(2)} - \varepsilon_{zz,xx}^{(1)}\right) \\ 2\varepsilon_{xz,x}^{(1)} - w_{0,xx} + z_2\left(\varepsilon_{zz,xx}^{(1)} - \varepsilon_{zz,xx}^{(2)}\right) \\ -\frac{1}{2}\varepsilon_{zz,xx}^{(1)} \\ v_{0,y} + 2z_2\left(\varepsilon_{yz,y}^{(2)} - \varepsilon_{yz,y}^{(1)}\right) + \frac{1}{2}z_2^2\left(\varepsilon_{zz,yy}^{(2)} - \varepsilon_{zz,yy}^{(1)}\right) \\ 2\varepsilon_{yz,y}^{(1)} - w_{0,yy} + z_2\left(\varepsilon_{zz,yy}^{(1)} - \varepsilon_{zz,yy}^{(2)}\right) \\ -\frac{1}{2}\varepsilon_{zz,yy}^{(1)} \\ u_{0,y} + v_{0,x} + 2z_2\left(\varepsilon_{xz,y}^{(2)} - \varepsilon_{xz,y}^{(1)} + \varepsilon_{yz,x}^{(2)} - \varepsilon_{yz,x}^{(1)}\right) + z_2^2\left(\varepsilon_{zz,xy}^{(2)} - \varepsilon_{zz,xy}^{(1)}\right) \\ 2\left(\varepsilon_{xz,y}^{(1)} + \varepsilon_{yz,x}^{(1)}\right) - 2w_{0,xy} + 2z_2\left(\varepsilon_{zz,xy}^{(1)} - \varepsilon_{zz,xy}^{(2)}\right) \\ -\varepsilon_{zz,xy}^{(1)} \\ 2\varepsilon_{xz}^{(1)} \\ 2\varepsilon_{yz}^{(1)} \\ \varepsilon_{zz}^{(1)} \end{array} \right\} +$$

$$+ \left\{ \begin{array}{c} \frac{1}{2}\left[w_{0,x} + z_2\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(1)}\right)\right]^2 \\ \left[w_{0,x} + z_2\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(1)}\right)\right]\varepsilon_{zz,x}^{(1)} \\ \frac{1}{2}\left(\varepsilon_{zz,x}^{(1)}\right)^2 \\ \frac{1}{2}\left[w_{0,y} + z_2\left(\varepsilon_{zz,y}^{(2)} - \varepsilon_{zz,y}^{(1)}\right)\right]^2 \\ \left[w_{0,y} + z_2\left(\varepsilon_{zz,y}^{(2)} - \varepsilon_{zz,y}^{(1)}\right)\right]\varepsilon_{zz,y}^{(1)} \\ \frac{1}{2}\left(\varepsilon_{zz,y}^{(1)}\right)^2 \\ \left[w_{0,x} + z_2\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(1)}\right)\right]\left[w_{0,y} + z_2\left(\varepsilon_{zz,y}^{(2)} - \varepsilon_{zz,y}^{(1)}\right)\right] \\ \varepsilon_{zz,y}^{(1)}\left[w_{0,x} + z_2\left(\varepsilon_{zz,x}^{(2)} - \varepsilon_{zz,x}^{(1)}\right)\right] + \varepsilon_{zz,x}^{(1)}\left[w_{0,y} + z_2\left(\varepsilon_{zz,y}^{(2)} - \varepsilon_{zz,y}^{(1)}\right)\right] \\ \varepsilon_{zz,x}^{(1)}\varepsilon_{zz,y}^{(1)} \\ 0 \\ 0 \\ 0 \end{array} \right\} =$$

$$
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & -2z_2\frac{\partial}{\partial x} & 0 & -\frac{1}{2}z_2^2\frac{\partial^2}{\partial x^2} & 2z_2\frac{\partial}{\partial x} & 0 & \frac{1}{2}z_2^2\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\
-\frac{\partial^2}{\partial x^2} & 0 & 0 & 2\frac{\partial}{\partial x} & 0 & z_2\frac{\partial^2}{\partial x^2} & 0 & 0 & -z_2\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 & 0 & -2z_2\frac{\partial}{\partial y} & -\frac{1}{2}z_2^2\frac{\partial^2}{\partial y^2} & 0 & 2z_2\frac{\partial}{\partial y} & \frac{1}{2}z_2^2\frac{\partial^2}{\partial y^2} & 0 & 0 & 0 \\
0 & 0 & -\frac{\partial^2}{\partial y^2} & 0 & 2\frac{\partial}{\partial y} & z_2\frac{\partial^2}{\partial y^2} & 0 & 0 & -z_2\frac{\partial^2}{\partial y^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\varepsilon^{(1)}_{zz,yy} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & -2z_2\frac{\partial}{\partial y} & -2z_2\frac{\partial}{\partial x} & -z_2^2\frac{\partial^2}{\partial x\partial y} & 2z_2\frac{\partial}{\partial y} & 2z_2\frac{\partial}{\partial x} & z_2^2\frac{\partial^2}{\partial x\partial y} & 0 & 0 & 0 \\
0 & 0 & -2\frac{\partial^2}{\partial x\partial y} & 2\frac{\partial}{\partial y} & 2\frac{\partial}{\partial x} & 2z_2\frac{\partial^2}{\partial x\partial y} & 0 & 0 & -2z_2\frac{\partial^2}{\partial x\partial y} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\partial^2}{\partial x\partial y} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
u_0 \\ v_0 \\ w_0 \\ \varepsilon^{(1)}_{xz} \\ \varepsilon^{(1)}_{yz} \\ \varepsilon^{(1)}_{zz} \\ \varepsilon^{(2)}_{xz} \\ \varepsilon^{(2)}_{yz} \\ \varepsilon^{(2)}_{zz} \\ \varepsilon^{(3)}_{xz} \\ \varepsilon^{(3)}_{yz} \\ \varepsilon^{(3)}_{zz}
\end{Bmatrix} +
$$

$$
+\begin{Bmatrix}
\frac{1}{2}\left[w_{0,x} + z_2\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(1)}_{zz,x}\right)\right]^2 \\
\left[w_{0,x} + z_2\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(1)}_{zz,x}\right)\right]\varepsilon^{(1)}_{zz,x} \\
\frac{1}{2}\left(\varepsilon^{(1)}_{zz,x}\right)^2 \\
\frac{1}{2}\left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right]^2 \\
\left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right]\varepsilon^{(1)}_{zz,y} \\
\frac{1}{2}\left(\varepsilon^{(1)}_{zz,y}\right)^2 \\
\left[w_{0,x} + z_2\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(1)}_{zz,x}\right)\right]\left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right] \\
\varepsilon^{(1)}_{zz,y}\left[w_{0,x} + z_2\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(1)}_{zz,x}\right)\right] + \varepsilon^{(1)}_{zz,x}\left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right] \\
\varepsilon^{(1)}_{zz,x}\varepsilon^{(1)}_{zz,y} \\
0 \\
0 \\
0
\end{Bmatrix}.
$$

So,

$$
\underset{(6\times 1)}{\left\{\varepsilon^{(1)}\right\}} = \underset{(6\times 12)}{[\,Z\,]}\left(\underbrace{\underset{(12\times 12)}{\left[\,\partial^{(1)}\,\right]}\underset{(12\times 1)}{\{f\}} + \underset{(12\times 1)}{\left\{\eta^{(1)}\right\}}}_{\{\varphi^{(1)}\}}\right), \tag{3.5.6}
$$

where

$$\underset{(12\times 12)}{\left[\partial^{(1)}\right]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & -2z_2\frac{\partial}{\partial x} & 0 & -\frac{1}{2}z_2^2\frac{\partial^2}{\partial x^2} & 2z_2\frac{\partial}{\partial x} & 0 & \frac{1}{2}z_2^2\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ -\frac{\partial^2}{\partial x^2} & 0 & 0 & 2\frac{\partial}{\partial x} & 0 & z_2\frac{\partial^2}{\partial x^2} & 0 & 0 & -z_2\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & -2z_2\frac{\partial}{\partial y} & -\frac{1}{2}z_2^2\frac{\partial^2}{\partial y^2} & 0 & 2z_2\frac{\partial}{\partial y} & \frac{1}{2}z_2^2\frac{\partial^2}{\partial y^2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial y^2} & 0 & 2\frac{\partial}{\partial y} & z_2\frac{\partial^2}{\partial y^2} & 0 & 0 & -z_2\frac{\partial^2}{\partial y^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\varepsilon^{(1)}_{zz,yy} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & -2z_2\frac{\partial}{\partial y} & -2z_2\frac{\partial}{\partial x} & -z_2^2\frac{\partial^2}{\partial x\partial y} & 2z_2\frac{\partial}{\partial y} & 2z_2\frac{\partial}{\partial x} & z_2^2\frac{\partial^2}{\partial x\partial y} & 0 & 0 & 0 \\ 0 & 0 & -2\frac{\partial^2}{\partial x\partial y} & 2\frac{\partial}{\partial y} & 2\frac{\partial}{\partial x} & 2z_2\frac{\partial^2}{\partial x\partial y} & 0 & 0 & -2z_2\frac{\partial^2}{\partial x\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\partial^2}{\partial x\partial y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.5.7}$$

$$\underset{(12\times 1)}{\left\{\eta^{(1)}\right\}} = \left\{\begin{matrix} \frac{1}{2}\left[w_{0,x} + z_2\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(1)}_{zz,x}\right)\right]^2 \\ \left[w_{0,x} + z_2\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(1)}_{zz,x}\right)\right]\varepsilon^{(1)}_{zz,x} \\ \frac{1}{2}\left(\varepsilon^{(1)}_{zz,x}\right)^2 \\ \frac{1}{2}\left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right]^2 \\ \left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right]\varepsilon^{(1)}_{zz,y} \\ \frac{1}{2}\left(\varepsilon^{(1)}_{zz,y}\right)^2 \\ \left[w_{0,x} + z_2\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(1)}_{zz,x}\right)\right]\left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right] \\ \varepsilon^{(1)}_{zz,y}\left[w_{0,x} + z_2\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(1)}_{zz,x}\right)\right] + \varepsilon^{(1)}_{zz,x}\left[w_{0,y} + z_2\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(1)}_{zz,y}\right)\right] \\ \varepsilon^{(1)}_{zz,x}\varepsilon^{(1)}_{zz,y} \\ 0 \\ 0 \\ 0 \end{matrix}\right\}, \tag{3.5.8}$$

matrix $[Z]$ is defined by equation (3.5.4) and matrix $\{f\}$-by equation (3.5.5).

Analogously we obtain expressions for strains in the second and the third sublaminates in terms

of the unknown functions in matrix form:

$$\underset{(6\times 1)}{\left\{\varepsilon^{(2)}\right\}} = \underset{(6\times 12)}{[\,Z\,]}\left(\underbrace{\underset{(12\times 12)}{\left[\,\partial^{(2)}\,\right]}\underset{(12\times 1)}{\{f\}} + \underset{(12\times 1)}{\left\{\eta^{(2)}\right\}}}_{\left\{\varphi^{(2)}\right\}}\right), \tag{3.5.9}$$

where

$$\underset{(12\times 12)}{\left[\,\partial^{(2)}\,\right]} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 & 2\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial y^2} & 0 & 0 & 0 & 0 & 2\frac{\partial}{\partial y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial^2}{\partial y^2} & 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\partial^2}{\partial x\partial y} & 0 & 0 & 0 & 2\frac{\partial}{\partial y} & 2\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\partial^2}{\partial x\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{3.5.10}$$

$$\underset{(12\times 1)}{\left\{\eta^{(2)}\right\}} = \begin{Bmatrix} \frac{1}{2}(w_{0,x})^2 \\ w_{0,x}\varepsilon^{(2)}_{zz,x} \\ \frac{1}{2}\left(\varepsilon^{(2)}_{zz,x}\right)^2 \\ \frac{1}{2}\left(w_{0,y}\right)^2 \\ w_{0,y}\varepsilon^{(2)}_{zz,y} \\ \frac{1}{2}\left(\varepsilon^{(2)}_{zz,y}\right)^2 \\ w_{0,x}w_{0,y} \\ w_{0,y}\varepsilon^{(2)}_{zz,x} + w_{0,x}\varepsilon^{(2)}_{zz,y} \\ \varepsilon^{(2)}_{zz,x}\varepsilon^{(2)}_{zz,y} \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{3.5.11}$$

and

$$\underset{(6\times 1)}{\left\{\varepsilon^{(3)}\right\}} = \underset{(6\times 12)}{\left[\,Z\,\right]} \left(\underbrace{\underset{(12\times 12)}{\left[\,\partial^{(3)}\,\right]} \underset{(12\times 1)}{\{f\}} + \underset{(12\times 1)}{\left\{\eta^{(3)}\right\}}}_{\left\{\varphi^{(3)}\right\}} \right), \tag{3.5.12}$$

where

$$\underset{(12\times 12)}{\left[\,\partial^{(3)}\,\right]} = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 2z_3\frac{\partial}{\partial x} & 0 & \frac{1}{2}z_3^2\frac{\partial^2}{\partial x^2} & -2z_3\frac{\partial}{\partial x} & 0 & -\frac{1}{2}z_3^2\frac{\partial^2}{\partial x^2} \\
0 & 0 & -\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 & 0 & 0 & -z_3\frac{\partial^2}{\partial x^2} & 2\frac{\partial}{\partial x} & 0 & z_3\frac{\partial^2}{\partial x^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial^2}{\partial x^2} \\
0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 & 2z_3\frac{\partial}{\partial y} & \frac{1}{2}z_3^2\frac{\partial^2}{\partial y^2} & 0 & -2z_3\frac{\partial}{\partial y} & -\frac{1}{2}z_3^2\frac{\partial^2}{\partial y^2} \\
0 & 0 & -\frac{\partial^2}{\partial y^2} & 0 & 0 & 0 & 0 & 0 & -z_3\frac{\partial^2}{\partial y^2} & 0 & 2\frac{\partial}{\partial y} & z_3\frac{\partial^2}{\partial y^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{\partial^2}{\partial y^2} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 2z_3\frac{\partial}{\partial y} & 2z_3\frac{\partial}{\partial x} & z_3^2\frac{\partial^2}{\partial x\partial y} & -2z_3\frac{\partial}{\partial y} & -2z_3\frac{\partial}{\partial x} & -z_3^2\frac{\partial^2}{\partial x\partial y} \\
0 & 0 & -2\frac{\partial^2}{\partial x\partial y} & 0 & 0 & 0 & 0 & 0 & -2z_3\frac{\partial^2}{\partial x\partial y} & 2\frac{\partial}{\partial y} & 2\frac{\partial}{\partial x} & z_3\frac{\partial^2}{\partial x\partial y} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\partial^2}{\partial x\partial y} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \tag{3.5.13}$$

$$\underset{(12\times 1)}{\left\{\eta^{(3)}\right\}} = \left\{\begin{array}{c} \frac{1}{2}\left[w_{0,x} + z_3\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(3)}_{zz,x}\right)\right]^2 \\ \left[w_{0,x} + z_3\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(3)}_{zz,x}\right)\right]\varepsilon^{(3)}_{zz,x} \\ \frac{1}{2}\left(\varepsilon^{(3)}_{zz,x}\right)^2 \\ \frac{1}{2}\left[w_{0,y} + z_3\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(3)}_{zz,y}\right)\right]^2 \\ \left[w_{0,y} + z_3\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(3)}_{zz,y}\right)\right]\varepsilon^{(3)}_{zz,y} \\ \frac{1}{2}\left(\varepsilon^{(3)}_{zz,y}\right)^2 \\ \left[w_{0,x} + z_3\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(3)}_{zz,x}\right)\right]\left[w_{0,y} + z_3\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(3)}_{zz,y}\right)\right] \\ \varepsilon^{(3)}_{zz,x}\left[w_{0,y} + z_3\left(\varepsilon^{(2)}_{zz,y} - \varepsilon^{(3)}_{zz,y}\right)\right] + \varepsilon^{(3)}_{zz,y}\left[w_{0,x} + z_3\left(\varepsilon^{(2)}_{zz,x} - \varepsilon^{(3)}_{zz,x}\right)\right] \\ \varepsilon^{(3)}_{zz,x}\varepsilon^{(3)}_{zz,y} \\ 0 \\ 0 \\ 0 \end{array}\right\}. \tag{3.5.14}$$

In summary, the column-matrices of strains $\left\{\varepsilon^{(1)}\right\}$, $\left\{\varepsilon^{(2)}\right\}$, $\left\{\varepsilon^{(3)}\right\}$ in each of the three sublaminates (the face sheets and the core) are defined by the expressions

$$\underset{(6\times 1)}{\left\{\varepsilon^{(1)}\right\}} = \underset{(6\times 12)}{[\,Z\,]}\left(\underbrace{\underset{(12\times 12)}{\left[\,\partial^{(1)}\,\right]}\underset{(12\times 1)}{\{f\}} + \underset{(12\times 1)}{\left\{\eta^{(1)}\right\}}}_{\left\{\varphi^{(1)}\right\}}\right), \qquad \text{(eqn 3.5.6)}$$

$$\underset{(6\times 1)}{\left\{\varepsilon^{(2)}\right\}} = \underset{(6\times 12)}{[\,Z\,]}\left(\underbrace{\underset{(12\times 12)}{\left[\,\partial^{(2)}\,\right]}\underset{(12\times 1)}{\{f\}} + \underset{(12\times 1)}{\left\{\eta^{(2)}\right\}}}_{\left\{\varphi^{(2)}\right\}}\right), \qquad \text{(eqn 3.5.9)}$$

$$\underset{(6\times 1)}{\left\{\varepsilon^{(3)}\right\}} = \underset{(6\times 12)}{[\,Z\,]}\left(\underbrace{\underset{(12\times 12)}{\left[\,\partial^{(3)}\,\right]}\underset{(12\times 1)}{\{f\}} + \underset{(12\times 1)}{\left\{\eta^{(3)}\right\}}}_{\left\{\varphi^{(3)}\right\}}\right), \qquad \text{(eqn 3.5.12)}$$

where $\underset{(6\times 12)}{[\,Z\,]}$ is a matrix that depends only on the z-coordinate, $\underset{(12\times 12)}{\left[\,\partial^{(1)}\,\right]}$, $\underset{(12\times 12)}{\left[\,\partial^{(2)}\,\right]}$, $\underset{(12\times 12)}{\left[\,\partial^{(3)}\,\right]}$ are the matrices of differential operators, $\underset{(12\times 1)}{\{f\}}$ is a column-matrix of the unknown functions of the problem,

$\underset{(12\times1)}{\{\eta^{(1)}\}}$, $\underset{(12\times1)}{\{\eta^{(2)}\}}$, $\underset{(12\times1)}{\{\eta^{(3)}\}}$ are the column-matrices of non-linear combinations of the unknown functions of the problem.

3.6 Stress-Strain Relations

The fiber-reinforced lamina of a composite material are orthotropic. In a material coordinate system (x_1, x_2, x_3), whose x_1-axis is parallel to the fiber direction of a lamina, the stress-strain and strain-stress relations have the form

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}, \tag{3.6.1}$$

and

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}. \tag{3.6.2}$$

Quantities C_{ij} and S_{ij} are the stiffness coefficients and compliance coefficients in the material coordinate system. The strain-stress relations in the principal material coordinate system can be written in terms of engineering constants as follows

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}. \tag{3.6.3}$$

If we invert the compliance matrix in equation (3), we receive the following expressions for the **stiffness coefficients C_{ij}, in material coordinate system, in terms of engineering constants:**

$$C_{11} = \frac{\left(E_2 - \nu_{23}^2 E_3\right) E_1^2}{E_2 E_1 - E_1 \nu_{23}^2 E_3 - \nu_{12}^2 E_2^2 - 2\nu_{12} E_2 \nu_{23} \nu_{13} E_3 - \nu_{13}^2 E_2 E_3}, \tag{3.6.4}$$

$$C_{12} = \frac{(\nu_{12}E_2 + \nu_{23}\nu_{13}E_3)\,E_1E_2}{E_2E_1 - E_1\nu_{23}^2E_3 - \nu_{12}^2E_2^2 - 2\nu_{12}E_2\nu_{23}\nu_{13}E_3 - \nu_{13}^2E_2E_3}, \tag{3.6.5}$$

$$C_{13} = \frac{(\nu_{12}\nu_{23} + \nu_{13})\,E_1E_2E_3}{E_2E_1 - E_1\nu_{23}^2E_3 - \nu_{12}^2E_2^2 - 2\nu_{12}E_2\nu_{23}\nu_{13}E_3 - \nu_{13}^2E_2E_3}, \tag{3.6.6}$$

$$C_{22} = \frac{(E_1 - \nu_{13}^2E_3)\,E_2^2}{E_2E_1 - E_1\nu_{23}^2E_3 - \nu_{12}^2E_2^2 - 2\nu_{12}E_2\nu_{23}\nu_{13}E_3 - \nu_{13}^2E_2E_3}, \tag{3.6.7}$$

$$C_{23} = \frac{(\nu_{23}E_1 + \nu_{13}\nu_{12}E_2)\,E_2E_3}{E_2E_1 - E_1\nu_{23}^2E_3 - \nu_{12}^2E_2^2 - 2\nu_{12}E_2\nu_{23}\nu_{13}E_3 - \nu_{13}^2E_2E_3}, \tag{3.6.8}$$

$$C_{33} = \frac{(E_1 - \nu_{12}^2E_2)\,E_2E_3}{E_2E_1 - E_1\nu_{23}^2E_3 - \nu_{12}^2E_2^2 - 2\nu_{12}E_2\nu_{23}\nu_{13}E_3 - \nu_{13}^2E_2E_3}, \tag{3.6.9}$$

$$C_{44} = G_{23}, \tag{3.6.10}$$

$$C_{55} = G_{13}, \tag{3.6.11}$$

$$C_{66} = G_{12}. \tag{3.6.12}$$

In the laminate coordinate system (x, y, z), whose axes are aligned with the sides of the plate (Figure 3.2), the stress-strain relations have the form (Reddy, 1996):

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & 0 & 0 & \overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{23} & 0 & 0 & \overline{C}_{26} \\ \overline{C}_{13} & \overline{C}_{23} & \overline{C}_{33} & 0 & 0 & \overline{C}_{36} \\ 0 & 0 & 0 & \overline{C}_{44} & \overline{C}_{45} & 0 \\ 0 & 0 & 0 & \overline{C}_{45} & \overline{C}_{55} & 0 \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{36} & 0 & 0 & \overline{C}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \\ 2\varepsilon_{xy} \end{Bmatrix}, \tag{3.6.13}$$

or

$$\{\sigma\} = \left[\,\overline{C}\,\right]\{\varepsilon\}, \tag{3.6.14}$$

where $\overline{C}_{ij}$ are the transformed elastic coefficients, referred to the laminate coordinate system, which are related to the elastic coefficients C_{ij} in the material coordinate system by the following formulas:

$$\overline{C}_{11} = C_{11}c^4 + 2\,(C_{12} + 2C_{66})\,c^2s^2 + C_{22}s^4, \tag{3.6.15}$$

$$\overline{C}_{12} = C_{12}c^4 + (C_{11} + C_{22} - 4C_{66})\, c^2 s^2 + C_{12}s^4, \tag{3.6.16}$$

$$\overline{C}_{13} = C_{13}c^2 + C_{23}s^2, \tag{3.6.17}$$

$$\overline{C}_{16} = (C_{11} - C_{12} - 2C_{66})c^3 s - 3C_{16}c^2 s^2 + (2C_{66} + C_{12} - C_{22})\, cs^3, \tag{3.6.18}$$

$$\overline{C}_{22} = C_{22}c^4 + 2\,(C_{12} + 2C_{66})\, c^2 s^2 + C_{11}s^4, \tag{3.6.19}$$

$$\overline{C}_{23} = C_{23}c^2 + C_{13}s^2, \tag{3.6.20}$$

$$\overline{C}_{26} = (C_{12} - C_{22} + 2C_{66})\, c^3 s + (C_{11} - C_{12} - 2C_{66})cs^3, \tag{3.6.21}$$

$$\overline{C}_{33} = C_{33}, \tag{3.6.22}$$

$$\overline{C}_{36} = (C_{13} - C_{23})\, cs, \tag{3.6.23}$$

$$\overline{C}_{44} = C_{44}c^2 + C_{55}s^2, \tag{3.6.24}$$

$$\overline{C}_{45} = (C_{55} - C_{44})cs, \tag{3.6.25}$$

$$\overline{C}_{55} = C_{55}c^2 + C_{44}s^2, \tag{3.6.26}$$

$$\overline{C}_{66} = (C_{11} + C_{22} - 2C_{12} - 2C_{66})\, c^2 s^2 + C_{66}\left(c^4 + s^4\right)\ , \tag{3.6.27}$$

where $c = \cos\theta$, $s = \sin\theta$, θ is an angle between a direction of fiber orientation in a lamina and the x-axis of a laminate coordinate system, measured counterclockwise (Figure 3.2).

3.7 Strain Energy of the Sandwich Composite Plate

In order to perform the finite element formulation, it is necessary to write the strain energy of the sandwich plate in matrix form in terms of the unknown functions. The strain energy of the whole sandwich plate consists of the strain energies of the face sheets and the core. Therefore, in this section the expressions for the strain energies of the face sheets and the core are derived

Strain Energy of the Lower Face Sheet

The face sheets of the sandwich platform are composite laminated plates, which are built up of fiber-reinforced plies. The orientation of the fibers can vary from ply to ply, and, therefore, values of the stiffness coefficients $\overline{C}_{ij}$ in the Hooke's law (referred to the laminate coordinate system) can vary from ply to ply in the face sheets. Let us introduce the following notation for a stiffness coefficient in the Hooke's law for a ply of the lower face sheet, in the laminate coordinate system:

$$ {}^{\alpha}\overline{C}_{ij}^{(1)} , \tag{3.7.1} $$

where the right superscript (1) denotes that a stiffness coefficient is associated with the first sub-laminate (i.e. the lower face sheet), the left superscript α is a number of a ply in a lower face sheet, subscripts i and j denote a position of the stiffness coefficient in the stiffness matrix. The stiffness matrix with components ${}^{\alpha}\overline{C}_{ij}^{(1)}$ will be denoted as

$$ \left[\overline{C}_{\alpha}^{(1)}\right] \equiv \begin{bmatrix} {}^{\alpha}\overline{C}_{11}^{(1)} & {}^{\alpha}\overline{C}_{12}^{(1)} & {}^{\alpha}\overline{C}_{13}^{(1)} & 0 & 0 & {}^{\alpha}\overline{C}_{16}^{(1)} \\ {}^{\alpha}\overline{C}_{12}^{(1)} & {}^{\alpha}\overline{C}_{22}^{(1)} & {}^{\alpha}\overline{C}_{23}^{(1)} & 0 & 0 & {}^{\alpha}\overline{C}_{26}^{(1)} \\ {}^{\alpha}\overline{C}_{13}^{(1)} & {}^{\alpha}\overline{C}_{23}^{(1)} & {}^{\alpha}\overline{C}_{33}^{(1)} & 0 & 0 & {}^{\alpha}\overline{C}_{36}^{(1)} \\ 0 & 0 & 0 & {}^{\alpha}\overline{C}_{44}^{(1)} & {}^{\alpha}\overline{C}_{45}^{(1)} & 0 \\ 0 & 0 & 0 & {}^{\alpha}\overline{C}_{45}^{(1)} & {}^{\alpha}\overline{C}_{55}^{(1)} & 0 \\ {}^{\alpha}\overline{C}_{16}^{(1)} & {}^{\alpha}\overline{C}_{26}^{(1)} & {}^{\alpha}\overline{C}_{36}^{(1)} & 0 & 0 & {}^{\alpha}\overline{C}_{66}^{(1)} \end{bmatrix} . \tag{3.7.2} $$

So, the strain energy of a lower face sheet's ply with a number α is

$$ U_{\alpha}^{(1)} = \frac{1}{2} \iiint\limits_{\left(V_{\alpha}^{(1)}\right)} \left\{\varepsilon^{(1)}\right\}^{T} \left[\overline{C}_{\alpha}^{(1)}\right] \left\{\varepsilon^{(1)}\right\} \, dV , \tag{3.7.3} $$

where $V_{\alpha}^{(1)}$ is volume of a ply with number α, of the lower face sheet (Figure 9.1), and the column-matrix of strains $\left\{\varepsilon^{(1)}\right\}$ is defined as follows:

$$ \underset{(6\times 1)}{\left\{\varepsilon^{(1)}\right\}} \equiv \left\lfloor \begin{matrix} \varepsilon_{xx}^{(1)} & \varepsilon_{yy}^{(1)} & \varepsilon_{zz}^{(1)} & 2\varepsilon_{yz}^{(1)} & 2\varepsilon_{xz}^{(1)} & 2\varepsilon_{xy}^{(1)} \end{matrix} \right\rfloor^{T} . \tag{3.7.4} $$

Unlike the material coefficients ${}^{\alpha}\overline{C}_{ij}^{(1)}$, the strains do not have a subscript α, which denotes the number of a ply of the lower face sheet, because assumptions about through-the-thickness variation of strains[4] are made for the whole lower face sheet, not for each individual ply of the lower face sheet. Therefore, each strain in the lower face sheet, as a function of z-coordinate, is represented with a single expression for all the domain $z_1 \leq z \leq z_2$ (Figure 2.3)

If one substitutes equation (3.5.6) into equation (3.7.3), one obtains

$$U_{\alpha}^{(1)} = \frac{1}{2} \iiint\limits_{\left(V_{\alpha}^{(1)}\right)} \underset{(1\times 12)}{\left\{\varphi^{(1)}\right\}^{T}} \underset{(12\times 6)}{[\,Z\,]^{T}} \underset{(6\times 6)}{\left[\overline{C}_{\alpha}^{(1)}\right]} \underset{(6\times 12)}{[\,Z\,]} \underset{(12\times 1)}{\left\{\varphi^{(1)}\right\}} \, dV \,. \tag{3.7.5}$$

Let n be a number of plies in the lower face sheet and let

$$\xi_1 = z_1,\, \xi_2,\, \xi_3,\, ...,\, \xi_n = z_2$$

be z-coordinates of the interfaces between the plies of the lower face sheet (Figure 3.3). A ply with a number α is enclosed between the planes $z = \xi_{\alpha}$ and $z = \xi_{\alpha+1}$. Then expression (3.7.5) can be written as

$$U_{\alpha}^{(1)} = \frac{1}{2} \int_0^B \int_0^L \left\{\varphi^{(1)}\right\}^{T} \left(\int_{\xi_{\alpha}}^{\xi_{\alpha}} [\,Z\,]^{T} \left[\overline{C}_{\alpha}^{(1)}\right] [\,Z\,] \; dz \right) \left\{\varphi^{(1)}\right\} \; dx \; dy \,. \tag{3.7.6}$$

The strain energy of the whole lower face sheet is

$$U^{(1)} = \sum_{\alpha=1}^{n} U_{\alpha}^{(1)} = \frac{1}{2} \int_0^B \int_0^L \left\{\varphi^{(1)}\right\}^{T} \left(\sum_{\alpha=1}^{n} \int_{\xi_{\alpha}}^{\xi_{\alpha}} [\,Z\,]^{T} \left[\overline{C}_{\alpha}^{(1)}\right] [\,Z\,] \; dz \right) \left\{\varphi^{(1)}\right\} \; dx \; dy =$$

$$= \frac{1}{2} \int_0^B \int_0^L \left\{\varphi^{(1)}\right\}^{T} \left[D^{(1)}\right] \left\{\varphi^{(1)}\right\} \; dx \; dy, \tag{3.7.7}$$

where

$$\underset{(12\times 12)}{\left[D^{(1)}\right]} = \sum_{\alpha=1}^{n} \int_{\xi_{\alpha}}^{\xi_{\alpha+1}} [\,Z\,]^{T} \left[\overline{C}_{\alpha}^{(1)}\right] [\,Z\,] \; dz \,. \tag{3.7.8}$$

Matrix $\underset{(12\times 12)}{\left[D^{(1)}\right]}$ is symmetric and its components are

[4] section 3.2 of the chapter 3

$$D_{11}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{11}^{(1)} \left(\xi_{\alpha+1} - \xi_{\alpha}\right), \; D_{12}^{(1)} = \frac{1}{2} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{11}^{(1)} \left(\xi_{\alpha+1}^2 - \xi_{\alpha}^2\right),$$

$$D_{13}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{11}^{(1)} \left(\xi_{\alpha+1}^3 - \xi_{\alpha}^3\right), \; D_{14}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{12}^{(1)} \left(\xi_{\alpha+1} - \xi_{\alpha}\right),$$

$$D_{15}^{(1)} = \frac{1}{2} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{12}^{(1)} \left(\xi_{\alpha+1}^2 - \xi_{\alpha}^2\right), \; D_{16}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{12}^{(1)} \left(\xi_{\alpha+1}^3 - \xi_{\alpha}^3\right),$$

$$D_{17}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{16}^{(1)} \left(\xi_{\alpha+1} - \xi_{\alpha}\right), \; D_{18}^{(1)} = \frac{1}{2} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{16}^{(1)} \left(\xi_{\alpha+1}^2 - \xi_{\alpha}^2\right),$$

$$D_{19}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{16}^{(1)} \left(\xi_{\alpha+1}^3 - \xi_{\alpha}^3\right), \; D_{1,10}^{(1)} = 0,$$

$$D_{1,11}^{(1)} = 0, \; D_{1,12}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{13}^{(1)} \left(\xi_{\alpha+1} - \xi_{\alpha}\right),$$

$$D_{22}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{11}^{(1)} \left(\xi_{\alpha+1}^3 - \xi_{\alpha}^3\right), \; D_{23}^{(1)} = \frac{1}{4} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{11}^{(1)} \left(\xi_{\alpha+1}^4 - \xi_{\alpha}^4\right),$$

$$D_{24}^{(1)} = \frac{1}{2} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{12}^{(1)} \left(\xi_{\alpha+1}^2 - \xi_{\alpha}^2\right), \; D_{25}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{12}^{(1)} \left(\xi_{\alpha+1}^3 - \xi_{\alpha}^3\right),$$

$$D_{26}^{(1)} = \frac{1}{4} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{12}^{(1)} \left(\xi_{\alpha+1}^4 - \xi_{\alpha}^4\right), \; D_{27}^{(1)} = \frac{1}{2} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{16}^{(1)} \left(\xi_{\alpha+1}^2 - \xi_{\alpha}^2\right),$$

$$D_{28}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{16}^{(1)} \left(\xi_{\alpha+1}^3 - \xi_{\alpha}^3\right), \; D_{29}^{(1)} = \frac{1}{4} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{16}^{(1)} \left(\xi_{\alpha+1}^4 - \xi_{\alpha}^4\right),$$

$$D_{2,10}^{(1)} = 0, \; D_{2,11}^{(1)} = 0, \; D_{2,12}^{(1)} = \frac{1}{2} \sum_{\alpha=1}^{n} \left(\xi_{\alpha+1}^2 - \xi_{\alpha}^2\right) \; {}^{\alpha}\overline{C}_{13}^{(1)},$$

$$D_{33}^{(1)} = \frac{1}{5} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{11}^{(1)} \left(\xi_{\alpha+1}^5 - \xi_{\alpha}^5\right), \; D_{34}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{12}^{(1)} \left(\xi_{\alpha+1}^3 - \xi_{\alpha}^3\right),$$

$$D_{35}^{(1)} = \frac{1}{4} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{12}^{(1)} \left(\xi_{\alpha+1}^{4} - \xi_{\alpha}^{4}\right), \; D_{36}^{(1)} = \frac{1}{5} \sum_{\alpha=1}^{n} \left(\xi_{\alpha+1}^{5} - \xi_{\alpha}^{5}\right) \; {}^{\alpha}\overline{C}_{12}^{(1)},$$

$$D_{37}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} \left(\xi_{\alpha+1}^{3} - \xi_{\alpha}^{3}\right) \; {}^{\alpha}\overline{C}_{16}^{(1)}, \; D_{38}^{(1)} = \frac{1}{4} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{16}^{(1)} \left(\xi_{\alpha+1}^{4} - \xi_{\alpha}^{4}\right),$$

$$D_{39}^{(1)} = \frac{1}{5} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{16}^{(1)} \left(\xi_{\alpha+1}^{5} - \xi_{\alpha}^{5}\right), \; D_{3,10}^{(1)} = 0,$$

$$D_{3,11}^{(1)} = 0, \; D_{3,12}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{13}^{(1)} \left(\xi_{\alpha+1}^{3} - \xi_{\alpha}^{3}\right),$$

$$D_{44}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{22}^{(1)} \left(\xi_{\alpha+1} - \xi_{\alpha}\right), \; D_{45}^{(1)} = \frac{1}{2} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{22}^{(1)} \left(\xi_{\alpha+1}^{2} - \xi_{\alpha}^{2}\right),$$

$$D_{46}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{22}^{(1)} \left(\xi_{\alpha+1}^{3} - \xi_{\alpha}^{3}\right), \; D_{47}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{26}^{(1)} \left(\xi_{\alpha+1} - \xi_{\alpha}\right),$$

$$D_{48}^{(1)} = \frac{1}{2} \sum_{\alpha=1}^{n} \overline{C}_{26}^{(1)} \left(\xi_{\alpha+1}^{2} - \xi_{\alpha}^{2}\right), \; D_{49}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{26}^{(1)} \left(\xi_{\alpha+1}^{3} - \xi_{\alpha}^{3}\right),$$

$$D_{4,10}^{(1)} = 0, \; D_{4,11}^{(1)} = 0, \; D_{4,12}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{23}^{(1)} \left(\xi_{3} - \xi_{2}\right),$$

$$D_{55}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{22}^{(1)} \left(\xi_{\alpha+1}^{3} - \xi_{\alpha}^{3}\right), \; D_{56}^{(1)} = \frac{1}{4} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{22}^{(1)} \left(\xi_{\alpha+1}^{4} - \xi_{\alpha}^{4}\right),$$

$$D_{57}^{(1)} = \frac{1}{2} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{26}^{(1)} \left(\xi_{\alpha+1}^{2} - \xi_{\alpha}^{2}\right), \; D_{58}^{(1)} = \frac{1}{3} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{26}^{(1)} \left(\xi_{\alpha+1}^{3} - \xi_{\alpha}^{3}\right),$$

$$D_{59}^{(1)} = \frac{1}{4} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{26}^{(1)} \left(\xi_{\alpha+1}^{4} - \xi_{\alpha}^{4}\right), \; D_{5,10}^{(1)} = 0, \; D_{5,11}^{(1)} = 0,$$

$$D_{5,12}^{(1)} = \frac{1}{2} \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{23}^{(1)} \left(\xi_{\alpha+1}^{2} - \xi_{\alpha}^{2}\right),$$

$$D_{66}^{(1)} = \frac{1}{5}\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{22}^{(1)} \left(\xi_{\alpha+1}^{5} - \xi_{\alpha}^{5}\right), \quad D_{67}^{(1)} = \frac{1}{3}\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{26}^{(1)} \left(\xi_{\alpha+1}^{3} - \xi_{\alpha}^{3}\right),$$

$$D_{68}^{(1)} = \frac{1}{4}\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{26}^{(1)} \left(\xi_{\alpha+1}^{4} - \xi_{\alpha}^{4}\right), \quad D_{69}^{(1)} = \frac{1}{5}\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{26}^{(1)} \left(\xi_{\alpha+1}^{5} - \xi_{\alpha}^{5}\right),$$

$$D_{6,10}^{(1)} = 0, \quad D_{6,11}^{(1)} = 0, \quad D_{6,12}^{(1)} = \frac{1}{3}\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{23}^{(1)} \left(\xi_{\alpha+1}^{3} - \xi_{\alpha}^{3}\right),$$

$$D_{77}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{66}^{(1)} \left(\xi_{\alpha+1} - \xi_{\alpha}\right), \quad D_{78}^{(1)} = \frac{1}{2}\sum_{\alpha=1}^{n} {}^{\alpha}\, \overline{C}_{66}^{(1)} \left(\xi_{\alpha+1}^{2} - \xi_{\alpha}^{2}\right),$$

$$D_{79}^{(1)} = \frac{1}{3}\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{66}^{(1)} \left(\xi_{\alpha+1}^{3} - \xi_{\alpha}^{3}\right), \quad D_{7,10}^{(1)} = 0,$$

$$D_{7,11}^{(1)} = 0, \quad D_{7,12}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{36}^{(1)} \left(\xi_{\alpha+1} - \xi_{\alpha}\right),$$

$$D_{88}^{(1)} = \frac{1}{3}\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{66}^{(1)} \left(\xi_{\alpha+1}^{3} - \xi_{\alpha}^{3}\right), \quad D_{89}^{(1)} = \frac{1}{4}\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{66}^{(1)} \left(\xi_{\alpha+1}^{4} - \xi_{\alpha}^{4}\right),$$

$$D_{8,10}^{(1)} = 0, \; D_{8,11}^{(1)} = 0, \quad D_{8,12}^{(1)} = \frac{1}{2}\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{36}^{(1)} \left(\xi_{\alpha+1}^{2} - \xi_{\alpha}^{2}\right),$$

$$D_{99}^{(1)} = \frac{1}{5}\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{66}^{(1)} \left(\xi_{\alpha+1}^{5} - \xi_{\alpha}^{5}\right), \quad D_{9,10}^{(1)} = 0,$$

$$D_{9,11}^{(1)} = 0, \; D_{9,12}^{(1)} = \frac{1}{3}\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{36}^{(1)} \left(\xi_{\alpha+1}^{3} - \xi_{\alpha}^{3}\right),$$

$$D_{10,10}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{55}^{(1)} \left(\xi_{\alpha+1} - \xi_{\alpha}\right), \quad D_{10,11}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{45}^{(1)} \left(\xi_{\alpha+1} - \xi_{\alpha}\right),$$

$$D_{10,12}^{(1)} = 0, \; D_{11,11}^{(1)} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{44}^{(1)} \left(\xi_{\alpha+1} - \xi_{\alpha}\right), \quad D_{11,12}^{(1)} = 0,$$

$$D^{(1)}_{12,12} = \sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}^{(1)}_{33} \left(\xi_{\alpha+1} - \xi_{\alpha}\right). \tag{3.7.9}$$

The quantities $D^{(1)}_{ij}$ characterize the averaged (through the thickness) material properties of the lower face sheet. If failure occurs, the material constants ${}^{\alpha}\overline{C}^{(1)}_{ij}$, that characterize each individual ply of the lower face sheet, change their values. Therefore, if the failure occurs, the averaged material properties $D^{(1)}_{ij}$ change their values too. The method of reducing the values of the material constants ${}^{\alpha}\overline{C}^{(k)}_{33}$ in case of the failure, is described in the subsequent sections.

So, the strain energy of the lower face sheet is

$$U^{(1)} = \frac{1}{2}\int_0^B\int_0^L \left\{\varphi^{(1)}\right\}^T \left[D^{(1)}\right] \left\{\varphi^{(1)}\right\} \, dV =$$

$$\frac{1}{2}\int_0^B\int_0^L \left(\underset{(12\times12)}{\left[\,\partial^{(1)}\,\right]} \underset{(12\times1)}{\{f\}} + \underset{(12\times1)}{\left\{\eta^{(1)}\right\}} \right)^T \underset{(12\times12)}{\left[D^{(1)}\right]} \left(\underset{(12\times12)}{\left[\,\partial^{(1)}\,\right]} \underset{(12\times1)}{\{f\}} + \underset{(12\times1)}{\left\{\eta^{(1)}\right\}} \right) dx\, dy\,, \tag{3.7.10}$$

where the matrix $\left[\,\partial^{(1)}\,\right]$ of differential operators is defined by equations (3.5.7) , the column-matrix $\{f\}$ of the unknown functions is defined by equation (3.5.5), $\left\{\eta^{(1)}\right\}$ is the column-matrix of the non-linear combinations of the unknown functions, and matrix $\left[D^{(1)}\right]$ is the matrix of material constants, averaged over the thickness of the lower face sheet.

Strain Energy of the Upper Face Sheet

Let us introduce the following notation for a stiffness coefficient in the Hooke's law for a ply of the upper face sheet in the laminate coordinate system:

$$ {}^{\alpha}\overline{C}^{(3)}_{ij}\,, \tag{3.7.11}$$

where the right superscript (3) denotes that a stiffness coefficient is associated with the third sub-laminate (i.e. the upper face sheet), the left superscript α is a number of a ply in the upper face sheet, subscripts i and j denote a position of the stiffness coefficient in the stiffness matrix. The stiffness matrix with components ${}^{\alpha}\overline{C}^{(3)}_{ij}$ will be denoted as

$$\left[\overline{C}^{(3)}_{\alpha}\right] \equiv \begin{bmatrix} {}^{\alpha}\overline{C}^{(3)}_{11} & {}^{\alpha}\overline{C}^{(3)}_{12} & {}^{\alpha}\overline{C}^{(3)}_{13} & 0 & 0 & {}^{\alpha}\overline{C}^{(3)}_{16} \\ {}^{\alpha}\overline{C}^{(3)}_{12} & {}^{\alpha}\overline{C}^{(3)}_{22} & {}^{\alpha}\overline{C}^{(3)}_{23} & 0 & 0 & {}^{\alpha}\overline{C}^{(3)}_{26} \\ {}^{\alpha}\overline{C}^{(3)}_{13} & {}^{\alpha}\overline{C}^{(3)}_{23} & {}^{\alpha}\overline{C}^{(3)}_{33} & 0 & 0 & {}^{\alpha}\overline{C}^{(3)}_{36} \\ 0 & 0 & 0 & {}^{\alpha}\overline{C}^{(3)}_{44} & {}^{\alpha}\overline{C}^{(3)}_{45} & 0 \\ 0 & 0 & 0 & {}^{\alpha}\overline{C}^{(3)}_{45} & {}^{\alpha}\overline{C}^{(3)}_{55} & 0 \\ {}^{\alpha}\overline{C}^{(3)}_{16} & {}^{\alpha}\overline{C}^{(3)}_{26} & {}^{\alpha}\overline{C}^{(3)}_{36} & 0 & 0 & {}^{\alpha}\overline{C}^{(3)}_{66} \end{bmatrix}. \tag{3.7.12}$$

So, the strain energy of a ply with a number α, of the upper face sheet, is

$$U_\alpha^{(3)} = \frac{1}{2} \iiint\limits_{\left(V_\alpha^{(3)}\right)} \left\{\varepsilon^{(3)}\right\}^T \left[\overline{C}_\alpha^{(3)}\right] \left\{\varepsilon^{(3)}\right\} \, dV \, , \tag{3.7.13}$$

where $V_\alpha^{(3)}$ is volume of a ply with number α, of the upper face sheet. Let m be a number of plies in the upper face sheet and let

$$\zeta_1 = z_3, \, \zeta_2, \, \zeta_3, \, ..., \, \zeta_m = z_4$$

be z-coordinates of the interfaces between the plies of the upper face sheet.

Then, performing the same derivations as for the lower face sheets, one can obtain the following expression for the strain energy of the upper face sheet

$$U^{(3)} = \frac{1}{2} \int_0^B \int_0^L \underset{(1\times 12)}{\left\{\varphi^{(3)}\right\}^T} \underset{(12\times 12)}{\left[D^{(3)}\right]} \underset{(12\times 1)}{\left\{\varphi^{(3)}\right\}} \, dx \, dy, \tag{3.7.14}$$

where matrix $\left[D^{(3)}\right]$ is symmetric and its components are defined by the formulas similar to the formulas that define the components of the matrix $\left[D^{(1)}\right]$, for example:

$$D_{11}^{(3)} = \sum_{\alpha=1}^{m} {}^{\alpha}\overline{C}_{11}^{(3)} \left(\zeta_{\alpha+1} - \zeta_\alpha\right), \; D_{12}^{(3)} = \frac{1}{2} \sum_{\alpha=1}^{m} {}^{\alpha}\overline{C}_{11}^{(3)} \left(\zeta_{\alpha+1}^2 - \zeta_\alpha^2\right), \; D_{13}^{(3)} = \frac{1}{3} \sum_{\alpha=1}^{m} {}^{\alpha}\overline{C}_{11}^{(3)} \left(\zeta_{\alpha+1}^3 - \zeta_\alpha^3\right). \tag{3.7.15}$$

So, the strain energy of the upper face sheet is

$$U^{(3)} = \frac{1}{2} \int_0^B \int_0^L \left\{\varphi^{(3)}\right\}^T \left[D^{(3)}\right] \left\{\varphi^{(3)}\right\} \, dx \, dy =$$

$$= \frac{1}{2} \int_0^B \int_0^L \left(\underset{(12\times 12)}{\left[\, \partial^{(3)} \, \right]} \underset{(12\times 1)}{\{f\}} + \underset{(12\times 1)}{\left\{\eta^{(3)}\right\}} \right)^T \underset{(12\times 12)}{\left[D^{(3)}\right]} \left(\underset{(12\times 12)}{\left[\, \partial^{(3)} \, \right]} \underset{(12\times 1)}{\{f\}} + \underset{(12\times 1)}{\left\{\eta^{(3)}\right\}} \right) \, dx \, dy \, , \tag{3.7.16}$$

where matrices $\left[\, \partial^{(3)} \, \right]$, $\{f\}$ and $\left\{\eta^{(3)}\right\}$ are defined by formulas (3.5.13), (3.5.5) and (3.5.14) respectively and matrix $\left[D^{(3)}\right]$ is a matrix of material constants, averaged over the thickness of the upper face sheet.

Strain energy of the core of the sandwich plate

The core of the sandwich plate is considered to be a homogeneous orthotropic medium. But the failure in the core can be distributed nonuniformly in the thickness direction. As a result of this, in the presence of failure the coefficients $\overline{C}_{ij}$ of the stress-strain relation of the core can vary in the

thickness direction. To take account of this, the core is nominally divided into the layers, parallel to the x-y–plane, such that within each layer the coefficients of the stress-strain relation do not vary in the thickness direction. Therefore, the core is treated as a laminated plate, the same way as the face sheets, and the expression for the strain energy of the core has the same form as the expressions for the strain energy of the face sheets:

$$U^{(2)} = \frac{1}{2}\int_0^B\int_0^L \left(\underset{(12\times12)}{\left[\,\partial^{(2)}\,\right]} \underset{(12\times1)}{\{f\}} + \underset{(12\times1)}{\left\{\eta^{(2)}\right\}} \right)^T \underset{(12\times12)}{\left[D^{(2)}\right]} \left(\underset{(12\times12)}{\left[\,\partial^{(2)}\,\right]} \underset{(12\times1)}{\{f\}} + \underset{(12\times1)}{\left\{\eta^{(2)}\right\}} \right) dx\,dy\,, \qquad (3.7.17)$$

where matrices $\left[\,\partial^{(2)}\,\right]$, $\{f\}$ and $\left\{\eta^{(2)}\right\}$ are defined by formulas (3.5.10), (3.5.5) and (3.5.11) respectively. The matrix $\left[D^{(2)}\right]$ is a matrix of material constants, averaged over the thickness of the core. It is defined analogously to the matrices $\left[D^{(1)}\right]$ and $\left[D^{(3)}\right]$.

Strain Energy of the Sandwich Plate

The strain energy of the sandwich plate is the sum of the strain energies of the core and the face sheets:

$$\begin{aligned} U_p &= U^{(1)} + U^{(2)} + U^{(3)} = \\ &= \frac{1}{2}\int_0^B\int_0^L \left(\underset{(12\times12)}{\left[\,\partial^{(1)}\,\right]} \underset{(12\times1)}{\{f\}} + \underset{(12\times1)}{\left\{\eta^{(1)}\right\}} \right)^T \underset{(12\times12)}{\left[D^{(1)}\right]} \left(\underset{(12\times12)}{\left[\,\partial^{(1)}\,\right]} \underset{(12\times1)}{\{f\}} + \underset{(12\times1)}{\left\{\eta^{(1)}\right\}} \right) dx\,dy\, + \\ &+ \frac{1}{2}\int_0^B\int_0^L \left(\underset{(12\times12)}{\left[\,\partial^{(2)}\,\right]} \underset{(12\times1)}{\{f\}} + \underset{(12\times1)}{\left\{\eta^{(2)}\right\}} \right)^T \underset{(12\times12)}{\left[D^{(2)}\right]} \left(\underset{(12\times12)}{\left[\,\partial^{(2)}\,\right]} \underset{(12\times1)}{\{f\}} + \underset{(12\times1)}{\left\{\eta^{(2)}\right\}} \right) dx\,dy\, + \\ &+ \frac{1}{2}\int_0^B\int_0^L \left(\underset{(12\times12)}{\left[\,\partial^{(3)}\,\right]} \underset{(12\times1)}{\{f\}} + \underset{(12\times1)}{\left\{\eta^{(3)}\right\}} \right)^T \underset{(12\times12)}{\left[D^{(3)}\right]} \left(\underset{(12\times12)}{\left[\,\partial^{(3)}\,\right]} \underset{(12\times1)}{\{f\}} + \underset{(12\times1)}{\left\{\eta^{(3)}\right\}} \right) dx\,dy\,, \qquad (3.7.18) \end{aligned}$$

where $\left[D^{(1)}\right]$, $\left[D^{(2)}\right]$, $\left[D^{(3)}\right]$ are the matrices of material constants, averaged over the thickness of the sublaminates; $\{f\}$ is the column-matrix of the unknown functions of the problem; $\left\{\eta^{(1)}\right\}$, $\left\{\eta^{(2)}\right\}$, $\left\{\eta^{(3)}\right\}$ are the column-matrices of the non-linear combinations of the unknown functions of the problem and their spatial derivatives. All the functions, that enter into the expression (3.7.18) for the strain energy of the sandwich plate, depend on coordinates x, y and time t. Therefore, this expression is suitable for construction of the two-dimensional plate theory of the sandwich composite platform.

3.8 Strain Energy of Elastic Foundation

We shall model the ground, on which the platform is dropped, as a Winkler elastic foundation, i.e. we shall take the reaction forces of the elastic foundation to be linearly proportional to the displacement of the platform in z-direction at the area of contact of the platform with the ground. In such a model, the force per unit area, resisting the displacement of the platform, is equal to $-s\, w^{(1)}\big|_{z=z_1}$, where the function $s(x,y)$ is usually referred to as the modulus of the foundation. Then the strain energy of the elastic foundation is

$$U_f = \frac{1}{2}\int_0^B\int_0^L s(x,y)\left[w^{(1)}(x,y,z_1,t)\right]^2\, dx\, dy \tag{3.8.1}$$

According to equation (3.3.14),

$$w^{(1)}(x,y,z_1,t) = w_0 + \varepsilon_{zz}^{(2)} z_2 + \varepsilon_{zz}^{(1)}(z_1 - z_2)\ , \tag{3.8.2}$$

or

$$w^{(1)}(x,y,z_1,t) = \left\lfloor\, 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad (z_1 - z_2) \quad 0 \quad 0 \quad z_2 \quad 0 \quad 0 \quad 0 \,\right\rfloor \{f\}, \tag{3.8.3}$$

where

$$\{f\} = \left\lfloor\, u_0 \quad v_0 \quad w_0 \quad \varepsilon_{xz}^{(1)} \quad \varepsilon_{yz}^{(1)} \quad \varepsilon_{zz}^{(1)} \quad \varepsilon_{xz}^{(2)} \quad \varepsilon_{yz}^{(2)} \quad \varepsilon_{zz}^{(2)} \quad \varepsilon_{xz}^{(3)} \quad \varepsilon_{yz}^{(3)} \quad \varepsilon_{zz}^{(3)} \,\right\rfloor^T$$

is a column-matrix of the unknown functions of the problem. Then

$$\left[w^{(1)}(x,y,z_1,t)\right]^2 = \{f\}^T \left\{\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ z_1 - z_2 \\ 0 \\ 0 \\ z_2 \\ 0 \\ 0 \\ 0 \end{array}\right\} \left\{\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ z_1 - z_2 \\ 0 \\ 0 \\ z_2 \\ 0 \\ 0 \\ 0 \end{array}\right\}^T \{f\} =$$

$$= \underset{(1\times12)}{\{f\}}^{T} \underset{(12\times12)}{[\,\overline{D}\,]} \underset{(12\times1)}{\{f\}} \;, \tag{3.8.4}$$

where

$$[\,\overline{D}\,] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & z_1 - z_2 & 0 & 0 & z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_1 - z_2 & 0 & 0 & (z_1 - z_2)^2 & 0 & 0 & (z_1 - z_2)\, z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_2 & 0 & 0 & (z_1 - z_2)\, z_2 & 0 & 0 & z_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{3.8.5}$$

The substitution of equation (11.4) into equation (11.1) yields

$$U_f = \frac{1}{2}\int_0^B\!\!\int_0^L s\,(x,y)\; \underset{(1\times12)}{\{f\}}^{T} \underset{(12\times12)}{[\,\overline{D}\,]} \underset{(12\times1)}{\{f\}} \; dx\; dy \;. \tag{3.8.6}$$

This is the expression for the strain energy of the elastic foundation in terms of the unknown functions

$$\{f\} = \left\lfloor\, u_0 \quad v_0 \quad w_0 \quad \varepsilon_{xz}^{(1)} \quad \varepsilon_{yz}^{(1)} \quad \varepsilon_{zz}^{(1)} \quad \varepsilon_{xz}^{(2)} \quad \varepsilon_{yz}^{(2)} \quad \varepsilon_{zz}^{(2)} \quad \varepsilon_{xz}^{(3)} \quad \varepsilon_{yz}^{(3)} \quad \varepsilon_{zz}^{(3)} \,\right\rfloor^{T} .$$

3.9 Potential Energy of the Platform and the Cargo in the Gravity Field

Potential energy of the platform in the gravity field

We take a zero level of the potential energy of the platform in the position, in which the platform touches the ground, but the ground is not compressed yet, or, in other words, it is assumed that the potential energy of the platform in the gravity field is equal to zero at the initial moment of interaction of the platform with the ground. Let us find an expression for the **potential energy of the lower face sheet**. The projection on the z-axis of the gravity force per unit volume, acting on the lower face sheet, is

$$G_z^{(1)} = -\rho^{(1)} g, \tag{3.9.1}$$

where $\rho^{(1)}$ is mass density of the material of the lower face sheet, and $g = 9.8\frac{m}{s^2}$ is the absolute value of acceleration of free fall (absolute value of gravity force per unit mass). The projection $G_z^{(1)}$ of the gravity force on the z-axis is negative because the gravity force is directed downward, while the z-axis is chosen to be directed upward. Therefore, we had to put the " $-$ " sign in the expression (3.9.1). When the platform deforms as a result of its interaction with the ground, the gravity force performs mechanical work, which for the lower face sheet has the form

$$W^{(1)} = \iiint\limits_{(V^{(1)})} G_z^{(1)}\ w^{(1)}\ dV = \iiint\limits_{(V^{(1)})} - \rho^{(1)} g w^{(1)}\ dV = -\rho^{(1)} g \int_0^B \int_0^L \int_{z_1}^{z_2} w^{(1)}\ dz\ dx\ dy. \tag{3.9.2}$$

Therefore, **the potential energy of the lower face sheet**, due to the gravity force, is

$$\Pi^{(1)} = -W^{(1)} = \rho^{(1)} g \int_0^B \int_0^L \int_{z_1}^{z_2} w^{(1)}\ dz\ dx\ dy\ . \tag{3.9.3}$$

According to equation (3.3.14), it was found that

$$w^{(1)}(x,y,z,t) = w_0(x,y,t) + \varepsilon_{zz}^{(2)}(x,y,t)\ z_2 + \varepsilon_{zz}^{(1)}(x,y,t)(z - z_2) \quad (z_1 \le z \le z_2)\,.$$

The substitution of the last expression into the expression (3.9.3) yields

$$\Pi^{(1)} = \rho^{(1)} g \int_0^B \int_0^L \int_{z_1}^{z_2} \left[w_0(x,y,t) + \varepsilon_{zz}^{(2)}(x,y,t)\ z_2 + \varepsilon_{zz}^{(1)}(x,y,t)(z - z_2) \right]\ dz\ dx\ dy\ .$$

The integrating of the last expression with respect to z leads to

$$\Pi^{(1)} = \rho^{(1)} g \int_0^B \int_0^L \left[w_0 \left(z_2 - z_1\right) + \varepsilon_{zz}^{(2)} \; z_2 \left(z_2 - z_1\right) - \frac{1}{2} \varepsilon_{zz}^{(1)} \left(z_1 - z_2\right)^2 \right] \; dx \; dy \; . \tag{3.9.4}$$

The potential energy of the core of the sandwich plate in the gravity field is

$$\Pi^{(2)} = \rho^{(2)} g \int_0^B \int_0^L \int_{z_2}^{z_3} w^{(2)} \; dz \; dx \; dy \; . \tag{3.9.5}$$

According to equation (3.3.8),

$$w^{(2)} \left(x, y, z, t\right) = w_0 \left(x, y, t\right) + \varepsilon_{zz}^{(2)} \left(x, y, t\right) \; z \; .$$

Substituting the last expression into expression (3.9.5.5) and integrating with respect to z, one obtains

$$\Pi^{(2)} = \rho^{(2)} g \int_0^B \int_0^L \left[w_0 \left(z_3 - z_2\right) + \frac{1}{2} \varepsilon_{zz}^{(2)} \left(z_3^2 - z_2^2\right) \right] \; dx \; dy \; . \tag{3.9.6}$$

The potential energy of the upper face sheet of the sandwich plate in the gravity field is

$$\Pi^{(3)} = \rho^{(3)} g \int_0^B \int_0^L \int_{z_3}^{z_4} w^{(3)} \; dz \; dx \; dy \; . \tag{3.9.7}$$

Then one can incorporate equation (3.3.15),

$$w^{(3)} \left(x, y, z, t\right) = w_0 \left(x, y, t\right) + \varepsilon_{zz}^{(2)} \left(x, y, t\right) \; z_3 + \varepsilon_{zz}^{(3)} \left(x, y, t\right) \left(z - z_3\right) \qquad \left(z_3 \leq z \leq z_4\right) .$$

into expression (3.9.7) and integrate with respect to z, yielding

$$\Pi^{(3)} = \rho^{(3)} g \int_0^B \int_0^L \left[w_0 \left(z_4 - z_3\right) + \varepsilon_{zz}^{(2)} \; z_3 \left(z_4 - z_3\right) + \varepsilon_{zz}^{(3)} \frac{1}{2} \left(z_4 - z_3\right)^2 \right] \; dx \; dy \; . \tag{3.9.8}$$

The potential energy of the whole sandwich plate in the gravity field is

$$\Pi_{platform} = \Pi^{(1)} + \Pi^{(2)} + \Pi^{(3)} =$$

$$= \rho^{(1)} g \int_0^B \int_0^L \left[w_0 \left(z_2 - z_1\right) + \varepsilon_{zz}^{(2)} \; z_2 \left(z_2 - z_1\right) - \frac{1}{2} \varepsilon_{zz}^{(1)} \left(z_1 - z_2\right)^2 \right] \; dx \; dy +$$

$$+\rho^{(2)}g\int_0^B\int_0^L\left[w_0\left(z_3-z_2\right)+\frac{1}{2}\varepsilon_{zz}^{(2)}\left(z_3^2-z_2^2\right)\right]\,dx\,dy+$$

$$+\rho^{(3)}g\int_0^B\int_0^L\left[w_0\left(z_4-z_3\right)+\varepsilon_{zz}^{(2)}\,z_3\left(z_4-z_3\right)+\varepsilon_{zz}^{(3)}\frac{1}{2}\left(z_4-z_3\right)^2\right]\,dx\,dy \tag{3.9.9}$$

This expression can be written in matrix form as

$$\Pi_{platform}=\int_0^B\int_0^L\{f\}^T\{\Gamma_p\}\,dx\,dy, \tag{3.9.10}$$

where

$$\{f\}^T=\left\lfloor\begin{array}{cccccccccccc} u_0 & v_0 & w_0 & \varepsilon_{xz}^{(1)} & \varepsilon_{yz}^{(1)} & \varepsilon_{zz}^{(1)} & \varepsilon_{xz}^{(2)} & \varepsilon_{yz}^{(2)} & \varepsilon_{zz}^{(2)} & \varepsilon_{xz}^{(3)} & \varepsilon_{yz}^{(3)} & \varepsilon_{zz}^{(3)}\end{array}\right\rfloor$$

is a row-matrix of the unknown functions of the problem and

$$\{\Gamma_p\}=g\left\{\begin{array}{c} 0\\ 0\\ \left[\rho^{(1)}\left(z_2-z_1\right)+\rho^{(2)}\left(z_3-z_2\right)+\rho^{(3)}\left(z_4-z_3\right)\right]\\ 0\\ 0\\ -\frac{1}{2}\rho^{(1)}\left(z_1-z_2\right)^2\\ 0\\ 0\\ \left[\rho^{(1)}z_2\left(z_2-z_1\right)+\frac{1}{2}\rho^{(2)}\left(z_3^2-z_2^2\right)+\rho^{(3)}\,z_3\left(z_4-z_3\right)\right]\\ 0\\ 0\\ \frac{1}{2}\rho^{(3)}\left(z_4-z_3\right)^2 \end{array}\right\}. \tag{3.9.11}$$

Potential energy of the cargo in the gravity field

Next, let the cargo of mass M on the upper surface occupy the region S_0 of area A_0. We assume that contact between the cargo and upper surface of the sandwich platform exists all the time. During interaction of the platform with the ground, the displacement of the cargo is equal to the

displacement of the region S_0 of the upper surface, which is in contact with the cargo, i.e. the displacement of the cargo is equal to $w^{(3)}(x, y, z_4, t)$, where x and y belong to the region S_0. When the platform deforms as a result of its interaction with the ground and the cargo, the gravity force, acting on the cargo, produces mechanical work

$$W_{\mathrm{cargo}} = -\iint\limits_{(S_0)} \mu\ g\ w^{(3)}(x, y, z_4, t)\ dx\ dy$$

$$= -\int_0^B\int_0^L \mu\ g\ H(x, y)\ w^{(3)}(x, y, z_4, t)\ dx\ dy\ , \tag{3.9.12}$$

where

$$H(x, y) \equiv \begin{cases} 1 \text{ in region } S_0 \\ 0 \text{ otherwise} \end{cases} \tag{3.9.13}$$

and μ is the mass of the cargo per unit area of contact with the platform, i.e. a quantity such that

$$M = \iint\limits_{(S_0)} \mu\ dx\ dy\ .$$

If the mass of the cargo is uniformly distributed over the surface of the contact, then

$$\mu = \frac{M}{A_0}\ .$$

Then the potential energy of the cargo in the gravity field is

$$\Pi_{\mathrm{cargo}} = -W_{\mathrm{cargo}} = \int_0^B\int_0^L \mu\ g\ H(x, y)\ w^{(3)}(x, y, z_4, t)\ dx\ dy\ . \tag{3.9.14}$$

According to equation (3.3.15),

$$w^{(3)}(x, y, z_4, t) = w_0(x, y, t) + \varepsilon_{zz}^{(2)}(x, y, t)\ z_3 + \varepsilon_{zz}^{(3)}(x, y, t)(z_4 - z_3)\,. \tag{3.9.15}$$

Substituting (3.9.15) into (3.9.14), one receives

$$\Pi_{\mathrm{cargo}} = \int_0^B\int_0^L \mu\ g\ H(x, y)\ \left[w_0 + \varepsilon_{zz}^{(2)}\ z_3 + \varepsilon_{zz}^{(3)}(z_4 - z_3)\right]\ dx\ dy\ . \tag{3.9.16}$$

Expression (3.9.16) can be written in matrix form as follows:

$$\Pi_{\mathrm{cargo}} = \int_0^B\int_0^L\ \{f\}^T\{\Gamma_c\}\ , \tag{3.9.17}$$

where $\{f\}^T = \lfloor \; u_0 \;\; v_0 \;\; w_0 \;\; \varepsilon_{xz}^{(1)} \;\; \varepsilon_{yz}^{(1)} \;\; \varepsilon_{zz}^{(1)} \;\; \varepsilon_{xz}^{(2)} \;\; \varepsilon_{yz}^{(2)} \;\; \varepsilon_{zz}^{(2)} \;\; \varepsilon_{xz}^{(3)} \;\; \varepsilon_{yz}^{(3)} \;\; \varepsilon_{zz}^{(3)} \; \rfloor$ is a row-matrix of the unknown functions of the problem and

$$\{\Gamma_c\} = \mu \; g \; H(x,y) \left\lfloor \; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; z_3 \;\; 0 \;\; 0 \;\; z_4 - z_3 \; \right\rfloor^T . \tag{3.9.18}$$

Thus, the total potential energy of the platform and the cargo in the gravity field is

$$\Pi_{platform} + \Pi_{c\,arg\,o} = \int_0^B \int_0^L \{f\}^T \left(\{\Gamma_p\} + \{\Gamma_c\}\right) \; dx \; dy =$$

$$= \int_0^B \int_0^L \{f\}^T \{\Gamma\} \; dx \; dy \; , \tag{3.9.19}$$

where

$$\{\Gamma\} = \{\Gamma_p\} + \{\Gamma_c\} =$$

$$\left\{ \begin{array}{c} 0 \\ 0 \\ g\left[\rho^{(1)}(z_2 - z_1) + \rho^{(2)}(z_3 - z_2) + \rho^{(3)}(z_4 - z_3) + \mu \; H(x,y)\right] \\ 0 \\ 0 \\ -\frac{1}{2}\rho^{(1)} g (z_1 - z_2)^2 \\ 0 \\ 0 \\ g\left[\rho^{(1)} z_2 (z_2 - z_1) + \frac{1}{2}\rho^{(2)}\left(z_3^2 - z_2^2\right) + \rho^{(3)} \; z_3 (z_4 - z_3) + \mu \; H(x,y) \, z_3\right] \\ 0 \\ 0 \\ g\left[\rho^{(3)} \frac{1}{2} (z_4 - z_3)^2 + \mu \; H(x,y)(z_4 - z_3)\right] \end{array} \right\} . \tag{3.9.20}$$

3.10 Kinetic Energy of the Platform and the Cargo

In order to perform the finite element formulation on the basis of the Hamilton's principle, it is necessary to have an expression for the kinetic energy in terms of the unknown functions. The kinetic energy of the system under consideration consists of kinetic energies of the platform and the cargo.

Kinetic energy of the platform

Considering the fact that the mass density of the face sheets is constant, **kinetic energy of the lower face sheet** can be written as follows:

$$K^{(1)} = \frac{1}{2}\rho^{(1)} \iiint\limits_{(V)} \left[\left(\dot{u}^{(1)}\right)^2 + \left(\dot{v}^{(1)}\right)^2 + \left(\dot{w}^{(1)}\right)^2 \right] dV =$$

$$= \frac{1}{2}\rho^{(1)} \iiint\limits_{(V)} \begin{Bmatrix} \dot{u}^{(1)} \\ \dot{v}^{(1)} \\ \dot{w}^{(1)} \end{Bmatrix}^T \begin{Bmatrix} \dot{u}^{(1)} \\ \dot{v}^{(1)} \\ \dot{w}^{(1)} \end{Bmatrix} dV \,, \tag{3.10.1}$$

where dots over letters denote partial derivatives with respect to time.

According to equation (3.3.62), the column-matrix $\left\lfloor \dot{u}^{(1)} \quad \dot{v}^{(1)} \quad \dot{w}^{(1)} \right\rfloor^T$ can be written in the form

$$\begin{Bmatrix} \dot{u}^{(1)} \\ \dot{v}^{(1)} \\ \dot{w}^{(1)} \end{Bmatrix} = \frac{\partial}{\partial t} \begin{Bmatrix} u^{(1)} \\ v^{(1)} \\ w^{(1)} \end{Bmatrix} = \underset{(3\times 8)}{\left[\tilde{Z} \right]} \underset{(8\times 12)}{\left[\tilde{\partial}^{(1)} \right]} \frac{\partial}{\partial t} \underset{(12\times 1)}{\{f\}} \,. \tag{3.10.2}$$

Therefore,

$$K^{(1)} = \frac{1}{2}\rho^{(1)} \iiint\limits_{(V)} \left(\underset{(8\times 12)}{\left[\tilde{\partial}^{(1)} \right]} \frac{\partial}{\partial t} \underset{(12\times 1)}{\{f\}} \right)^T \underset{(8\times 3)}{\left[\tilde{Z} \right]^T} \underset{(3\times 8)}{\left[\tilde{Z} \right]} \left(\underset{(8\times 12)}{\left[\tilde{\partial}^{(1)} \right]} \frac{\partial}{\partial t} \underset{(12\times 1)}{\{f\}} \right) dV =$$

$$= \frac{1}{2}\rho^{(1)} \iiint\limits_{(V)} \left(\underset{(8\times 12)}{\left[\tilde{\partial}^{(1)} \right]} \frac{\partial}{\partial t} \underset{(12\times 1)}{\{f\}} \right)^T \underset{(8\times 8)}{\left[\tilde{\tilde{Z}} \right]} \left(\underset{(8\times 12)}{\left[\tilde{\partial}^{(1)} \right]} \frac{\partial}{\partial t} \underset{(12\times 1)}{\{f\}} \right) dV \,, \tag{3.10.3}$$

where

$$\underset{(8\times 8)}{\left[\tilde{\tilde{Z}} \right]} = \underset{(8\times 3)}{\left[\tilde{Z} \right]^T} \underset{(3\times 8)}{\left[\tilde{Z} \right]} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ z & 0 & 0 \\ z^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & z & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & z & z^2 & 0 & 0 & 0 & 0 & 0 \\ z & z^2 & z^3 & 0 & 0 & 0 & 0 & 0 \\ z^2 & z^3 & z^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z & z^2 & 0 & 0 \\ 0 & 0 & 0 & z & z^2 & z^3 & 0 & 0 \\ 0 & 0 & 0 & z^2 & z^3 & z^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & z & z^2 \end{bmatrix} . \tag{3.10.4}$$

Now the kinetic energy can be written in the form

$$K^{(1)} = \frac{1}{2}\rho^{(1)} \int_0^B \int_0^L \left(\underset{(8\times12)}{\left[\widetilde{\partial}^{(1)} \right]} \frac{\partial}{\partial t} \underset{(12\times1)}{\{f\}} \right)^T \left(\int_{z_1}^{z_2} \underset{(8\times8)}{\left[\widetilde{\widetilde{Z}} \right]} dz \right) \left(\underset{(8\times12)}{\left[\widetilde{\partial}^{(1)} \right]} \frac{\partial}{\partial t} \underset{(12\times1)}{\{f\}} \right) dx\, dy =$$

$$= \frac{1}{2}\rho^{(1)} \int_0^B \int_0^L \left(\underset{(8\times12)}{\left[\widetilde{\partial}^{(1)} \right]} \frac{\partial}{\partial t} \underset{(12\times1)}{\{f\}} \right)^T \left[\widetilde{D}^{(1)} \right] \left(\underset{(8\times12)}{\left[\widetilde{\partial}^{(1)} \right]} \frac{\partial}{\partial t} \underset{(12\times1)}{\{f\}} \right) dx\, dy \,, \tag{3.10.5}$$

where

$$\left[\widetilde{D}^{(1)} \right] = \int_{z_1}^{z_2} \underset{(8\times8)}{\left[\widetilde{\widetilde{Z}} \right]} dz =$$

$$= \int_{z_1}^{z_2} \begin{bmatrix} 1 & z & z^2 & 0 & 0 & 0 & 0 & 0 \\ z & z^2 & z^3 & 0 & 0 & 0 & 0 & 0 \\ z^2 & z^3 & z^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z & z^2 & 0 & 0 \\ 0 & 0 & 0 & z & z^2 & z^3 & 0 & 0 \\ 0 & 0 & 0 & z^2 & z^3 & z^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z \\ 0 & 0 & 0 & 0 & 0 & 0 & z & z^2 \end{bmatrix} dz =$$

$$\begin{bmatrix} z_2 - z_1 & \frac{1}{2}\left(z_2^2 - z_1^2\right) & \frac{1}{3}\left(z_2^3 - z_1^3\right) & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\left(z_2^2 - z_1^2\right) & \frac{1}{3}\left(z_2^3 - z_1^3\right) & \frac{1}{4}\left(z_2^4 - z_1^4\right) & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3}\left(z_2^3 - z_1^3\right) & \frac{1}{4}\left(z_2^4 - z_1^4\right) & \frac{1}{5}\left(z_2^5 - z_1^5\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_2 - z_1 & \frac{1}{2}\left(z_2^2 - z_1^2\right) & \frac{1}{3}\left(z_2^3 - z_1^3\right) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(z_2^2 - z_1^2\right) & \frac{1}{3}\left(z_2^3 - z_1^3\right) & \frac{1}{4}\left(z_2^4 - z_1^4\right) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3}\left(z_2^3 - z_1^3\right) & \frac{1}{4}\left(z_2^4 - z_1^4\right) & \frac{1}{5}\left(z_2^5 - z_1^5\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_2 - z_1 & \frac{1}{2}\left(z_2^2 - z_1^2\right) \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(z_2^2 - z_1^2\right) & \frac{1}{3}\left(z_2^3 - z_1^3\right) \end{bmatrix} . \tag{3.10.6}$$

Analogously, we can write the **kinetic energy of the core and the upper face sheet**:

$$K^{(2)} = \frac{1}{2}\rho^{(2)} \int_0^B \int_0^L \left(\underset{(8\times12)}{\left[\widetilde{\partial}^{(2)} \right]} \frac{\partial}{\partial t} \underset{(12\times1)}{\{f\}} \right)^T \left[\widetilde{D}^{(2)} \right] \left(\underset{(8\times12)}{\left[\widetilde{\partial}^{(2)} \right]} \frac{\partial}{\partial t} \underset{(12\times1)}{\{f\}} \right) dx\, dy \ , \tag{3.10.7}$$

$$K^{(3)} = \frac{1}{2}\rho^{(3)} \int_0^B \int_0^L \left(\underset{(8\times12)}{\left[\widetilde{\partial}^{(3)} \right]} \frac{\partial}{\partial t} \underset{(12\times1)}{\{f\}} \right)^T \left[\widetilde{D}^{(3)} \right] \left(\underset{(8\times12)}{\left[\widetilde{\partial}^{(3)} \right]} \frac{\partial}{\partial t} \underset{(12\times1)}{\{f\}} \right) dx\, dy \ , \tag{3.10.8}$$

where

$$\left[\widetilde{D}^{(2)} \right] = \int_{z_2}^{z_3} \underset{(8\times8)}{\left[\widetilde{\widetilde{Z}} \right]} dz =$$

$$
\begin{bmatrix}
z_3-z_2 & \frac{1}{2}\left(z_3^2-z_2^2\right) & \frac{1}{3}\left(z_3^3-z_2^3\right) & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}\left(z_3^2-z_2^2\right) & \frac{1}{3}\left(z_3^3-z_2^3\right) & \frac{1}{4}\left(z_3^4-z_2^4\right) & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3}\left(z_3^3-z_2^3\right) & \frac{1}{4}\left(z_3^4-z_2^4\right) & \frac{1}{5}\left(z_3^5-z_2^5\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z_3-z_2 & \frac{1}{2}\left(z_3^2-z_2^2\right) & \frac{1}{3}\left(z_3^3-z_2^3\right) & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}\left(z_3^2-z_2^2\right) & \frac{1}{3}\left(z_3^3-z_2^3\right) & \frac{1}{4}\left(z_3^4-z_2^4\right) & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3}\left(z_3^3-z_2^3\right) & \frac{1}{4}\left(z_3^4-z_2^4\right) & \frac{1}{5}\left(z_3^5-z_2^5\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z_3-z_2 & \frac{1}{2}\left(z_3^2-z_2^2\right) \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(z_3^2-z_2^2\right) & \frac{1}{3}\left(z_3^3-z_2^3\right)
\end{bmatrix},
\tag{3.10.9}
$$

$$
\left[\, \widetilde{D}^{(3)} \,\right] = \int\limits_{z_3}^{z_4} \underset{(8\times 8)}{\left[\, \widetilde{\widetilde{Z}} \,\right]} dz =
$$

$$
\begin{bmatrix}
z_4-z_3 & \frac{1}{2}\left(z_4^2-z_3^2\right) & \frac{1}{3}\left(z_4^3-z_3^3\right) & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2}\left(z_4^2-z_3^2\right) & \frac{1}{3}\left(z_4^3-z_3^3\right) & \frac{1}{4}\left(z_4^4-z_3^4\right) & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3}\left(z_4^3-z_3^3\right) & \frac{1}{4}\left(z_4^4-z_3^4\right) & \frac{1}{5}\left(z_4^5-z_3^5\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z_4-z_3 & \frac{1}{2}\left(z_4^2-z_3^2\right) & \frac{1}{3}\left(z_4^3-z_3^3\right) & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}\left(z_4^2-z_3^2\right) & \frac{1}{3}\left(z_4^3-z_3^3\right) & \frac{1}{4}\left(z_4^4-z_3^4\right) & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3}\left(z_4^3-z_3^3\right) & \frac{1}{4}\left(z_4^4-z_3^4\right) & \frac{1}{5}\left(z_4^5-z_3^5\right) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z_4-z_3 & \frac{1}{2}\left(z_4^2-z_3^2\right) \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(z_4^2-z_3^2\right) & \frac{1}{3}\left(z_4^3-z_3^3\right)
\end{bmatrix},
\tag{3.10.10}
$$

and $\left[\, \widetilde{\partial}^{(2)} \,\right]$ and $\left[\, \widetilde{\partial}^{(3)} \,\right]$ are the matrices of differential operators, defined by formulas (3.3.59) and (3.3.61).

So, the kinetic energy of the sandwich plate is

$$
K_p = K^{(1)} + K^{(2)} + K^{(3)} =
$$

$$
= \frac{1}{2}\rho^{(1)} \int\limits_0^B \int\limits_0^L \left(\underset{(8\times 12)}{\left[\, \widetilde{\partial}^{(1)} \,\right]} \frac{\partial}{\partial t} \underset{(12\times 1)}{\{f\}} \right)^T \underset{(8\times 8)}{\left[\, \widetilde{D}^{(1)} \,\right]} \left(\underset{(8\times 12)}{\left[\, \widetilde{\partial}^{(1)} \,\right]} \frac{\partial}{\partial t} \underset{(12\times 1)}{\{f\}} \right) dx\, dy +
$$

$$
+ \frac{1}{2}\rho^{(2)} \int\limits_0^B \int\limits_0^L \left(\underset{(8\times 12)}{\left[\, \widetilde{\partial}^{(2)} \,\right]} \frac{\partial}{\partial t} \underset{(12\times 1)}{\{f\}} \right)^T \underset{(8\times 8)}{\left[\, \widetilde{D}^{(2)} \,\right]} \left(\underset{(8\times 12)}{\left[\, \widetilde{\partial}^{(2)} \,\right]} \frac{\partial}{\partial t} \underset{(12\times 1)}{\{f\}} \right) dx\, dy +
$$

$$+\frac{1}{2}\rho^{(3)}\int\limits_0^B\int\limits_0^L\left(\underset{(8\times 12)}{\left[\,\widetilde{\partial}^{(3)}\right]}\frac{\partial}{\partial t}\underset{(12\times 1)}{\{f\}}\right)^T\underset{(8\times 8)}{\left[\,\widetilde{D}^{(3)}\,\right]}\left(\underset{(8\times 12)}{\left[\,\widetilde{\partial}^{(3)}\right]}\frac{\partial}{\partial t}\underset{(12\times 1)}{\{f\}}\right)\,dx\,dy\,. \tag{3.10.11}$$

where matrices of differential operators $\left[\,\widetilde{\partial}^{(1)}\,\right]$, $\left[\,\widetilde{\partial}^{(2)}\,\right]$, $\left[\,\widetilde{\partial}^{(3)}\right]$ are defined by equations (3.3.56), (3.3.59), and (3.3.61); $\{f\}$ is a column-matrix of the unknown functions, defined by equation (3.3.57); and $\left[\,\widetilde{D}^{(1)}\right]$, $\left[\,\widetilde{D}^{(2)}\right]$ and $\left[\,\widetilde{D}^{(3)}\right]$ are matrices of constants, defined by equations (3.10.6) (3.10.9) and (3.10.10).

Kinetic energy of the cargo

The cargo of mass M on the upper surface is said to occupy the region S_0 of area A_0. We assume that a contact between the cargo and upper surface of the sandwich platform exists all the time. During interaction of the platform with the ground, the velocity of the cargo is equal to the velocity of the region S_0 of the upper surface, which is in contact with the cargo, i.e. the velocity of the cargo is equal to $\frac{\partial}{\partial t}w^{(3)}\left(x,y,z_4,t\right)$, where x and y belong to the region S_0. Therefore, the kinetic energy of the cargo is equal to

$$K_c=\frac{1}{2}\iint\limits_{(S_0)}\mu\left(\frac{\partial w^{(3)}\left(x,y,z_4,t\right)}{\partial t}\right)^2\,dx\,dy\;, \tag{3.10.12}$$

where μ is the mass of the cargo per unit area of contact with the platform, i.e. a quantity such that

$$M=\iint\limits_{(S_0)}\mu\,dx\,dy\;. \tag{3.10.13}$$

If the mass of the cargo is uniformly distributed over the surface of the contact, then

$$\mu=\frac{M}{A_0}\;.$$

Equation (3.10.12) can also be written in the form

$$K_c=\frac{1}{2}\int\limits_0^B\int\limits_0^L\mu\;H\left(x,y\right)\left(\frac{\partial w^{(3)}\left(x,y,z_4,t\right)}{\partial t}\right)^2\,dx\,dy\;, \tag{3.10.14}$$

where

$$H(x,y)=\begin{cases}1\text{ in region }S_0\\0\text{ otherwise}\end{cases} \tag{3.10.15}$$

According to equation (3.3.15),

$$w^{(3)}(x,y,z_4,t) = w_0 + \varepsilon_{zz}^{(2)} z_3 + \varepsilon_{zz}^{(3)}(z_4 - z_3) \tag{3.10.16}$$

or

$$w^{(3)}(x,y,z_4,t) = \{f\}^T \{\omega\} \ , \tag{3.10.17}$$

where

$$\{f\}^T = \lfloor\, u_0 \quad v_0 \quad w_0 \quad \varepsilon_{xz}^{(1)} \quad \varepsilon_{yz}^{(1)} \quad \varepsilon_{zz}^{(1)} \quad \varepsilon_{xz}^{(2)} \quad \varepsilon_{yz}^{(2)} \quad \varepsilon_{zz}^{(2)} \quad \varepsilon_{xz}^{(3)} \quad \varepsilon_{yz}^{(3)} \quad \varepsilon_{zz}^{(3)} \,\rfloor$$

is the row-matrix of the unknown functions of the problem and

$$\{\omega\} = \lfloor\, 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad z_3 \quad 0 \quad 0 \quad z_4 - z_3 \,\rfloor^T . \tag{3.10.18}$$

Then

$$\frac{\partial}{\partial t} w^{(3)}(x,y,z_4,t) = \left(\frac{\partial}{\partial t}\{f\}\right)^T \{\omega\}$$

and

$$\left(\frac{\partial w^{(3)}(x,y,z_4,t)}{\partial t}\right)^2 = \underset{(1\times 12)}{\left(\frac{\partial}{\partial t}\{f\}\right)^T} \underset{(12\times 1)(1\times 12)}{\{\omega\}\ \{\omega\}^T} \underset{(12\times 1)}{\left(\frac{\partial}{\partial t}\{f\}\right)} =$$

$$= \underset{(1\times 12)}{\left(\frac{\partial}{\partial t}\{f\}\right)^T} \underset{(12\times 12)}{\left[\widetilde{D}_c\right]} \underset{(12\times 1)}{\left(\frac{\partial}{\partial t}\{f\}\right)} , \tag{3.10.19}$$

where

$$\underset{(12\times 12)}{\left[\widetilde{D}_c\right]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & z_3 & 0 & 0 & z_4 - z_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_3 & 0 & 0 & 0 & 0 & 0 & z_3^2 & 0 & 0 & z_3(z_4 - z_3) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_4 - z_3 & 0 & 0 & 0 & 0 & 0 & z_3(z_4 - z_3) & 0 & 0 & (z_4 - z_3)^2 \end{bmatrix} . \tag{3.10.20}$$

Substitution of expression (18) into the expression (13) yields

$$K_c = \frac{1}{2}\int_0^B\int_0^L \mu\ H(x,y)\ \underset{(1\times 12)}{\left(\frac{\partial}{\partial t}\{f\}\right)^T}\ \underset{(12\times 12)}{\left[\ \widetilde{D}_c\ \right]}\ \underset{(12\times 1)}{\left(\frac{\partial}{\partial t}\{f\}\right)} dx\ dy\ . \tag{3.10.21}$$

So, the **total kinetic energy of the platform and the cargo** is

$$K = K_p + K_c =$$

$$= \frac{1}{2}\rho^{(1)}\int_0^B\int_0^L \left(\underset{(8\times 12)}{\left[\ \widetilde{\partial}^{(1)}\right]}\frac{\partial}{\partial t}\underset{(12\times 1)}{\{f\}}\right)^T \underset{(8\times 8)}{\left[\ \widetilde{D}^{(1)}\ \right]} \left(\underset{(8\times 12)}{\left[\ \widetilde{\partial}^{(1)}\right]}\frac{\partial}{\partial t}\underset{(12\times 1)}{\{f\}}\right)\ dx\ dy +$$

$$+ \frac{1}{2}\rho^{(2)}\int_0^B\int_0^L \left(\underset{(8\times 12)}{\left[\ \widetilde{\partial}^{(2)}\right]}\frac{\partial}{\partial t}\underset{(12\times 1)}{\{f\}}\right)^T \underset{(8\times 8)}{\left[\ \widetilde{D}^{(2)}\ \right]} \left(\underset{(8\times 12)}{\left[\ \widetilde{\partial}^{(2)}\right]}\frac{\partial}{\partial t}\underset{(12\times 1)}{\{f\}}\right)\ dx\ dy +$$

$$+ \frac{1}{2}\rho^{(3)}\int_0^B\int_0^L \left(\underset{(8\times 12)}{\left[\ \widetilde{\partial}^{(3)}\right]}\frac{\partial}{\partial t}\underset{(12\times 1)}{\{f\}}\right)^T \underset{(8\times 8)}{\left[\ \widetilde{D}^{(3)}\ \right]} \left(\underset{(8\times 12)}{\left[\ \widetilde{\partial}^{(3)}\right]}\frac{\partial}{\partial t}\underset{(12\times 1)}{\{f\}}\right)\ dx\ dy +$$

$$+ \frac{1}{2}\int_0^B\int_0^L \mu\ H(x,y)\ \underset{(1\times 12)}{\left(\frac{\partial}{\partial t}\{f\}\right)^T}\ \underset{(12\times 12)}{\left[\ \widetilde{D}_c\ \right]}\ \underset{(12\times 1)}{\left(\frac{\partial}{\partial t}\{f\}\right)} dx\ dy\ , \tag{3.10.22}$$

where $\left[\ \widetilde{\partial}^{(1)}\right]$, $\left[\ \widetilde{\partial}^{(2)}\right]$, $\left[\ \widetilde{\partial}^{(3)}\right]$ are the matrices of differential operators, defined by formulas (3.3.56), (3.3.59) and (3.3.61); $\{f\}$ is a column-matrix of the unknown functions of the problem, defined by formula (3.3.57); $\left[\ \widetilde{D}^{(1)}\ \right]$, $\left[\ \widetilde{D}^{(2)}\ \right]$, $\left[\ \widetilde{D}^{(3)}\ \right]$, $\left[\ \widetilde{D}_c\ \right]$ are the matrices of constants, defined by formulas (3.10.6), (3.10.9) and (3.10.10); μ is the mass of the cargo per unit area of the contact with the platform, defined by formula (3.10.13); $H(x,y)$ is a function, defined by formula (3.10.15); $\rho^{(1)}$, $\rho^{(2)}$, $\rho^{(3)}$ are the mass densities of the lower face sheet, the core and the upper face sheet.

3.11 Hamilton's Principle for the Sandwich Composite Platform with the Cargo on its Upper Surface, Dropped on Elastic Foundation

As it was discussed in the chapter 2, the virtual work principle[5]

$$\iiint\limits_{(V)} \sigma_{ij}\ \delta\varepsilon_{ij}\ dV = \iiint\limits_{(V)} F_i\ \delta u_i\ dV + \iint\limits_{(S)} q_i\ \delta u_i\ dS \tag{3.11.1}$$

contains information that the transverse stresses, obtained by integration of the pointwise equilibrium equations (second form of the transverse stresses), satisfy the stress boundary conditions on both the upper and lower surfaces of the plate[6], i.e. the transverse stresses at the upper and lower surfaces are equal to the externally applied loads per unit area. Therefore, the finite element formulation, based on the virtual work principle, guarantees that the values of the unknown functions, computed by the finite element method, are such that the second forms of the transverse stresses, expressed in terms of the unknown functions, satisfy the stress boundary conditions on both the upper and lower surfaces. In the chapter 2, the finite element formulation, on the basis of the virtual work principle, for a plate in cylindrical bending was performed for a static problem. For the problem of the cargo platform, dropped on the elastic foundation, which is essentially a dynamic problem, the dynamic form of the virtual work principle will be used for the finite element formulation. In the dynamic problems, by the use of the d'Alembert's principle which states that a system can be considered to be in equilibrium if inertial forces are taken into account, the principle of virtual work can be derived in a manner similar to the static problems, except that the terms representing the virtual work done by the inertial forces are now included (Washizu, 1982). The virtual work principle for the dynamic problems has the form:

$$\iiint\limits_{(V)} \sigma_{ij}\ \delta\varepsilon_{ij}\ dV = \iiint\limits_{(V)} (F_i - \rho\ddot{u}_i)\,\delta u_i\ dV + \iint\limits_{(S)} q_i\ \delta u_i\ dS. \tag{3.11.2}$$

[5] where F_i are components of the body force per unit volume, q_i are components of the surface force per unit volume

[6] in addition to satisfaction of the conditions of continuity of the transverse stresses across the interfaces between the plies of a laminated plate; this continuity is assured by the process of integration of the pointwise equilibrium equations.

In case of elastic bodies, the virtual work $\iiint\limits_{(V)} \sigma_{ij}\ \delta\varepsilon_{ij}\ dV$ of internal forces can be written as a variation of the strain energy

$$U = \iiint\limits_{(V)} \frac{1}{2}\sigma_{ij}\ \varepsilon_{ij}\ dV = \iiint\limits_{(V)} \frac{1}{2}C_{ijmn}\varepsilon_{ij}\varepsilon_{mn}\ dV \equiv \iiint\limits_{(V)} \hat{U}\ dV, \tag{3.11.3}$$

where C_{ijmn} are elastic constants. Besides, suppose that the body forces F_i and the surface forces q_i have the conservative and nonconservative parts:

$$F_i = F_i^{(c)} + F_i^{(nc)} = -\frac{\partial \tilde{V}}{\partial u_i} + F_i^{(nc)}, \qquad q_i = q_i^{(c)} + q_i^{(nc)} = -\frac{\partial \hat{V}}{\partial u_i} + F_i^{(nc)}, \tag{3.11.4}$$

where $\tilde{V}$ is a potential energy density due to the body forces, and $\hat{V}$ is a potential energy density due to the surface forces. Then, the virtual work principle (3.11.2) can be written in the form:

$$\delta\Pi = \iiint\limits_{(V)} \left(F_i^{(nc)} - \rho\ddot{u}_i\right)\delta u_i\ dV + \iint\limits_{(S)} q_i^{(nc)}\ \delta u_i\ dS, \tag{3.11.5}$$

where

$$\Pi \equiv \iiint\limits_{(V)} \hat{U}\ dV + \iiint\limits_{(V)} \tilde{V}\ dV + \iint\limits_{(S)} \hat{V}\ dS \tag{3.11.6}$$

is the total potential energy of the system. The dynamic virtual work principle (3.11.5) can be integrated with respect to time between two limits $t = t_1$ and $t = t_2$. Through integration by parts and by the use of the convention that the virtual displacements vanish at the limits, one can write the dynamic virtual work principle in the form of the extended Hamilton's principle (Meirovich, 1970)

$$\delta\int_{t_1}^{t_2} (T - \Pi)\ dt + \int_{t_1}^{t_2} \delta' W_{nc}\ dt = 0, \tag{3.11.7}$$

where

$$T = \frac{1}{2}\iiint\limits_{(V)} \rho\ \dot{u}_i\dot{u}_i\ dV \tag{3.11.8}$$

is kinetic energy of the system, and

$$\delta' W_{nc} = \iiint\limits_{(V)} F_i^{(nc)}\delta u_i\ dV + \iint\limits_{(S)} q_i^{(nc)}\ \delta u_i\ dS \tag{3.11.9}$$

is the virtual work of the external nonconservative forces. In the notation $\delta' W_{nc}$ the prime is used in order to make it understood, that $\delta' W_{nc}$ is not a variation of some state function W'_{nc} (Washizu, 1982).

The mechanical system under consideration consists of the sandwich platform, the cargo on its upper surface and elastic foundation. This system is not acted upon by any nonconservative surface forces. The nonconservative body forces are the forces of internal friction that cause damping (damping forces).

So, the Hamilton's principle for the system that consists of the sandwich platform, the cargo on its upper surface and the elastic foundation can be written as follows:

$$\delta \int_{t_1}^{t_2} [(\text{strain energy of platform}) + (\text{ strain energy of elastic foundation}) +$$

$$+ (\text{potential energy of platform in gravity field}) + (\text{potential energy of cargo in gravity field})$$

$$- (\text{kinetic energy of platform}) - (\text{kinetic energy of cargo})] \; dt$$

$$- \int_{t_1}^{t_2} (\text{virtual work of damping forces }) \; dt = 0. \tag{3.11.10}$$

In order to perform the finite element formulation, the Hamilton's principle (3.11.10) needs to be written in terms of the unknown functions for a finite element, and that allows to derive the element stiffness matrix, mass matrix, damping matrix and load vector. In a finite element model of the whole structure, these element matrices and vectors need to be assembled into the global matrices and vectors. In general, the global damping matrix can not be constructed from the element damping matrices, the same way as it is done for the mass and stiffness matrices, mainly because the damping properties of the separate finite elements are difficult to measure experimentally, and because the energy dissipation in a system depends on the properties of the whole system. Therefore, it is a common practice to construct the global damping matrix as a linear combination of the mass matrix and stiffness matrix of the complete element assemblage (Bathe, 1995). Therefore, for the purpose of developing the finite element formulation, there is no need to include the virtual work of the damping forces into the Hamilton's principle, written for a finite element. The components of all terms of the equation (3.11.10), except for the virtual work of the damping forces, were written in

terms of the column-matrix of the unknown functions

$$\{f\} = \left\lfloor \begin{array}{cccccccccccc} u_0 & v_0 & w_0 & \varepsilon_{xz}^{(1)} & \varepsilon_{yz}^{(1)} & \varepsilon_{zz}^{(1)} & \varepsilon_{xz}^{(2)} & \varepsilon_{yz}^{(2)} & \varepsilon_{zz}^{(2)} & \varepsilon_{xz}^{(3)} & \varepsilon_{yz}^{(3)} & \varepsilon_{zz}^{(3)} \end{array} \right\rfloor^T$$

in the previous sections of this chapter.

The Hamilton's principle, written in the form of equation (3.11.10), is convenient for the finite element formulation of the problem. A method of performing the finite element formulation for the problem under consideration will be discussed in the following section.

3.12 Some Considerations Regarding Finite Element Formulation

The maximum order of derivatives of w_0 and $\varepsilon_{zz}^{(k)}$ with respect to x and y in the Hamilton's principle is 2. Hence, the convergence of the finite element model will be ensured if, along the interelement boundaries, interpolation polynomials for w_0 and $\varepsilon_{zz}^{(k)}$ and their first derivatives in the directions normal to the element boundaries ($\frac{\partial w_0}{\partial n}$ and $\frac{\partial \varepsilon_{zz}^{(k)}}{\partial n}$) are continuous. If finite elements satisfy these requirements, they are called **conforming elements with C^1 continuity**. If interpolating polynomials for w_0 and $\varepsilon_{zz}^{(k)}$ are continuous at the interelement boundaries, and their first derivatives with respect to x and y are continuous at the nodes, but the normal derivatives $\frac{\partial w_0}{\partial n}$ and $\frac{\partial \varepsilon_{zz}^{(k)}}{\partial n}$ at the interelement boundaries are not continuous, then the elements are called **nonconforming elements with C^1 continuity**. Conformity of an element is not an indispensable requirement: the non-conforming elements (i.e. the elements that lack the required level of continuity in order to make the convergence most plausible) can still be successful (MacNeal, 1994). The nonconforming elements can be even needed to model discontinuities of the first derivatives of the unknown functions, that can appear in places of abrupt changes of plate thickness, or in places of abrupt in-plane changes of material properties of a plate. But since the cargo platform, that we are modelling, does not have such discontinuities, we expect that the conforming finite elements will produce more accurate results. Therefore, we will use the conforming finite elements for the unknown functions w_0 and $\varepsilon_{zz}^{(k)}$. Besides, our finite elements will be rectangular, since the cargo platform has the rectangular shape.

Let us consider, for example, interpolation of w_0. The interpolation polynomial for rectangular four-node element, that provides continuity of w_0 and $\frac{\partial w_0}{\partial n}$ along the interelement boundaries, developed by Bogner, Fox and Schmidt (1965), has the form (Figure 3.4):

$$w_0 = \frac{1}{16}\left(\frac{\overline{x}}{a}-1\right)^2\left(\frac{\overline{x}}{a}+2\right)\left(\frac{\overline{y}}{b}-1\right)^2\left(\frac{\overline{y}}{b}+2\right)w_0\left(A_1\right)+$$

$$+\frac{1}{16}\left(\frac{\overline{x}}{a}-1\right)^2\left(1+\frac{\overline{x}}{a}\right)\left(\frac{\overline{y}}{b}-1\right)^2\left(\frac{\overline{y}}{b}+2\right)a\frac{\partial w_0}{\partial x}\left(A_1\right)+$$

$$+\frac{1}{16}\left(\frac{\overline{x}}{a}-1\right)^2\left(\frac{\overline{x}}{a}+2\right)\left(\frac{\overline{y}}{b}-1\right)^2\left(1+\frac{\overline{y}}{b}\right)b\frac{\partial w_0}{\partial y}\left(A_1\right)+$$

$$+\frac{1}{16}\left(\frac{\overline{x}}{a}-1\right)^2\left(1+\frac{\overline{x}}{a}\right)\left(\frac{\overline{y}}{b}-1\right)^2\left(1+\frac{\overline{y}}{b}\right)ab\frac{\partial^2 w_0}{\partial x\partial y}\left(A_1\right)+$$

$$+\frac{1}{16}\left(1+\frac{\overline{x}}{a}\right)^2\left(2-\frac{\overline{x}}{a}\right)\left(\frac{\overline{y}}{b}-1\right)^2\left(\frac{\overline{y}}{b}+2\right)w_0\,(A_2)+$$

$$+\frac{1}{16}\left(1+\frac{\overline{x}}{a}\right)^2\left(\frac{\overline{x}}{a}-1\right)\left(\frac{\overline{y}}{b}-1\right)^2\left(\frac{\overline{y}}{b}+2\right)a\frac{\partial w_0}{\partial x}\,(A_2)+$$

$$+\frac{1}{16}\left(1+\frac{\overline{x}}{a}\right)^2\left(2-\frac{\overline{x}}{a}\right)\left(\frac{\overline{y}}{b}-1\right)^2\left(1+\frac{\overline{y}}{b}\right)b\frac{\partial w_0}{\partial y}\,(A_2)+$$

$$+\frac{1}{16}\left(1+\frac{\overline{x}}{a}\right)^2\left(\frac{\overline{x}}{a}-1\right)\left(\frac{\overline{y}}{b}-1\right)^2\left(1+\frac{\overline{y}}{b}\right)ab\frac{\partial^2 w_0}{\partial x\partial y}\,(A_2)+$$

$$+\frac{1}{16}\left(\frac{\overline{x}}{a}+1\right)^2\left(\frac{\overline{x}}{a}-2\right)\left(\frac{\overline{y}}{b}+1\right)^2\left(\frac{\overline{y}}{b}-2\right)w_0\,(A_3)+$$

$$+\frac{1}{16}\left(\frac{\overline{x}}{a}+1\right)^2\left(1-\frac{\overline{x}}{a}\right)\left(\frac{\overline{y}}{b}+1\right)^2\left(\frac{\overline{y}}{b}-2\right)a\frac{\partial w_0}{\partial x}\,(A_3)+$$

$$+\frac{1}{16}\left(\frac{\overline{x}}{a}+1\right)^2\left(\frac{\overline{x}}{a}-2\right)\left(\frac{\overline{y}}{b}+1\right)^2\left(1-\frac{\overline{y}}{b}\right)b\frac{\partial w_0}{\partial y}\,(A_3)+$$

$$+\frac{1}{16}\left(\frac{\overline{x}}{a}+1\right)^2\left(\frac{\overline{x}}{a}-1\right)\left(\frac{\overline{y}}{b}+1\right)^2\left(\frac{\overline{y}}{b}-1\right)ab\frac{\partial^2 w_0}{\partial x\partial y}\,(A_3)+$$

$$+\frac{1}{16}\left(\frac{\overline{x}}{a}-1\right)^2\left(\frac{\overline{x}}{a}+2\right)\left(\frac{\overline{y}}{b}+1\right)^2\left(2-\frac{\overline{y}}{b}\right)w_0\,(A_4)+$$

$$+\frac{1}{16}\left(\frac{\overline{x}}{a}-1\right)^2\left(\frac{\overline{x}}{a}+1\right)\left(\frac{\overline{y}}{b}+1\right)^2\left(2-\frac{\overline{y}}{b}\right)a\frac{\partial w_0}{\partial x}\,(A_4)+$$

$$+\frac{1}{16}\left(\frac{\overline{x}}{a}-1\right)^2\left(\frac{\overline{x}}{a}+2\right)\left(\frac{\overline{y}}{b}+1\right)^2\left(\frac{\overline{y}}{b}-1\right)b\frac{\partial w_0}{\partial y}\,(A_4)+$$

$$+\frac{1}{16}\left(\frac{\overline{x}}{a}-1\right)^2\left(\frac{\overline{x}}{a}+1\right)\left(\frac{\overline{y}}{b}+1\right)^2\left(\frac{\overline{y}}{b}-1\right)ab\frac{\partial^2 w_0}{\partial x\partial y}\,(A_4), \tag{3.12.1}$$

where $\overline{x}$ and $\overline{y}$ are the coordinates in the element (local) coordinate system. Each node of this element has the following degrees of freedom: w_0, $\frac{\partial w_0}{\partial x}$, $\frac{\partial w_0}{\partial y}$, $\frac{\partial^2 w_0}{\partial x\partial y}$.

The interpolation polynomial w_0 and its boundary-normal derivative $\frac{\partial w_0}{\partial n}$ are continuous along a common boundary with another element because w_0 and $\frac{\partial w_0}{\partial n}$ depend only on the degrees of freedom of the nodes, that belong to the boundary. To verify this, let us find w_0 and $\frac{\partial w_0}{\partial n}$ at the element boundaries, i.e. at $\overline{x} = \pm a$, $\overline{y} = \pm b$.

At the edge A_1A_2 ($\overline{y} = -b$)

$$w_0|_{y=-b} = \frac{1}{4}\left(\frac{\overline{x}}{a}-1\right)^2\left(\frac{\overline{x}}{a}+2\right) w_0(A_1) + \frac{1}{4}\left(\frac{\overline{x}}{a}-1\right)^2\left(1+\frac{\overline{x}}{a}\right) a\frac{\partial w_0}{\partial x}(A_1) +$$

$$+\frac{1}{4}\left(1+\frac{\overline{x}}{a}\right)^2\left(2-\frac{\overline{x}}{a}\right) w_0(A_2) + \frac{1}{4}\left(1+\frac{\overline{x}}{a}\right)^2\left(\frac{\overline{x}}{a}-1\right) a\frac{\partial w_0}{\partial x}(A_2), \tag{3.12.2}$$

$$\left.\frac{\partial w_0}{\partial \overline{y}}\right|_{\overline{y}=-b} = \left(\frac{1}{4a}\overline{x}^2 - \frac{1}{4}a - \frac{1}{4}\overline{x} + \frac{1}{4a^2}\overline{x}^3\right)\frac{\partial^2 w_0}{\partial \overline{x}\partial \overline{y}}(A_2) + \left(\frac{1}{4}a - \frac{1}{4}\overline{x} - \frac{1}{4a}\overline{x}^2 + \frac{1}{4a^2}\overline{x}^3\right)\frac{\partial^2 w_0}{\partial \overline{x}\partial \overline{y}}(A_1) +$$

$$+\left(\frac{3}{4a}\overline{x} + \frac{1}{2} - \frac{1}{4a^3}\overline{x}^3\right)\frac{\partial w_0}{\partial \overline{y}}(A_2) + \left(\frac{1}{4a^3}\overline{x}^3 - \frac{3}{4a}\overline{x} + \frac{1}{2}\right)\frac{\partial w_0}{\partial \overline{y}}(A_1). \tag{3.12.3}$$

At the edge A_2A_3 ($\overline{x} = a$):

$$w_0|_{\overline{x}=a} = +\frac{1}{4}\left(\frac{\overline{y}}{b}-1\right)^2\left(\frac{\overline{y}}{b}+2\right) w_0(A_2) + \frac{1}{4}\left(\frac{\overline{y}}{b}-1\right)^2\left(1+\frac{\overline{y}}{b}\right) b\frac{\partial w_0}{\partial \overline{y}}(A_2)$$

$$-\frac{1}{4}\left(\frac{\overline{y}}{b}+1\right)^2\left(\frac{\overline{y}}{b}-2\right) w_0(A_3) - \frac{1}{4}\left(\frac{\overline{y}}{b}+1\right)^2\left(1-\frac{\overline{y}}{b}\right) b\frac{\partial w_0}{\partial \overline{y}}(A_3), \tag{3.12.4}$$

$$\left.\frac{\partial w_0}{\partial \overline{x}}\right|_{\overline{x}=a} = \left(\frac{1}{4b^3}\overline{y}^3 - \frac{3}{4b}\overline{y} + \frac{1}{2}\right)\frac{\partial w_0}{\partial \overline{x}}(A_2) + \left(\frac{1}{4}b - \frac{1}{4b}\overline{y}^2 + \frac{1}{4b^2}\overline{y}^3 - \frac{1}{4}\overline{y}\right)\frac{\partial^2 w_0}{\partial \overline{x}\partial \overline{y}}(A_2)$$

$$+\left(\frac{1}{2} - \frac{1}{4b^3}\overline{y}^3 + \frac{3}{4b}\overline{y}\right)\frac{\partial w_0}{\partial \overline{x}}(A_3) + \left(\frac{1}{4b^2}\overline{y}^3 - \frac{1}{4}\overline{y} - \frac{1}{4}b + \frac{1}{4b}\overline{y}^2\right)\frac{\partial^2 w_0}{\partial \overline{x}\partial \overline{y}}(A_3). \tag{3.12.5}$$

At the edge A_3A_4 ($\overline{y} = b$)

$$w_0|_{\overline{y}=b} = \frac{1}{4}\left(\frac{\overline{x}}{a}+1\right)^2\left(2-\frac{\overline{x}}{a}\right) w_0(A_3) + \frac{1}{4}\left(\frac{\overline{x}}{a}+1\right)^2\left(\frac{\overline{x}}{a}-1\right) a\frac{\partial w_0}{\partial \overline{x}}(A_3)$$

$$+\frac{1}{4}\left(\frac{\overline{x}}{a}-1\right)^2\left(\frac{\overline{x}}{a}+2\right) w_0(A_4) + \frac{1}{4}\left(\frac{\overline{x}}{a}-1\right)^2\left(\frac{\overline{x}}{a}+1\right) a\frac{\partial w_0}{\partial \overline{x}}(A_4), \tag{3.12.6}$$

$$\left.\frac{\partial w_0}{\partial \overline{y}}\right|_{\overline{y}=b} = \left(\frac{1}{4a^3}\overline{x}^3 - \frac{3}{4a}\overline{x} + \frac{1}{2}\right)\frac{\partial w_0}{\partial \overline{y}}(A_4) + \left(\frac{1}{4a^2}\overline{x}^3 + \frac{1}{4a}\overline{x}^2 - \frac{1}{4}\overline{x} - \frac{1}{4}a\right)\frac{\partial^2 w_0}{\partial \overline{x}\partial \overline{y}}(A_3) +$$

$$+\left(\frac{1}{2} - \frac{1}{4a^3}\overline{x}^3 + \frac{3}{4a}\overline{x}\right)\frac{\partial w_0}{\partial \overline{y}}(A_3) + \left(\frac{1}{4a^2}\overline{x}^3 - \frac{1}{4a}\overline{x}^2 + \frac{1}{4}a - \frac{1}{4}\overline{x}\right)\frac{\partial^2 w_0}{\partial \overline{x}\partial \overline{y}}(A_4). \tag{3.12.7}$$

At the edge A_1A_4 $(\overline{x} = -a)$

$$w_0|_{x=-a} = \frac{1}{4}\left(\frac{\overline{y}}{b} - 1\right)^2\left(\frac{\overline{y}}{b} + 2\right)w_0(A_1) + \frac{1}{4}\left(\frac{\overline{y}}{b} - 1\right)^2\left(1 + \frac{\overline{y}}{b}\right)b\frac{\partial w_0}{\partial \overline{y}}(A_1) +$$

$$+\frac{1}{4}\left(\frac{\overline{y}}{b} + 1\right)^2\left(2 - \frac{\overline{y}}{b}\right)w_0(A_4) + \frac{1}{4}\left(\frac{\overline{y}}{b} + 1\right)^2\left(\frac{\overline{y}}{b} - 1\right)b\frac{\partial w_0}{\partial \overline{y}}(A_4). \tag{3.12.8}$$

$$\left.\frac{\partial w_0}{\partial \overline{x}}\right|_{\overline{x}=-a} = \left(\frac{1}{4b^2}\overline{y}^3 - \frac{1}{4b}\overline{y}^2 - \frac{1}{4}\overline{y} + \frac{1}{4}b\right)\frac{\partial^2 w_0}{\partial \overline{x}\partial \overline{y}}(A_1) + \left(\frac{1}{2} + \frac{1}{4b^3}\overline{y}^3 - \frac{3}{4b}\overline{y}\right)\frac{\partial w_0}{\partial \overline{x}}(A_1) +$$

$$+\left(\frac{1}{2} - \frac{1}{4b^3}\overline{y}^3 + \frac{3}{4b}\overline{y}\right)\frac{\partial w_0}{\partial \overline{x}}(A_4) + \left(\frac{1}{4b^2}\overline{y}^3 - \frac{1}{4}\overline{y} - \frac{1}{4}b + \frac{1}{4b}\overline{y}^2\right)\frac{\partial^2 w_0}{\partial \overline{x}\partial \overline{y}}(A_4). \tag{3.12.9}$$

We see that, indeed, w_0 and $\frac{\partial w_0}{\partial n}$ on the element boundaries depend only on nodal variables of those boundaries at which w_0 and $\frac{\partial w_0}{\partial n}$ are evaluated. Therefore, w_0 and $\frac{\partial w_0}{\partial n}$ are continuous on a common boundary with another element.

For the unknown functions $\varepsilon_{zz}^{(k)}$ $(k = 1, 2, 3)$, we will use the interpolation polynomial of the same type as (16.1), i.e. at each node the degrees of freedom will be $\varepsilon_{zz}^{(k)}$, $\frac{\partial \varepsilon_{zz}^{(k)}}{\partial x}$, $\frac{\partial \varepsilon_{zz}^{(k)}}{\partial y}$, $\frac{\partial^2 \varepsilon_{zz}^{(k)}}{\partial x \partial y}$ and the shape functions will be the same as in polynomial (16.1).

The maximum order of derivatives of the unknown functions u_0, $v_0, \varepsilon_{xz}^{(k)}$ and $\varepsilon_{yz}^{(k)}$ $(k = 1, 2, 3)$ in the Hamilton's principle is 1. Therefore, it is necessary that interpolation polynomials for these functions are continuous at the interelement boundaries, but the derivatives of the interpolation polynomials of these unknown functions at the interelement boundaries are not required to be continuous. The four-node rectangular element, that has these properties, is called the **bilinear Lagrange element** (Cook, Malkus, Plesha, 1989). Let us consider, for example, the unknown function u_0. The bilinear Lagrange element for u_0 has the form (Figure 3.4):

$$u_0 = \frac{1}{4}\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)u_0(A_1) + \frac{1}{4}\left(1 + \frac{x}{a}\right)\left(1 - \frac{y}{b}\right)u_0(A_2) +$$
$$+\frac{1}{4}\left(1 + \frac{x}{a}\right)\left(1 + \frac{y}{b}\right)u_0(A_3) + \frac{1}{4}\left(1 - \frac{x}{a}\right)\left(1 + \frac{y}{b}\right)u_0(A_4). \tag{3.12.10}$$

This element has four degrees of freedom: the values of the interpolated functions at the nodes.

The combined finite element for all the unknown functions of the problem will have 96 degrees of freedom: 4 degrees of freedom must be used for interpolation of each of the functions u_0, $v_0, \varepsilon_{xz}^{(k)}$, $\varepsilon_{yz}^{(k)}$ $(k = 1, 2, 3)$, and 16 degrees of freedom must be used for interpolation of each of the functions $w_0, \varepsilon_{zz}^{(k)}$ $(k = 1, 2, 3)$. Each node of the combined finite element has 24 degrees of freedom The nodal variables of each node of the combined finite element are u_0,v_0, w_0, $\frac{\partial w_0}{\partial x}$, $\frac{\partial w_0}{\partial y}$, $\frac{\partial^2 w_0}{\partial x \partial y}$, $\varepsilon_{xz}^{(1)}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{xz}^{(3)}$, $\varepsilon_{yz}^{(1)}$, $\varepsilon_{yz}^{(2)}$, $\varepsilon_{yz}^{(3)}$, $\varepsilon_{zz}^{(1)}$, $\frac{\partial \varepsilon_{zz}^{(1)}}{\partial x}$, $\frac{\partial \varepsilon_{zz}^{(1)}}{\partial y}$, $\frac{\partial^2 \varepsilon_{zz}^{(1)}}{\partial x \partial y}$, $\varepsilon_{zz}^{(2)}$, $\frac{\partial \varepsilon_{zz}^{(2)}}{\partial x}$, $\frac{\partial \varepsilon_{zz}^{(2)}}{\partial y}$, $\frac{\partial^2 \varepsilon_{zz}^{(2)}}{\partial x \partial y}$, $\varepsilon_{zz}^{(3)}$, $\frac{\partial \varepsilon_{zz}^{(3)}}{\partial x}$, $\frac{\partial \varepsilon_{zz}^{(3)}}{\partial y}$, $\frac{\partial^2 \varepsilon_{zz}^{(3)}}{\partial x \partial y}$.

The finite element model, based on the layerwise plate theory presented in this chapter, allows to analyze the sandwich composite plates with fewer degrees of freedom than the finite models constructed with the use of three-dimensional finite elements. This is due to the fact that in the three-dimensional finite element models it is necessary to represent the thickness of one ply of the face sheets with a thickness of at least one three-dimensional finite element, in order to compute accurately the through-the-thickness variation of displacements and stresses and in order to determine damage in each ply; On the other hand, in the layerwise plate theory, discussed in this chapter, the number of the finite elements, required to represent properly the through-the-thickness variation of displacements and stresses and the damage in each ply, does not depend on the number of plies in the composite face sheets[7].

Let us consider an example problem and compare the number of the degrees of freedom in the three-dimensional and layerwise plate finite element models. We will consider an example of a sandwich plate with the following characteristics: thickness of the lower face sheet $0.01m$, thickness of the upper face sheet $0.005m$, thickness of the core $0.05m$, number of plies in the lower face sheets is 100, number of plies in the upper face sheets is 50, in-plane dimensions $1m \times 1m$. Each ply of the face sheets has the thickness of $1 \times 10^{-4}m$.

Suppose this sandwich plate is modelled with the linear solid elements, i.e. the eight-node brick elements. Each node of such an element has three degrees of freedom: the nodal displacements. In order to avoid ill-conditioning of the finite element equations, the in-plane dimensions of these finite elements must be not much larger than their size in the thickness direction. For the same reason,

[7]Though, with the increase of the number of plies in the face sheets, the number of the finite elements, required to achieve convergence, increases. But this increase of the number of the elements in the layerwise plate model, dictated by the convergence requirement, is not proportional to the number of plies and is very small as compared to the increase of the number of the three-dimensional elements in the three-dimensional finite element models, dictated by the requirement of representing the thickness of one ply with a thickness of at least one three-dimensional element.

the sizes of adjacent elements must not be much different. Besides, the mesh in the core must be sufficiently fine in order to determine the damage in the core, that can be distributed nonuniformly in the thickness direction and in the in-plane directions.

Therefore, for the purpose of estimating a number of elements in this example problem, the finite elements will be considered with in-plane dimensions five times larger than their thickness, and all the elements will be chosen to be of the same size. If in the thickness of one ply there is one such element, then the size of each element is $0.5mm \times 0.5mm \times 0.1mm$, and the total number of the elements in the whole model of the plate is $2000 \times 2000 \times 650 = 2.6 \times 10^9$. The total number of the nodes in this model is $2001 \times 2001 \times 651 \approx 2.6 \times 10^9$, and the total number of degrees of freedom in the whole three-dimensional model is $2.6 \times 10^9 \times 3 = 7.8 \times 10^9$.

Now, let us evaluate the number of degrees of freedom in the layerwise plate FE model with a 50×50 FE mesh. The number of nodes in such a two-dimensional FE model is $51 \times 51 = 2601$, and the number of degrees of freedom is $2601 \times 24 = 62\,424$. As it will be shown in the chapter 5, the stresses, computed by the use of the layerwise plate FE model of the sandwich plates, including the transverse stresses, are sufficiently accurate as compared with the stresses of exact elasticity solutions , if the transverse stresses are computed by integration of the equilibrium equations.

So, we see that the use of the two-dimensional layerwise FE model of the sandwich plates, presented in this chapter, allows to achieve a tremendous decrease of the number of degrees of freedom, as compared to the three-dimensional FE model, without decrease of the accuracy of stress computation.

3.13 Post-processing Stage of the Finite Element Analysis: Expressions for the In-Plane Stress Components and the Second Forms of the Transverse Stress Components in Terms of the Unknown Functions u_0, v_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{yz}^{(k)}$, $\varepsilon_{zz}^{(k)}$.

After the finite element solution for the unknown functions is obtained, the components of the stress tensor need to be computed . As it was mentioned previously, the in-plane stress components will be computed from the constitutive relations, i.e. by substituting the in-plane strains, expressed in terms of the unknown functions, into the Hooke's law for the in-plane stresses. The transverse stress components will be computed not from the Hooke's law, but by integration of the equations of motion (3.1.21)-(3.1.23). The transverse stress components, obtained by integration of the equations of motion (the second form of the transverse stresses) are more accurate than those obtained from the Hooke's law (the first form of the transverse stresses), because, as it was shown in chapter 2, the second forms of the transverse stresses, unlike the first forms, satisfy the boundary conditions at the upper and lower surfaces of the sandwich plate and at the interfaces between the face sheets and the core.

The expressions (3.4.1) for the in-plane strains in terms of the unknown functions, written here again, are the following

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}^{(k)} = \begin{Bmatrix} \varphi_{xx0} \\ \varphi_{yy0} \\ \varphi_{xy0} \end{Bmatrix}^{(k)} + \begin{Bmatrix} \varphi_{xx1} \\ \varphi_{yy1} \\ \varphi_{xy1} \end{Bmatrix}^{(k)} z + \begin{Bmatrix} \varphi_{xx2} \\ \varphi_{yy2} \\ \varphi_{xy2} \end{Bmatrix}^{(k)} z^2 , \qquad (3.13.1)$$

where the functions φ in the right-hand side of the equation (3.13.1) are expressed in terms of the unknown functions by equations (3.4.2)-(3.4.28). The Hooke's law for an orthotropic material (in a

coordinate system, whose coordinate planes do not coincide with planes of elastic symmetry), is

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\end{array}\right\}=\left[\begin{array}{cccccc}\overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & 0 & 0 & \overline{C}_{16}\\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{23} & 0 & 0 & \overline{C}_{26}\\ \overline{C}_{13} & \overline{C}_{23} & \overline{C}_{33} & 0 & 0 & \overline{C}_{36}\\ 0 & 0 & 0 & \overline{C}_{44} & \overline{C}_{45} & 0\\ 0 & 0 & 0 & \overline{C}_{45} & \overline{C}_{55} & 0\\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{36} & 0 & 0 & \overline{C}_{66}\end{array}\right]\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ 2\varepsilon_{yz}\\ 2\varepsilon_{xz}\\ 2\varepsilon_{xy}\end{array}\right\}. \tag{3.13.2}$$

Therefore, the Hooke's law for only the in-plane stresses is

$$\left\{\begin{array}{c}{}^{H}\sigma_{xx}\\ {}^{H}\sigma_{yy}\\ {}^{H}\sigma_{xy}\end{array}\right\}^{(k)}=\left[\begin{array}{cccc}\overline{C}_{11} & \overline{C}_{12} & \overline{C}_{16} & \overline{C}_{13}\\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{26} & \overline{C}_{23}\\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{66} & \overline{C}_{36}\end{array}\right]^{(k)}\left\{\begin{array}{c}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\\ \varepsilon_{zz}\end{array}\right\}^{(k)}, \tag{3.13.3}$$

where the left superscript H in notations for stresses means that the stresses are computed by the Hooke's law (in contrast to the second forms of transverse stresses σ_{xz}, σ_{yz} and σ_{zz}, that will be computed by integrating the 3-D equations of motion). Substitution of equation (3.13.1) into equation (3.13.3) yields

$$\left\{\begin{array}{c}{}^{H}\sigma_{xx}\\ {}^{H}\sigma_{yy}\\ {}^{H}\sigma_{xy}\end{array}\right\}^{(k)}=$$

$$=\left[\begin{array}{cccc}\overline{C}_{11} & \overline{C}_{12} & \overline{C}_{16} & \overline{C}_{13}\\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{26} & \overline{C}_{23}\\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{66} & \overline{C}_{36}\end{array}\right]^{(k)}\left(\left\{\begin{array}{c}\varphi_{xx0}\\ \varphi_{yy0}\\ \varphi_{xy0}\\ \varepsilon_{zz}\end{array}\right\}^{(k)}+\left\{\begin{array}{c}\varphi_{xx1}\\ \varphi_{yy1}\\ \varphi_{xy1}\\ 0\end{array}\right\}^{(k)}z+\left\{\begin{array}{c}\varphi_{xx2}\\ \varphi_{yy2}\\ \varphi_{xy2}\\ 0\end{array}\right\}^{(k)}z^2\right), \tag{3.13.4}$$

where the functions φ in the right-hand side of the equation (3.13.4) are expressed in terms of the unknown functions by equations (3.4.2)-(3.4.28).

Now let us express the transverse stresses σ_{xz}, σ_{yz} and σ_{zz} in terms of the in-plane stresses and displacements by integrating equations of motion (3.1.21)-(3.1.23). Then we can substitute into the

resulting equations expressions (3.13.4) for the in-plane stresses in terms of the unknown functions and expressions (3.3.27) for displacements in terms of the unknown functions. Thus, the transverse stresses can be expressed in terms of the unknown functions.

Performing integration of equation of motion (3.1.21) with k=1 (for the lower face sheet)

$$\sigma_{xx,x}^{(1)} + \sigma_{xy,y}^{(1)} + \sigma_{xz,z}^{(1)} = \rho^{(1)}\ddot{u}^{(1)}$$

with respect to z in the positive direction of the z-axis, we receive:

$$\sigma_{xz}^{(1)} = \underbrace{\sigma_{xz}^{(1)}(z_1)}_{0} + \int_{z_1}^{z} \left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma_{xx,x}^{(1)} - {}^{H}\sigma_{xy,y}^{(1)}\right) dz \quad (z_1 \le z \le z_2), \tag{3.13.5}$$

where $\sigma_{xz}^{(1)}(z_1) = 0$ due to the fact that tangential components of the surface traction at the lower surface of the platform is equal to zero (boundary conditions (3.1.24)). From equation (3.13.5) it follows that

$$\sigma_{xz}^{(1)}(z_2) = \int_{z_1}^{z_2} \left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma_{xx,x}^{(1)} - {}^{H}\sigma_{xy,y}^{(1)}\right) dz \tag{3.13.6}$$

or

$$\sigma_{xz}^{(2)}(z_2) = \int_{z_1}^{z_2} \left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma_{xx,x}^{(1)} - {}^{H}\sigma_{xy,y}^{(1)}\right) dz, \tag{3.13.7}$$

because

$$\sigma_{xz}^{(1)}(z_2) = \sigma_{xz}^{(2)}(z_2)$$

due to the first continuity condition (3.1.33).

Integrating equation of motion (3.1.21) with k=2 (for the core)

$$\sigma_{xx,x}^{(2)} + \sigma_{xy,y}^{(2)} + \sigma_{xz,z}^{(2)} = \rho^{(2)}\ddot{u}^{(2)}$$

from z_2 to z, where z belongs to the interval $z_2 \le z \le z_3$, one can receive

$$\sigma_{xz}^{(2)} = \sigma_{xz}^{(2)}(z_2) + \int_{z_2}^{z} \left(\rho^{(2)}\ddot{u}^{(2)} - {}^{H}\sigma_{xx,x}^{(2)} - {}^{H}\sigma_{xy,y}^{(2)}\right) dz \quad (z_2 \le z \le z_3). \tag{3.13.8}$$

The substitution of equation (3.13.7) into equation (3.13.8) yields:

$$\sigma_{xz}^{(2)} = \int_{z_1}^{z_2} \left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma_{xx,x}^{(1)} - {}^{H}\sigma_{xy,y}^{(1)}\right) dz +$$

$$+\int\limits_{z_2}^{z}\left(\rho^{(2)}\ddot{u}^{(2)} - {}^{H}\sigma^{(2)}_{xx,x} - {}^{H}\sigma^{(2)}_{xy,y}\right)\,dz \quad (z_2 \le z \le z_3)\,. \tag{3.13.9}$$

One can receive in the same way the following for the upper face sheets (k=3):

$$\sigma_{xz}{}^{(3)} = \int\limits_{z_1}^{z_2}\left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma^{(1)}_{xx,x} - {}^{H}\sigma^{(1)}_{xy,y}\right)\,dz+$$

$$+\int\limits_{z_2}^{z_3}\left(\rho^{(2)}\ddot{u}^{(2)} - {}^{H}\sigma^{(2)}_{xx,x} - {}^{H}\sigma^{(2)}_{xy,y}\right)\,dz+$$

$$+\int\limits_{z_3}^{z}\left(\rho^{(3)}\ddot{u}^{(3)} - {}^{H}\sigma^{(3)}_{xx,x} - {}^{H}\sigma^{(3)}_{xy,y}\right)\,dz \quad (z_3 \le z \le z_4)\,. \tag{3.13.10}$$

Analogously, integration of equation of motion (3.1.22)

$$\sigma^{(k)}_{yx,x} + \sigma^{(k)}_{yy,y} + \sigma^{k}_{yz,z} = \rho^{(k)}\ddot{v}^{(k)}$$

with respect to z gives expressions for $\sigma^{(k)}_{yz}$:

$$\sigma^{(1)}_{yz} = \underbrace{\sigma^{(1)}_{yz}(z_1)}_{0} + \int\limits_{z_1}^{z}\left(\rho^{(1)}\ddot{v}^{(1)} - {}^{H}\sigma^{(1)}_{yx,x} - {}^{H}\sigma^{(1)}_{yy,y}\right)\,dz \quad (z_1 \le z \le z_2)\,, \tag{3.13.11}$$

$$\sigma^{(2)}_{yz} = \int\limits_{z_1}^{z_2}\left(\rho^{\ (1)}\ddot{v}^{(2)} - {}^{H}\sigma^{(1)}_{yx,x} - {}^{H}\sigma^{(1)}_{yy,y}\right)\,dz+$$

$$+\int\limits_{z_2}^{z}\left(\rho^{(2)}\ddot{v}^{(3)} - {}^{H}\sigma^{(2)}_{yx,x} - {}^{H}\sigma^{(2)}_{yy,y}\right)\,dz \quad (z_2 \le z \le z_3)\,, \tag{3.13.12}$$

$$\sigma^{(3)}_{yz} = \int\limits_{z_1}^{z_2}\left(\rho^{(1)}\ddot{v}^{(1)} - {}^{H}\sigma^{(1)}_{yx,x} - {}^{H}\sigma^{(1)}_{yy,y}\right)\,dz+$$

$$+\int\limits_{z_2}^{z_3}\left(\rho^{(2)}\ddot{v}^{(2)} - {}^{H}\sigma^{(2)}_{yx,x} - {}^{H}\sigma^{(2)}_{yy,y}\right)\,dz+$$

$$+\int_{z_3}^{z} \left(\rho^{(3)}\ddot{v}^{(3)} - {}^{H}\sigma^{(3)}_{yx,x} - {}^{H}\sigma^{(3)}_{yy,y}\right)\, dz \qquad (z_3 \le z \le z_4)\,. \tag{3.13.13}$$

Expressions (3.13.5), (3.13.9)-(3.13.13) for $\sigma^{(k)}_{xz}$ and $\sigma^{(k)}_{yz}$ in terms of displacements and in-plane stresses can be written in tensor notations as[8]:

$$\sigma^{(k)}_{\alpha 3} = \sum_{m=1}^{k-1} \int_{z_m}^{z_{m+1}} \left(\rho^{(m)}\ddot{u}^{(m)}_{\alpha} - {}^{H}\sigma^{(m)}_{\alpha\beta,\beta}\right)\, dz + \int_{z_k}^{z} \left(\rho^{(k)}\ddot{u}^{(k)}_{\alpha} - {}^{H}\sigma^{(k)}_{\alpha\beta,\beta}\right)\, dz \quad (\ \alpha = 1, 2;\ \beta = 1, 2) \tag{3.13.14}$$

in the interval $z_k \le z \le z_{k+1}$,

where the sum is considered to be equal to zero, if the upper value of the summation index m is smaller than the lower value, i.e. if $k = 1$.

Let us integrate equations of motion (3.1.23)

$$\sigma^{(k)}_{zx,x} + \sigma^{(k)}_{zy,y} + \sigma^{(k)}_{zz,z} + \frac{\partial}{\partial x}\left(\sigma^{(k)}_{xx} w^{(k)}_{,x} + \sigma^{(k)}_{yx} w^{(k)}_{,y}\right) +$$

$$+\frac{\partial}{\partial y}\left(\sigma^{(k)}_{xy}\ w^{(k)}_{,x} + \sigma^{(k)}_{yy}\ w^{(k)}_{,y}\right) - \rho^{(k)} g = \rho^{(k)}\ddot{w}^{(k)} \qquad (k = 1, 2, 3)\,.$$

Doing this, one needs to take account of continuity conditions at the interfaces between the face sheets and the core $\sigma^{(1)}_{zz}(z_2) = \sigma^{(2)}_{zz}(z_2)$, $\sigma^{(2)}_{zz}(z_3) = \sigma^{(3)}_{zz}(z_3)$ (equations (3.1.33), (3.1.34)) and of the boundary condition at the lower surface of the platform $\sigma^{(1)}_{zz}(z_1) = -t_z(z_1)$ (the third equation (3.1.24)). The surface force per unit area $t_z(z_1)$ in this problem is equal to $-sw^{(1)}(z_1)$, where s is a modulus of the elastic foundation. As a result, we receive the following expressions for stresses $\sigma^{(k)}_{zz}$:

$$\sigma^{(1)}_{zz} = \underbrace{\sigma^{(1)}_{zz}(z_1)}_{sw^{(1)}(z_1)} + \int_{z_1}^{z} \left[\rho^{(1)}\left(\ddot{w}^{(1)} + g\right) - \frac{\partial}{\partial x}\left(\sigma^{(1)}_{xx} w^{(1)}_{,x} + \sigma^{(1)}_{yx} w^{(1)}_{,y}\right)\right.$$
$$\left. -\frac{\partial}{\partial y}\left(\sigma^{(1)}_{xy} w^{(1)}_{,x} + \sigma^{(1)}_{yy} w^{(1)}_{,y}\right) - \sigma^{(1)}_{zx,x} - \sigma^{(1)}_{zy,y}\right] dz \tag{3.13.15}$$

[8]In equation (3.3.14) the following notations are implied: $\sigma^{(k)}_{13} \equiv \sigma^{(k)}_{xz}$, $\sigma^{(k)}_{23} \equiv \sigma^{(k)}_{yz}$, and the upper index denotes the number of a sublaminate (k=1 means the lower face sheet, k=2 means the core, k=3 means the upper face sheet).

$$\sigma_{zz}^{(2)} = sw^{(1)}(z_1) + \int_{z_1}^{z_2} \left[\rho^{(1)}\left(\ddot{w}^{(1)} + g\right) - \frac{\partial}{\partial x}\left(\sigma_{xx}^{(1)} w_{,x}^{(1)} + \sigma_{yx}^{(1)} w_{,y}^{(1)}\right)\right.$$
$$\left. - \frac{\partial}{\partial y}\left(\sigma_{xy}^{(1)} w_{,x}^{(1)} + \sigma_{yy}^{(1)} w_{,y}^{(1)}\right) - \sigma_{zx,x}^{(1)} - \sigma_{zy,y}^{(1)}\right] dz +$$
$$+ \int_{z_2}^{z} \left[\rho^{(2)}\left(\ddot{w}^{(2)} + g\right) - \frac{\partial}{\partial x}\left(\sigma_{xx}^{(2)} w_{,x}^{(2)} + \sigma_{yx}^{(2)} w_{,y}^{(2)}\right)\right.$$
$$\left. - \frac{\partial}{\partial y}\left(\sigma_{xy}^{(2)} w_{,x}^{(2)} + \sigma_{yy}^{(2)} w_{,y}^{(2)}\right) - \sigma_{zx,x}^{(2)} - \sigma_{zy,y}^{(2)}\right] dz \tag{3.13.16}$$

$$\sigma_{zz}^{(3)} = sw^{(1)}(z_1) + \int_{z_1}^{z_2} \left[\rho^{(1)}\left(\ddot{w}^{(1)} + g\right) - \frac{\partial}{\partial x}\left(\sigma_{xx}^{(1)} w_{,x}^{(1)} + \sigma_{yx}^{(1)} w_{,y}^{(1)}\right)\right.$$
$$\left. - \frac{\partial}{\partial y}\left(\sigma_{xy}^{(1)} w_{,x}^{(1)} + \sigma_{yy}^{(1)} w_{,y}^{(1)}\right) - \sigma_{zx,x}^{(1)} - \sigma_{zy,y}^{(1)}\right] dz$$
$$+ \int_{z_2}^{z_3} \left[\rho^{(2)}\left(\ddot{w}^{(2)} + g\right) - \frac{\partial}{\partial x}\left(\sigma_{xx}^{(2)} w_{,x}^{(2)} + \sigma_{yx}^{(2)} w_{,y}^{(2)}\right)\right.$$
$$\left. - \frac{\partial}{\partial y}\left(\sigma_{xy}^{(2)} w_{,x}^{(2)} + \sigma_{yy}^{(2)} w_{,y}^{(2)}\right) - \sigma_{zx,x}^{(2)} - \sigma_{zy,y}^{(2)}\right] dz$$
$$+ \int_{z_3}^{z} \left[\rho^{(3)}\left(\ddot{w}^{(3)} + g\right) - \frac{\partial}{\partial x}\left(\sigma_{xx}^{(3)} w_{,x}^{(3)} + \sigma_{yx}^{(3)} w_{,y}^{(3)}\right)\right.$$
$$\left. - \frac{\partial}{\partial y}\left(\sigma_{xy}^{(3)} w_{,x}^{(3)} + \sigma_{yy}^{(3)} w_{,y}^{(3)}\right) - \sigma_{zx,x}^{(3)} - \sigma_{zy,y}^{(3)}\right] dz \tag{3.13.17}$$

Equations (3.13.15)-(3.13.17) can be written in tensor notations as follows:

$$\sigma_{33}^{(k)} = sw^{(1)}(z_1) + \sum_{m=1}^{k-1} \int_{z_m}^{z_{m+1}} \left[\rho^{(m)}\left(\ddot{u}_3^{(m)} + g\right) - \left({}^{H}\sigma_{\alpha\beta}^{(m)}\, u_{3,\alpha}^{(m)}\right)_{,\beta} - \sigma_{3\alpha,\alpha}^{(m)}\right] dz +$$

$$+ \int_{z_k}^{z} \left[\rho^{(k)}\left(\ddot{u}_3^{(k)} + g\right) - \left({}^{H}\sigma_{\alpha\beta}^{(k)}\, u_{3,\alpha}^{(k)}\right)_{,\beta} - \sigma_{3\alpha,\alpha}^{(k)}\right] dz \quad (\alpha = 1,2;\ \beta = 1,2) \tag{3.13.18}$$

in the interval $z_k \le z \le z_{k+1}$

Substitution of expressions (3.13.14) for $\sigma_{3\alpha}^{(k)}$ $(\alpha = 1,2)$ into equation (3.13.18) for $\sigma_{33}^{(k)}$ gives expressions for $\sigma_{33}^{(k)}$ in terms of displacements and in-plane stresses:

$$\sigma_{33}^{(k)} = sw^{(1)}(z_1) + \left\{ \sum_{m=1}^{k-1} \int_{z_m}^{z_{m+1}} \left[\rho^{(m)}\left(\ddot{u}_3^{(m)} + g\right) - \left({}^{H}\sigma_{\alpha\beta}^{(m)}\, u_{3,\alpha}^{(m)}\right)_{,\beta} - \sum_{n=1}^{m-1} \int_{z_n}^{z_{n+1}} \left(\rho^{(n)} \ddot{u}_{\alpha,\alpha}^{(n)} - {}^{H}\sigma_{\alpha\beta,\alpha\beta}^{(n)}\right) dz\right.\right.$$

$$-\int_{z_m}^{z} \left(\rho^{(m)}\ddot{u}^{(m)}_{\alpha,\alpha} - {}^{H}\sigma^{(m)}_{\alpha\beta,\alpha\beta}\right)\, dz\Bigg]\, dz\Bigg\} + \int_{z_k}^{z}\Bigg[\rho^{(k)}\left(\ddot{u}^{(k)}_3 + g\right) - \left({}^{H}\sigma^{(k)}_{\alpha\beta}\, u^{(k)}_{3,\alpha}\right)_{,\beta}$$

$$-\sum_{n=1}^{k-1}\int_{z_n}^{z_{n+1}} \left(\rho^{(n)}\,\ddot{u}^{(n)}_{\alpha,\alpha} - {}^{H}\sigma^{(n)}_{\alpha\beta,\alpha\beta}\right) dz - \int_{z_k}^{z}\left(\rho^{(k)}\,\ddot{u}^{(k)}_{\alpha,\alpha} - {}^{H}\sigma^{(k)}_{\alpha\beta,\alpha\beta}\right) dz\Bigg]\, dz \quad (\alpha = 1,2;\ \beta = 1,2) \tag{3.13.19}$$

in the interval $z_k \le z \le z_{k+1}$

or

$$\sigma^{(k)}_{33} = sw^{(1)}(z_1) + \sum_{m=1}^{k-1}\int_{z_m}^{z_{m+1}} \rho^{(m)}\left(\ddot{u}^{(m)}_3 + g\right) dz - \sum_{m=1}^{k-1}\int_{z_m}^{z_{m+1}}\left({}^{H}\sigma^{(m)}_{\alpha\beta}\, u^{(m)}_{3,\alpha}\right)_{,\beta} dz$$

$$-\sum_{m=1}^{k-1}\sum_{n=1}^{m-1}\int_{z_m}^{z_{m+1}}\int_{z_n}^{z_{n+1}}\left(\rho^{(n)}\ddot{u}^{(n)}_{\alpha,\alpha} - {}^{H}\sigma^{(n)}_{\alpha\beta,\alpha\beta}\right) dz\, dz$$

$$-\sum_{m=1}^{k-1}\int_{z_m}^{z_{m+1}}\int_{z_m}^{z}\left(\rho^{(m)}\ddot{u}^{(m)}_{\alpha,\alpha} - {}^{H}\sigma^{(m)}_{\alpha\beta,\alpha\beta}\right) dz\, dz$$

$$+\int_{z_k}^{z}\rho^{(k)}\left(\ddot{u}^{(k)}_3 + g\right) dz - \int_{z_k}^{z}\left({}^{H}\sigma^{(k)}_{\alpha\beta}\, u^{(k)}_{3,\alpha}\right)_{,\beta} dz$$

$$-\sum_{n=1}^{k-1}\int_{z_k}^{z}\int_{z_n}^{z_{n+1}}\left(\rho^{(n)}\,\ddot{u}^{(n)}_{\alpha,\alpha} - {}^{H}\sigma^{(n)}_{\alpha\beta,\alpha\beta}\right) dz\, dz$$

$$-\int_{z_k}^{z}\int_{z_k}^{z}\left(\rho^{(k)}\,\ddot{u}^{(k)}_{\alpha,\alpha} - {}^{H}\sigma^{(k)}_{\alpha\beta,\alpha\beta}\right) dz\, dz \quad (\alpha = 1,2;\ \beta = 1,2) \tag{3.13.20}$$

in the interval $z_k \le z \le z_{k+1}$.

So, equations (3.13.14) and (3.13.20) express the transverse stresses in terms of displacements $u^{(k)}$, $v^{(k)}$, $w^{(k)}$ and in-plane stresses ${}^{H}\sigma^{(k)}_{xx}$, ${}^{H}\sigma^{(k)}_{xy}$, ${}^{H}\sigma^{(k)}_{yy}$, which, in turn, are expressed in terms of

the unknown functions u_0, v_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{yz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ by equations (3.3.27)-(3.3.51) and by equations (3.13.4) together with equations (3.4.2)-(3.4.28). The explicit expressions for the transverse stresses in terms of the unknown functions u_0, v_0, w_0, $\varepsilon_{xz}^{(k)}$, $\varepsilon_{yz}^{(k)}$, $\varepsilon_{zz}^{(k)}$ are not shown here because of their large size.

The values of the in-plane and transverse stresses, computed by the formulas, derived in this chapter, can be substituted into the failure criteria in order to take account of damage progression in the sandwich platform. The methods of failure analysis will be discussed in chapter 5.

Figure 3.1

Cross-sections of the face sheets and the core

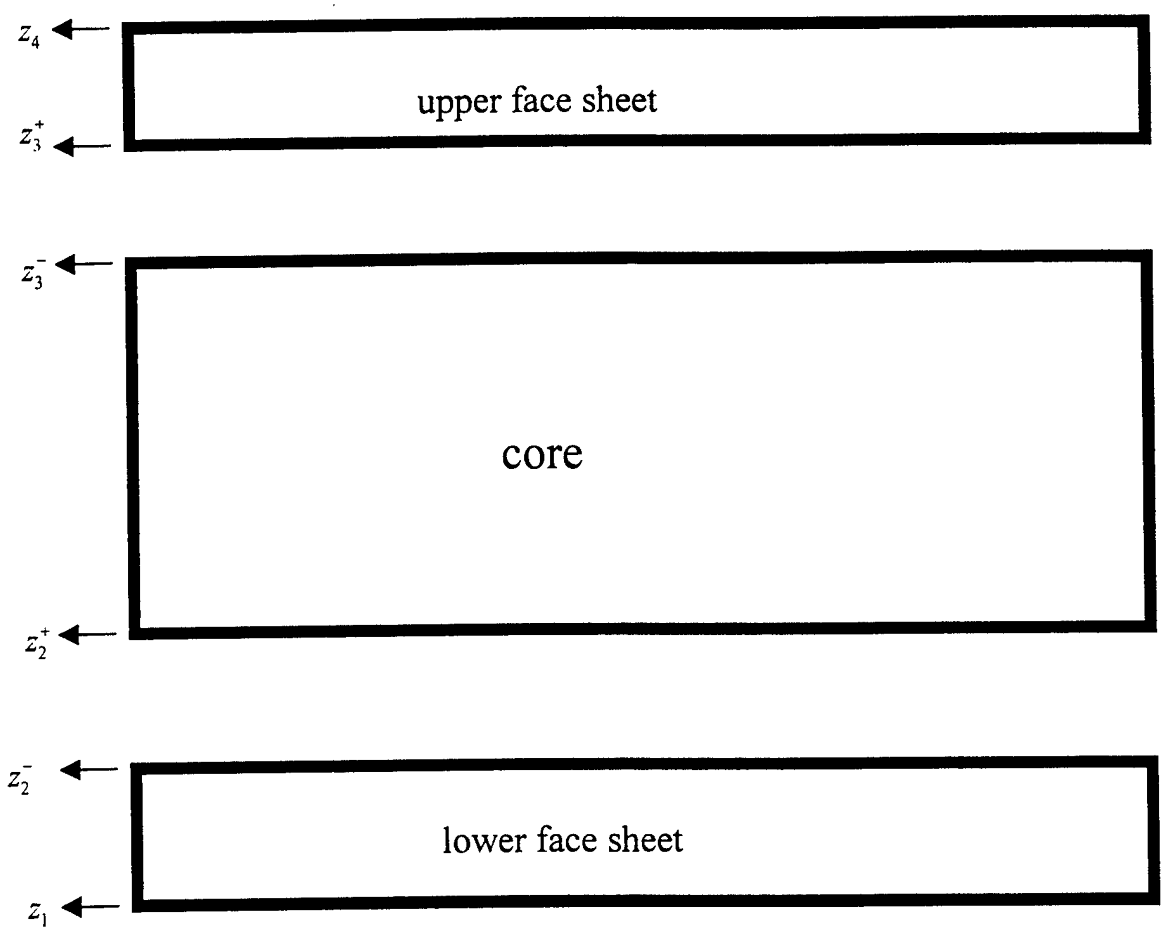


Figure 3.2

A fiber-reinforced lamina with global and material coordinate systems

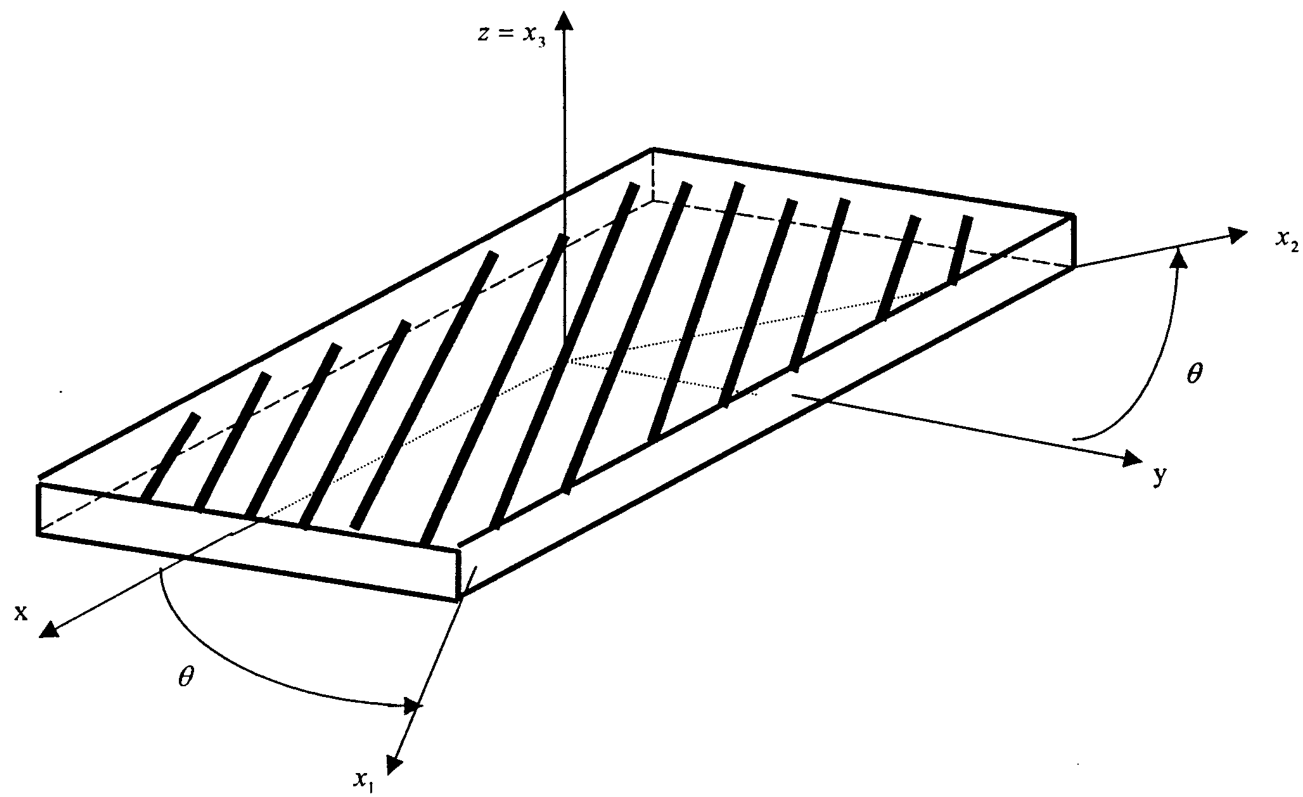


Figure 3.3

Notations for z-coordinates of interfaces between the plies

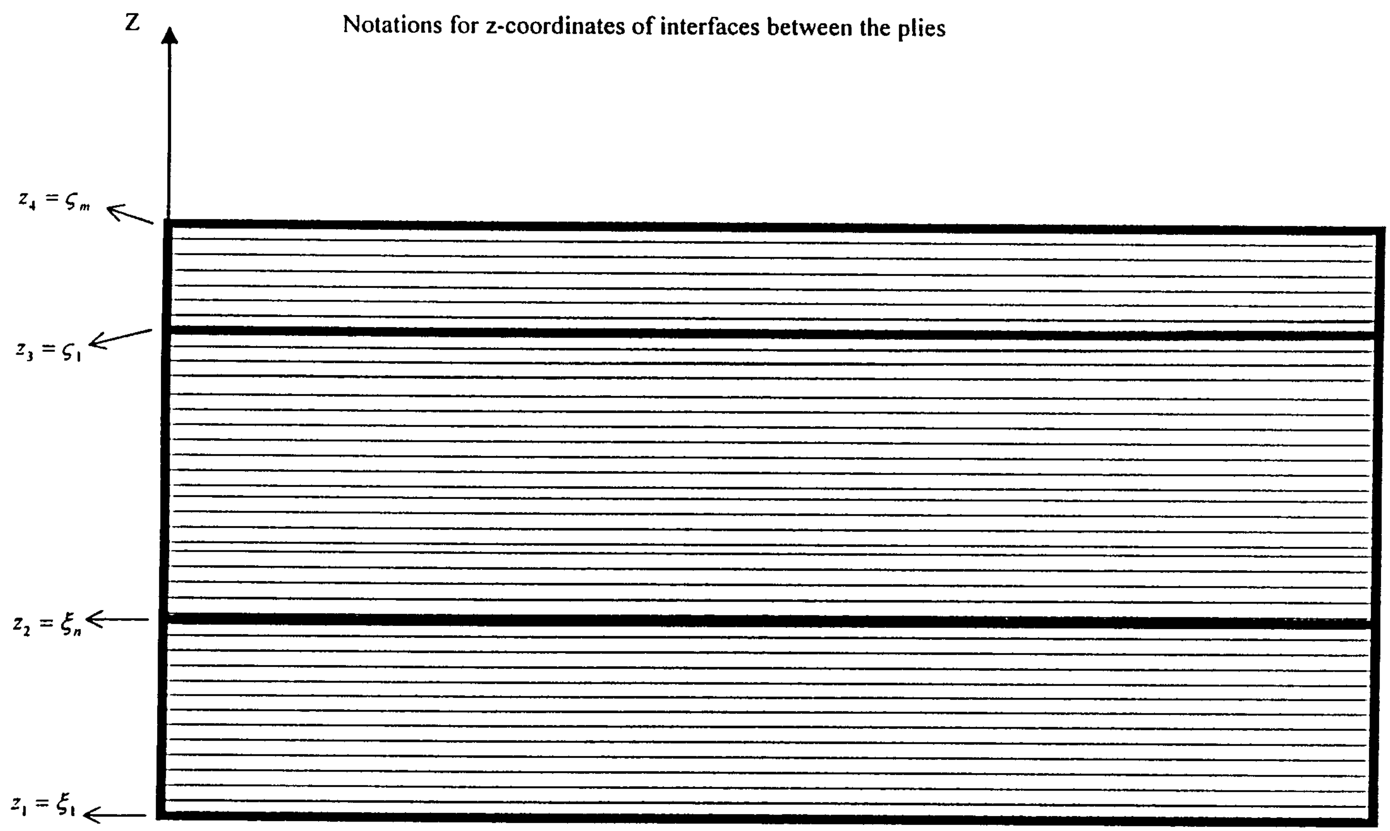
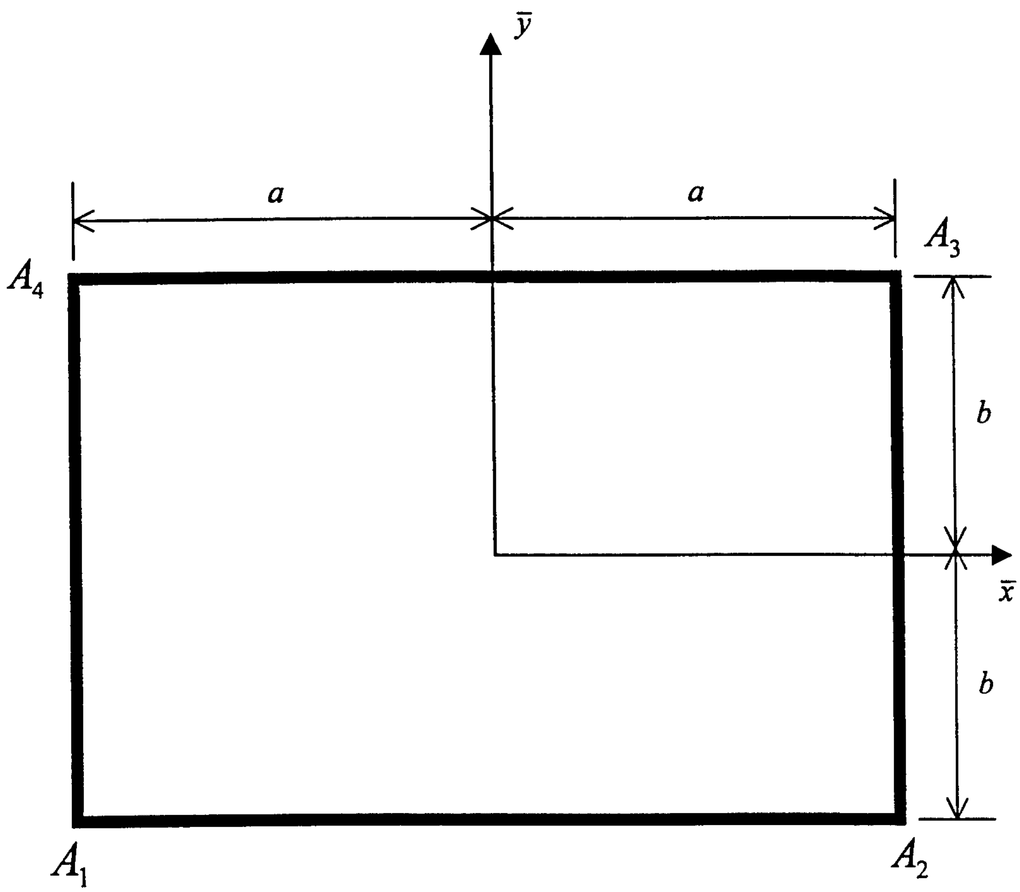


Figure 3.4
A rectangular finite element and the element (local) coordinate system



Chapter 4

A Simplified Approach to the Analysis of Sandwich Plates

In this chapter, a simplified approach to modelling the sandwich plates will be considered. This simplified approach is similar to the one presented in chapter 2, section 2.4, for the sandwich plate in cylindrical bending with homogeneous isotropic face sheets and the core. It is based on assuming that in the in the expression for the strain energy, the transverse strains in the face sheets are negligibly small. The transverse stresses are computed by integration of equilibrium equations, and they can be substituted into the strain-stress relations to obtain the second form of the transverse strains, that are not equal to zero. As it was shown in section 2.4 of the chapter 2 for a sandwich plate with homogeneous isotropic face sheets and core, the stresses produced by the simplified layerwise model are sufficiently close to the stresses obtained from the exact elasticity solutions, though the accuracy of stress computation is slightly lower than in the nonsimplified model presented in chapter 3. The advantage of this simplified model is that it has fewer unknown functions and fewer degrees of freedom in the finite element formulation.

4.1 Simplifying assumptions and the unknown functions

We will assume that in the expression for the strain energy of the core the transverse strains do not depend on the z-coordinate, and the **transverse strains of the face sheets are negligibly**

small:

$$\left.\begin{array}{r} \varepsilon_{xz}^{(1)}=0,\ \varepsilon_{yz}^{(1)}=0,\ \varepsilon_{zz}^{(1)}=0\ , \\ \varepsilon_{xz}^{(3)}=0,\ \varepsilon_{yz}^{(3)}=0,\ \varepsilon_{zz}^{(3)}=0\ , \\ \varepsilon_{xz}^{(2)}=\varepsilon_{xz}^{(2)}\left(x,y,t\right),\ \varepsilon_{yz}^{(2)}=\varepsilon_{yz}^{(2)}\left(x,y,t\right),\ \varepsilon_{zz}^{(2)}=\varepsilon_{zz}^{(2)}\left(x,y,t\right)\ . \end{array}\right\} \tag{4.1.1}$$

It is assumed also that at each point of the sandwich plate **there is a plane of elastic symmetry parallel to x-z plane**. This occurs if the sublaminates of the sandwich plate are cross-ply, specially orthotropic or isotropic. Besides, an account will be taken of the fact that in the problem of the cargo platform dropped on elastic foundation, there are **no external in-plane forces**, acting on the platform, and the **Poisson ratio of the core is small.** Due to the last three limitations of the problem, described in bold type, the in-plane middle surface displacements can be set equal to zero:

$$u_0=0,\ v_0=0\ . \tag{4.1.2}$$

So, the unknown functions of the problem are:

$$w_0\left(x,y,t\right),\ \varepsilon_{xz}^{(2)}\left(x,y,t\right),\ \varepsilon_{yz}^{(2)}\left(x,y,t\right),\ \varepsilon_{zz}^{(2)}\left(x,y,t\right)\ . \tag{4.1.3}$$

4.2 Displacements in terms of the unknown functions

Setting the transverse strains in the face sheets ($\varepsilon_{xz}^{(1)}$, $\varepsilon_{yz}^{(1)}$, $\varepsilon_{zz}^{(1)}$, $\varepsilon_{xz}^{(3)}$, $\varepsilon_{yz}^{(3)}$, $\varepsilon_{zz}^{(3)}$) equal to zero, one can obtain from formulas of chapter 5 the following expressions:

$$w^{(1)}\left(x,y,t\right)=w_0\left(x,y,t\right)+\varepsilon_{zz}^{(2)}\left(x,y,t\right)z_2 \qquad \left(z_1\le z\le z_2\right)\ , \tag{4.2.1}$$

$$w^{(2)}\left(x,y,z,t\right)=w_0\left(x,y,t\right)+\varepsilon_{zz}^{(2)}\left(x,y,t\right)z \qquad \left(z_2\le z\le z_3\right)\ , \tag{4.2.2}$$

$$w^{(3)}\left(x,y,t\right)=w_0\left(x,y,t\right)+\varepsilon_{zz}^{(2)}\left(x,y,t\right)z_3 \qquad \left(z_3\le z\le z_4\right), \tag{4.2.3}$$

$$u^{(1)}\left(x,y,z,t\right)=\left(2\varepsilon_{xz}^{(2)}-w_{0,x}\right)z_2-\frac{1}{2}\varepsilon_{zz,x}^{(2)}\,z_2^2-\left(\varepsilon_{zz,x}^{(2)}\,z_2+w_{0,x}\right)\left(z-z_2\right) \qquad \left(z_1\le z\le z_2\right)\ , \tag{4.2.4}$$

$$u^{(2)}(x,y,z,t) = \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right) z - \frac{1}{2}\varepsilon_{zz,x}^{(2)}\, z^2 \qquad (z_2 \le z \le z_3)\ , \tag{4.2.5}$$

$$u^{(3)}(x,y,z,t) = \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right) z_3 - \frac{1}{2}\varepsilon_{zz,x}^{(2)}\, z_3^2 - \left(w_{0,x} + \varepsilon_{zz,x}^{(2)} z_3\right)(z - z_3) \qquad (z_3 \le z \le z_4)\ , \tag{4.2.6}$$

$$v^{(1)}(x,y,z,t) = \left(2\varepsilon_{yz}^{(2)} - w_{0,y}\right) z_2 - \frac{1}{2}\varepsilon_{zz,y}^{(2)}\, z_2^2 - \left(w_{0,y} + \varepsilon_{zz,y}^{(2)}\, z_2\right)(z - z_2) \qquad (z_1 \le z \le z_2)\ , \tag{4.2.7}$$

$$v^{(2)}(x,y,z,t) = \left(2\varepsilon_{yz}^{(2)} - w_{0,y}\right) z - \frac{1}{2}\varepsilon_{zz,y}^{(2)}\, z^2 \qquad (z_2 \le z \le z_3)\ , \tag{4.2.8}$$

$$v^{(3)}(x,y,z,t) = \left(2\varepsilon_{yz}^{(2)} - w_{0,y}\right) z_3 - \frac{1}{2}\varepsilon_{zz,y}^{(2)}\, z_3^2 - \left(w_{0,y} + \varepsilon_{zz,y}^{(2)} z_3\right)(z - z_3) \qquad (z_3 \le z \le z_4)\ . \tag{4.2.9}$$

These relations can be written in the form:

$$\begin{Bmatrix} u^{(k)} \\ v^{(k)} \\ w^{(k)} \end{Bmatrix} = \left[\ \widetilde{Z}^{(k)}\right]\left[\ \widetilde{\partial}^{(k)}\ \right] \underset{(4\times 1)}{\{f\}} \qquad (k = 1,2,3)\ , \tag{4.2.10}$$

where

$$\{f\} = \begin{Bmatrix} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{yz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{Bmatrix}, \tag{4.2.11}$$

$$\underset{(3\times 5)}{\left[\ \widetilde{Z}^{(1)}\right]} = \underset{(3\times 5)}{\left[\ \widetilde{Z}^{(3)}\right]} = \begin{bmatrix} 1 & z & 0 & 0 & 0 \\ 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.2.12}$$

$$\underset{(3\times 6)}{\left[\ \widetilde{Z}^{(2)}\right]} = \underset{(3\times 6)}{\begin{bmatrix} z & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & z \end{bmatrix}}, \tag{4.2.13}$$

$$\underset{(5\times4)}{\left[\,\widetilde{\partial}^{(1)}\right]} = \begin{bmatrix} 0 & 2z_2 & 0 & \frac{1}{2}z_2^2\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x} & 0 & 0 & -z_2\frac{\partial}{\partial x} \\ 0 & 0 & 2z_2 & \frac{1}{2}z_2^2\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial y} & 0 & 0 & -z_2\frac{\partial}{\partial y} \\ 1 & 0 & 0 & z_2 \end{bmatrix}, \tag{4.2.14}$$

$$\underset{(6\times4)}{\left[\,\widetilde{\partial}^{(2)}\right]} = \underset{(6\times4)}{\begin{bmatrix} -\frac{\partial}{\partial x} & 2 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & 0 & 2 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\frac{\partial}{\partial y} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}, \tag{4.2.15}$$

$$\underset{(5\times4)}{\left[\,\widetilde{\partial}^{(3)}\right]} = \underset{(5\times4)}{\begin{bmatrix} 0 & 2z_3 & 0 & \frac{1}{2}z_3^2\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial x} & 0 & 0 & -z_3\frac{\partial}{\partial x} \\ 0 & 0 & 2z_3 & \frac{1}{2}z_3^2\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial y} & 0 & 0 & -z_3\frac{\partial}{\partial y} \\ 1 & 0 & 0 & z_3 \end{bmatrix}}. \tag{4.2.16}$$

4.3 Strains in terms of the unknown functions

Setting $\varepsilon_{xz}^{(1)} = 0$, $\varepsilon_{yz}^{(1)} = 0$, $\varepsilon_{zz}^{(1)} = 0$, $\varepsilon_{xz}^{(3)} = 0$, $\varepsilon_{yz}^{(3)} = 0$, $\varepsilon_{zz}^{(3)} = 0$ in expressions (3.5.6), (3.5.9) and (3.5.12) we obtain

$$\left\{\varepsilon^{(k)}\right\} = \left[Z^{(k)}\right]\left(\left[\partial^{(k)}\right]\underset{(4\times 1)}{\{f\}} + \left\{\eta^{(k)}\right\}\right) \qquad (k = 1,2,3) \ , \tag{4.3.1}$$

where

$$\{f\} = \left\{\begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{yz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array}\right\}, \tag{4.3.2}$$

$$\underset{(3\times 1)}{\left\{\varepsilon^{(1)}\right\}} \equiv \left\{\begin{array}{c} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ 2\varepsilon_{xy}^{(1)} \end{array}\right\}, \tag{4.3.3}$$

$$\underset{(3\times 8)}{\left[Z^{(1)}\right]} = \underset{(3\times 8)}{\left[Z^{(3)}\right]} = \begin{bmatrix} 1 & z & z^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z \end{bmatrix}, \tag{4.3.4}$$

$$\underset{(8\times 4)}{\left[\partial^{(1)}\right]} = \begin{bmatrix} 0 & 2z_2\frac{\partial}{\partial x} & 0 & \frac{1}{2}z_2^2\frac{\partial^2}{\partial x^2} \\ 0 & 0 & 0 & -z_2\frac{\partial^2}{\partial x^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2z_2\frac{\partial}{\partial y} & \frac{1}{2}z_2^2\frac{\partial^2}{\partial y^2} \\ -\frac{\partial^2}{\partial y^2} & 0 & 0 & -z_2\frac{\partial^2}{\partial y^2} \\ 0 & 0 & 0 & 0 \\ 0 & 2z_2\frac{\partial}{\partial y} & 2z_2\frac{\partial}{\partial x} & z_2^2\frac{\partial^2}{\partial x\partial y} \\ -2\frac{\partial^2}{\partial x\partial y} & 0 & 0 & -2z_2\frac{\partial^2}{\partial x\partial y} \end{bmatrix}, \tag{4.3.5}$$

$$\underset{(8\times 1)}{\left\{\eta^{(1)}\right\}} = \left\{\begin{array}{c} \frac{1}{2}\left(w_{0,x} + z_2\ \varepsilon^{(2)}_{zz,x}\right)^2 \\ 0 \\ 0 \\ \frac{1}{2}\left(w_{0,y} + z_2\ \varepsilon^{(2)}_{zz,y}\right)^2 \\ 0 \\ 0 \\ \left(w_{0,x} + z_2\ \varepsilon^{(2)}_{zz,x}\right)\left(w_{0,y} + z_2\ \varepsilon^{(2)}_{zz,y}\right) \\ 0 \end{array}\right\}, \tag{4.3.6}$$

$$\underset{(6\times 1)}{\left\{\varepsilon^{(2)}\right\}} = \left\{\begin{array}{c} \varepsilon^{(2)}_{xx} \\ \varepsilon^{(2)}_{yy} \\ \varepsilon^{(2)}_{zz} \\ 2\varepsilon^{(2)}_{yz} \\ 2\varepsilon^{(2)}_{xz} \\ 2\varepsilon^{(2)}_{xy} \end{array}\right\}, \tag{4.3.7}$$

$$\underset{(6\times 12)}{\left[Z^{(2)}\right]} = \underset{(6\times 12)}{\left[\begin{array}{cccccccccccc} 1 & z & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z & z^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & z & z^2 & 0 & 0 & 0 \end{array}\right]}, \tag{4.3.8}$$

$$\underset{(12\times4)}{\left[\,\partial^{(2)}\,\right]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{\partial^2}{\partial x^2} & 2\frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\frac{\partial^2}{\partial x^2} \\ 0 & 0 & 0 & 0 \\ -\frac{\partial^2}{\partial y^2} & 0 & 2\frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\frac{\partial^2}{\partial y^2} \\ 0 & 0 & 0 & 0 \\ -\frac{\partial^2}{\partial x\partial y} & 2\frac{\partial}{\partial y} & 2\frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & -\frac{\partial^2}{\partial x\partial y} \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.3.9}$$

$$\underset{(12\times1)}{\left\{\eta^{(2)}\right\}} = \underset{(12\times1)}{\left\{ \begin{array}{c} \frac{1}{2}(w_{0,x})^2 \\ w_{0,x}\varepsilon^{(2)}_{zz,x} \\ \frac{1}{2}\left(\varepsilon^{(2)}_{zz,x}\right)^2 \\ \frac{1}{2}\left(w_{0,y}\right)^2 \\ w_{0,y}\varepsilon^{(2)}_{zz,y} \\ \frac{1}{2}\left(\varepsilon^{(2)}_{zz,y}\right)^2 \\ w_{0,x}w_{0,y} \\ w_{0,y}\varepsilon^{(2)}_{zz,x} + w_{0,x}\varepsilon^{(2)}_{zz,y} \\ \varepsilon^{(2)}_{zz,x}\varepsilon^{(2)}_{zz,y} \\ 0 \\ 0 \\ 0 \end{array} \right\}}, \tag{4.3.10}$$

$$\left\{\varepsilon^{(3)}\right\} = \underset{(3\times1)}{\left\{ \begin{array}{c} \varepsilon^{(3)}_{xx} \\ \varepsilon^{(3)}_{yy} \\ 2\varepsilon^{(3)}_{xy} \end{array} \right\}}, \tag{4.3.11}$$

$$\underset{(8\times4)}{\left[\partial^{(3)}\right]} = \underset{(8\times4)}{\begin{bmatrix} 0 & 2z_3\frac{\partial}{\partial x} & 0 & \frac{1}{2}z_3^2\frac{\partial^2}{\partial x^2} \\ -\frac{\partial^2}{\partial x^2} & 0 & 0 & -z_3\frac{\partial^2}{\partial x^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2z_3\frac{\partial}{\partial y} & \frac{1}{2}z_3^2\frac{\partial^2}{\partial y^2} \\ -\frac{\partial^2}{\partial y^2} & 0 & 0 & -z_3\frac{\partial^2}{\partial y^2} \\ 0 & 0 & 0 & 0 \\ 0 & 2z_3\frac{\partial}{\partial y} & 2z_3\frac{\partial}{\partial x} & z_3^2\frac{\partial^2}{\partial x\partial y} \\ -2\frac{\partial^2}{\partial x\partial y} & 0 & 0 & -2z_3\frac{\partial^2}{\partial x\partial y} \end{bmatrix}}, \tag{4.3.12}$$

$$\left[\eta^{(3)}\right] = \underset{(8\times1)}{\begin{Bmatrix} \frac{1}{2}\left(w_{0,x} + z_3\ \varepsilon^{(2)}_{zz,x}\right)^2 \\ 0 \\ 0 \\ \frac{1}{2}\left(w_{0,y} + z_3\ \varepsilon^{(2)}_{zz,y}\right)^2 \\ 0 \\ 0 \\ \left(w_{0,x} + z_3\ \varepsilon^{(2)}_{zz,x}\right)\left(w_{0,y} + z_3\ \varepsilon^{(2)}_{zz,y}\right) \\ 0 \end{Bmatrix}}. \tag{4.3.13}$$

The transverse shear strains in the core $\varepsilon^{(2)}_{xz}$ and $\varepsilon^{(2)}_{yz}$, that enter into the expressions for the strain energy (the first form of the transverse shear strains) are assumed to be constant through the thickness of core. Therefore, the transverse shear stresses, computed from the stress-strain relations (first form of the transverse shear stresses) are also constant in the thickness direction. On the other hand, the same stresses computed in the post-processing stage by integration of the equations of motion, vary nonlinearly in the thickness direction. Besides, it is well known from elementary theory of homogeneous beams that the transverse shear stress varies parabolically through the beam thickness. In composite laminated beams and plates, the transverse shear stresses vary at least quadratically through layer thickness. This discrepancy between the transverse stresses computed from the Hooke's law on the one hand and from the equations of motion or exact solutions on the other hand, is often corrected (especially in the first order shear deformation theory) by multiplying the transverse shear strain energy by the shear correction coefficient. In the theory of the sandwich plate discussed in this chapter, we will introduce a shear correction coefficient, which, at first, will be

set equal to unity. If, with the shear correction coefficient equal to one, the results of the sandwich plate theory for the transverse stresses obtained by integration of equilibrium equations turn out to be close enough to the known exact elasticity solutions, then the further search for an optimal value of the shear correction factor may not be necessary. Otherwise, the shear correction coefficient can be determined by a method, presented in the paper of Whitney (1973).

In order to introduce the shear correction coefficient, it is convenient to divide the column-matrix of strains in the core $\underset{(6\times1)}{\left\{\varepsilon^{(2)}\right\}} = \left\lfloor \varepsilon_{xx}^{(2)} \quad \varepsilon_{yy}^{(2)} \quad \varepsilon_{zz}^{(2)} \quad 2\varepsilon_{yz}^{(2)} \quad 2\varepsilon_{xz}^{(2)} \quad 2\varepsilon_{xy}^{(2)} \right\rfloor^T$ into two parts: a part that contains the transverse shear strains: $\left\lfloor 2\varepsilon_{yz}^{(2)} \quad 2\varepsilon_{xz}^{(2)} \right\rfloor^T$, and the part that contains all the other strains in the core: $\left\lfloor \varepsilon_{xx}^{(2)} \quad \varepsilon_{yy}^{(2)} \quad \varepsilon_{zz}^{(2)} \quad 2\varepsilon_{xy}^{(2)} \right\rfloor^T$. Then, equation (4.3.1) with k=2, i.e. equation $\left\{\varepsilon^{(2)}\right\} = \left[Z^{(2)}\right]\left(\left[\partial^{(2)}\right]\{f\} + \left\{\eta^{(2)}\right\}\right)$ can be written as two separate matrix equations:

$$\left\{ \begin{array}{c} 2\varepsilon_{yz}^{(2)} \\ 2\varepsilon_{xz}^{(2)} \end{array} \right\} = \left[\begin{array}{cccc} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array} \right] \underset{(4\times1)}{\{f\}} \tag{4.3.14}$$

and

$$\left\{ \begin{array}{c} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ 2\varepsilon_{xy}^{(2)} \end{array} \right\} = \underset{(4\times10)}{\left[\widehat{Z}^{(2)}\right]} \left(\underset{(10\times4)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(4\times1)}{\{f\}} + \underset{(10\times1)}{\left[\widehat{\eta}^{(2)}\right]} \right), \tag{4.3.15}$$

where

$$\underset{(4\times10)}{\left[\widehat{Z}^{(2)}\right]} = \left[\begin{array}{cccccccccc} 1 & z & z^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & z & z^2 & 0 & 0 \end{array} \right], \tag{4.3.16}$$

$$
\underset{(10\times 4)}{\left[\widehat{\partial}^{(2)}\right]} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-\frac{\partial^2}{\partial x^2} & 2\frac{\partial}{\partial x} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2}\frac{\partial^2}{\partial x^2} \\
0 & 0 & 0 & 0 \\
-\frac{\partial^2}{\partial y^2} & 0 & 2\frac{\partial}{\partial y} & 0 \\
0 & 0 & 0 & -\frac{1}{2}\frac{\partial^2}{\partial y^2} \\
0 & 0 & 0 & 0 \\
-\frac{\partial^2}{\partial x \partial y} & 2\frac{\partial}{\partial y} & 2\frac{\partial}{\partial x} & 0 \\
0 & 0 & 0 & -\frac{\partial^2}{\partial x \partial y} \\
0 & 0 & 0 & 1
\end{bmatrix}, \tag{4.3.17}
$$

$$
\underset{(10\times 1)}{\left[\widehat{\eta}^{(2)}\right]} = \left\{ \begin{array}{c}
\frac{1}{2}(w_{0,x})^2 \\
w_{0,x}\varepsilon^{(2)}_{zz,x} \\
\frac{1}{2}\left(\varepsilon^{(2)}_{zz,x}\right)^2 \\
\frac{1}{2}\left(w_{0,y}\right)^2 \\
w_{0,y}\varepsilon^{(2)}_{zz,y} \\
\frac{1}{2}\left(\varepsilon^{(2)}_{zz,y}\right)^2 \\
w_{0,x}w_{0,y} \\
w_{0,y}\varepsilon^{(2)}_{zz,x} + w_{0,x}\varepsilon^{(2)}_{zz,y} \\
\varepsilon^{(2)}_{zz,x}\varepsilon^{(2)}_{zz,y} \\
0
\end{array} \right\}. \tag{4.3.18}
$$

4.4 Stress-strain relations

For the **lower and upper face sheets** (k=1 and k=3), where, according to our assumptions, $\varepsilon_{zz}^{(k)} = \varepsilon_{xz}^{(k)} = \varepsilon_{yz}^{(k)} = 0$ (k=1,3), the constitutive equations (3.6.13) take the form

$$\left\{\begin{array}{c} {}^H\sigma_{xx} \\ {}^H\sigma_{yy} \\ {}^H\sigma_{xy} \end{array}\right\}^{(k)} = \left[\begin{array}{ccc} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{array}\right]^{(k)} \left\{\begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{array}\right\}^{(k)} \qquad (k=1,3)\,,$$

$${}^H\sigma_{zz}^{(k)} = \left\lfloor \begin{array}{ccc} \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{36} \end{array} \right\rfloor^{(k)} \left\{\begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{array}\right\}^{(k)} \qquad k=(1,3), \tag{4.4.1}$$

$${}^H\sigma_{xz}^{(k)} = {}^H\sigma_{yz}^{(k)} = 0 \qquad (k=1,3)\,.$$

The constitutive equations for the **core** are

$$\left\{\begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \end{array}\right\}^{(2)} = \left[\begin{array}{cccc} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & \bar{C}_{36} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & \bar{C}_{66} \end{array}\right]^{(2)} \left\{\begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \end{array}\right\}^{(2)}, \tag{4.4.2}$$

$$\left\{\begin{array}{c} \sigma_{yz} \\ \sigma_{xz} \end{array}\right\}^{(2)} = \left[\begin{array}{cc} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{array}\right]^{(2)} \left\{\begin{array}{c} 2\varepsilon_{yz} \\ 2\varepsilon_{xz} \end{array}\right\}^{(2)}. \tag{4.4.3}$$

4.5 Strain energy of the core

The strain energy of the core is defined by expression

$$U^{(2)} = \frac{1}{2}\int_0^B\int_0^L\int_{z_2}^{z_3} \begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ 2\varepsilon_{xy}^{(2)} \end{Bmatrix}^T \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & \bar{C}_{36} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & \bar{C}_{66} \end{bmatrix}^{(2)} \begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \varepsilon_{zz}^{(2)} \\ 2\varepsilon_{xy}^{(2)} \end{Bmatrix} dz\ dx\ dy +$$

$$+\frac{1}{2}k_c\int_0^B\int_0^L\int_{z_2}^{z_3} \begin{Bmatrix} 2\varepsilon_{yz}^{(2)} \\ 2\varepsilon_{xz}^{(2)} \end{Bmatrix}^T \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(2)} \begin{Bmatrix} 2\varepsilon_{yz}^{(2)} \\ 2\varepsilon_{xz}^{(2)} \end{Bmatrix} dz\ dx\ dy\ , \qquad (4.5.1)$$

where k_c is shear correction factor.

Substitution of expressions (4.3.13) and (4.3.14) into the last expression yields

$$U^{(2)} = \frac{1}{2}\int_0^B\int_0^L \underset{(1\times 10)}{\left(\underset{(10\times 4)}{\left[\widehat{\partial}^{(2)}\right]}\underset{(4\times 1)}{\{f\}} + \underset{(10\times 1)}{\left[\widehat{\eta}^{(2)}\right]}\right)^T} \times$$

$$\times \left(\int_{z_2}^{z_3} \underset{(10\times 4)}{\left[\widehat{Z}^{(2)}\right]^T} \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & \bar{C}_{36} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & \bar{C}_{66} \end{bmatrix}^{(2)} \underset{(4\times 10)}{\left[\widehat{Z}^{(2)}\right]} dz\right) \times$$

$$\times \left(\underset{(10\times 4)}{\left[\widehat{\partial}^{(2)}\right]}\underset{(4\times 1)}{\{f\}} + \underset{(10\times 1)}{\left[\widehat{\eta}^{(2)}\right]}\right)\ dx\ dy +$$

$$+\frac{1}{2}k_c\int_0^B\int_0^L \{f\}^T \left(\int_{z_2}^{z_3} \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(2)} \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} dz\right) \underset{(4\times 1)}{\{f\}}\ dx\ dy\ , \qquad (4.5.2)$$

where matrix $\left[\ \widehat{Z}^{(2)}\ \right]$ is defined by expression (4.3.15), matrix $\left[\widehat{\partial}^{(2)}\right]$ - by expression (4.3.16) and matrix $\left[\ \widehat{\eta}^{(2)}\ \right]$ - by expression (4.3.17). So,

$$U^{(2)} = \frac{1}{2}\int\limits_0^B\int\limits_0^L \underset{(1\times 10)}{\left(\underset{(10\times 4)}{\left[\widehat{\partial}^{(2)}\right]}\underset{(4\times 1)}{\{f\}} + \underset{(10\times 1)}{\left[\ \widehat{\eta}^{(2)}\ \right]}\right)^T} \left[\widehat{D}^{(2)}\right] \left(\underset{(10\times 4)}{\left[\widehat{\partial}^{(2)}\right]}\underset{(4\times 1)}{\{f\}} + \underset{(10\times 1)}{\left[\ \widehat{\eta}^{(2)}\ \right]}\right)\ dx\ dy +$$

$$+\frac{1}{2}k_c\int\limits_0^B\int\limits_0^L \{f\}^T \left[\check{D}^{(2)}\right] \underset{(4\times 1)}{\{f\}}\ dx\ dy\ , \tag{4.5.3}$$

where

$$\underset{(10\times 10)}{\left[\widehat{D}^{(2)}\right]} = \int\limits_{z_2}^{z_3} \underset{(10\times 4)}{\left[\ \widehat{Z}^{(2)}\ \right]^T} \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{26} \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & \bar{C}_{36} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & \bar{C}_{66} \end{bmatrix}^{(2)} \underset{(4\times 10)}{\left[\ \widehat{Z}^{(2)}\ \right]} dz\ , \tag{4.5.4}$$

$$\left[\check{D}^{(2)}\right] = \int\limits_{z_2}^{z_3} \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(2)} \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} dz\ . \tag{4.5.5}$$

4.6 Strain energy of the face sheets

The strain energy of the face sheets is defined by expression

$$U^{(k)} = \frac{1}{2}\int_0^B\int_0^L\int_{z_k}^{z_{k+1}} \begin{Bmatrix} \varepsilon_{xx}^{(k)} \\ \varepsilon_{yy}^{(k)} \\ 2\varepsilon_{xy}^{(k)} \end{Bmatrix}^T \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^{(k)} \\ \varepsilon_{yy}^{(k)} \\ 2\varepsilon_{xy}^{(k)} \end{Bmatrix} dz\, dx\, dy \qquad (k=1,3) \qquad (4.6.1)$$

or

$$U^{(k)} = \frac{1}{2}\int_0^B\int_0^L\int_{z_k}^{z_{k+1}} \underset{(1\times 3)}{\left\{\varepsilon^{(k)}\right\}^T} \underset{(3\times 3)}{\left[C^{(k)}\right]} \underset{(3\times 1)}{\left\{\varepsilon^{(k)}\right\}}\, dz\, dx\, dy \qquad (k=1,3)\ . \qquad (4.6.2)$$

Substituting (4.3.1) into (4.6.2), we obtain

$$U^{(k)} = \frac{1}{2}\int_0^B\int_0^L \left(\underset{(8\times 4)}{\left[\partial^{(k)}\right]} \underset{(4\times 1)}{\{f\}} + \underset{(8\times 1)}{\left\{\eta^{(k)}\right\}} \right)^T \times$$

$$\times \int_{z_k}^{z_{k+1}} \underset{(8\times 3)}{\left[Z^{(k)}\right]}^T \underset{(3\times 3)}{\left[C^{(k)}\right]} \underset{(3\times 8)}{\left[Z^{(k)}\right]} dz \times$$

$$\times \left(\underset{(8\times 4)}{\left[\partial^{(k)}\right]} \underset{(4\times 1)}{\{f\}} + \underset{(8\times 1)}{\left\{\eta^{(k)}\right\}} \right) dx\, dy \qquad (k=1,3) \qquad (4.6.3)$$

or

$$U^{(k)} = \frac{1}{2}\int_0^B\int_0^L \left(\underset{(8\times 4)}{\left[\partial^{(k)}\right]} \underset{(4\times 1)}{\{f\}} + \underset{(8\times 1)}{\left\{\eta^{(k)}\right\}} \right)^T \underset{(8\times 8)}{\left[D^{(k)}\right]} \left(\underset{(8\times 4)}{\left[\partial^{(k)}\right]} \underset{(4\times 1)}{\{f\}} + \underset{(8\times 1)}{\left\{\eta^{(k)}\right\}} \right) dx\, dy \qquad (k=1,3)\,, \qquad (4.6.4)$$

where matrices $\left[Z^{(1)}\right]$and $\left[Z^{(3)}\right]$ are defined by equation (4.3.4), matrix $\left[\partial^{(1)}\right]$ - by equation (4.3.5), matrix $\left[\partial^{(3)}\right]$ - by equation (4.3.12), $\left[\eta^{(1)}\right]$ - by (4.3.6), $\left[\eta^{(3)}\right]$ - by (4.3.13).

Like in chapter 3, a stiffness coefficient in the Hooke's law for a ply of the lower face sheet, in the laminate coordinate system, will be denoted by ${}^{\alpha}\overline{C}_{ij}^{(1)}$,where the right superscript (1) denotes that a stiffness coefficient is associated with the 1-st sublaminate (i.e. the lower face sheet), the left superscript α is a number of a ply in the lower face sheet, subscripts i and j denote a position of

the stiffness coefficient in the stiffness matrix. The stiffness matrix with components ${}^{\alpha}C_{ij}^{(1)}$ will be denoted as $\left[\overline{C}_{\alpha}^{(1)}\right]$ i. e.

$$\left[\overline{C}_{\alpha}^{(1)}\right] = \begin{bmatrix} {}^{\alpha}\bar{C}_{11}^{(1)} & {}^{\alpha}\bar{C}_{12}^{(1)} & {}^{\alpha}\bar{C}_{16}^{(1)} \\ {}^{\alpha}\bar{C}_{12}^{(1)} & {}^{\alpha}\bar{C}_{22}^{(1)} & {}^{\alpha}\bar{C}_{26}^{(1)} \\ {}^{\alpha}\bar{C}_{16}^{(1)} & {}^{\alpha}\bar{C}_{26}^{(1)} & {}^{\alpha}\bar{C}_{66}^{(1)} \end{bmatrix}.$$

Analogously, a stiffness coefficient in the Hooke's law for a ply of the upper face sheet will be denoted by ${}^{\alpha}\overline{C}_{ij}^{(3)}$, and the matrix of these coefficients - by $\left[\overline{C}_{\alpha}^{(3)}\right]$, i. e.

$$\left[\overline{C}_{\alpha}^{(3)}\right] = \begin{bmatrix} {}^{\alpha}\bar{C}_{11}^{(3)} & {}^{\alpha}\bar{C}_{12}^{(3)} & {}^{\alpha}\bar{C}_{16}^{(3)} \\ {}^{\alpha}\bar{C}_{12}^{(3)} & {}^{\alpha}\bar{C}_{22}^{(3)} & {}^{\alpha}\bar{C}_{26}^{(3)} \\ {}^{\alpha}\bar{C}_{16}^{(3)} & {}^{\alpha}\bar{C}_{26}^{(3)} & {}^{\alpha}\bar{C}_{66}^{(3)} \end{bmatrix}.$$

Let n be a number of plies in the lower face sheet and let

$$\xi_1 = z_1,\ \xi_2,\ \xi_3,\ ...,\ \xi_n = z_2$$

be z-coordinates of the interfaces between the plies of the lower face sheet (Figure 3.3). Also, let m be a number of plies in the upper face sheet and let

$$\zeta_1 = z_3,\ \zeta_2,\ \zeta_3,\ ...,\ \zeta_m = z_4$$

be z-coordinates of the interfaces between the plies of the upper face sheet. Then

$$\underset{(8\times8)}{\left[D^{(1)}\right]} = \int_{z_1}^{z_2} \underset{(8\times3)}{\left[Z^{(1)}\right]^T} \underset{(3\times3)}{\left[\overline{C}^{(1)}\right]} \underset{(3\times8)}{\left[Z^{(1)}\right]} dz = \sum_{\alpha=1}^{n} \int_{\xi_\alpha}^{\xi_{\alpha+1}} \underset{(8\times3)}{\left[Z^{(1)}\right]^T} \underset{(3\times3)}{\left[\overline{C}^{(1)}\right]} \underset{(3\times8)}{\left[Z^{(1)}\right]} dz =$$

$$= \sum_{\alpha=1}^{n} \int_{\xi_\alpha}^{\xi_{\alpha+1}} \left[\ Z^{(1)}\ \right]^T \left[\ \overline{C}_{\alpha}^{(1)}\right] \left[\ Z^{(1)}\ \right]\ dz \tag{4.6.5}$$

and

$$\underset{(8\times8)}{\left[D^{(3)}\right]} = \int_{z_3}^{z_4} \underset{(8\times3)}{\left[Z^{(3)}\right]^T} \underset{(3\times3)}{\left[\overline{C}^{(3)}\right]} \underset{(3\times8)}{\left[Z^{(3)}\right]} dz = \sum_{\alpha=1}^{m} \int_{\zeta_\alpha}^{\zeta_{\alpha+1}} \underset{(8\times3)}{\left[Z^{(3)}\right]^T} \underset{(3\times3)}{\left[\overline{C}^{(3)}\right]} \underset{(3\times8)}{\left[Z^{(3)}\right]} dz =$$

$$= \sum_{\alpha=1}^{m} \int_{\zeta_\alpha}^{\zeta_{\alpha+1}} \underset{(8\times3)}{\left[Z^{(3)}\right]^T} \underset{(3\times3)}{\left[\overline{C}_{\alpha}^{(3)}\right]} \underset{(3\times8)}{\left[Z^{(3)}\right]} dz. \tag{4.6.6}$$

4.7 Strain energy of the plate

Strain energy of the sandwich plate is the sum of the strain energies of the core and the face sheets:

$$U_p = U^{(1)} + U^{(2)} + U^{(3)} =$$

$$= \frac{1}{2}\int_0^B\int_0^L \left(\underset{(8\times4)}{\left[\partial^{(1)}\right]} \underset{(4\times1)}{\{f\}} + \underset{(8\times1)}{\left\{\eta^{(1)}\right\}} \right)^T \underset{(8\times8)}{\left[D^{(1)}\right]} \left(\underset{(8\times4)}{\left[\partial^{(1)}\right]} \underset{(4\times1)}{\{f\}} + \underset{(8\times1)}{\left\{\eta^{(1)}\right\}} \right) dx\ dy +$$

$$+ \frac{1}{2}\int_0^B\int_0^L \underset{(1\times10)}{\left(\underset{(10\times4)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(4\times1)}{\{f\}} + \underset{(10\times1)}{\left[\widehat{\eta}^{(2)}\right]} \right)^T} \left[\widehat{D}^{(2)}\right] \left(\underset{(10\times4)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(4\times1)}{\{f\}} + \underset{(10\times1)}{\left[\widehat{\eta}^{(2)}\right]} \right) dx\ dy +$$

$$+ \frac{1}{2}\int_0^B\int_0^L \{f\}^T \left[\check{D}^{(2)}\right] \underset{(4\times1)}{\{f\}}\ dx\ dy +$$

$$+ \frac{1}{2}\int_0^B\int_0^L \left(\underset{(8\times4)}{\left[\partial^{(3)}\right]} \underset{(4\times1)}{\{f\}} + \underset{(8\times1)}{\left\{\eta^{(3)}\right\}} \right)^T \underset{(8\times8)}{\left[D^{(3)}\right]} \left(\underset{(8\times4)}{\left[\partial^{(3)}\right]} \underset{(4\times1)}{\{f\}} + \underset{(8\times1)}{\left\{\eta^{(3)}\right\}} \right) dx\ dy. \tag{4.7.1}$$

4.8 Strain energy of elastic foundation

The strain energy of the elastic foundation is defined by expression

$$U_f = \frac{1}{2}\int_0^B\int_0^L s(x,y)\left[w^{(1)}(x,y,t)\right]^2\ dx\ dy. \tag{4.8.1}$$

According to equation (4.2.1),

$$w^{(1)} = w_0 + \varepsilon_{zz}^{(2)} z_2 \quad (z_1 \le z \le z_2)$$

or

$$w^{(1)} = \left\lfloor\, 1 \quad 0 \quad 0 \quad z_2 \,\right\rfloor \left\{ \begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{yz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\} = \left\lfloor\, 1 \quad 0 \quad 0 \quad z_2 \,\right\rfloor \{f\}. \tag{4.8.2}$$

Then

$$\left(w^{(1)}\right)^2 = \{f\}^T \left\{ \begin{array}{c} 1 \\ 0 \\ 0 \\ z_2 \end{array} \right\} \left\lfloor \begin{array}{cccc} 1 & 0 & 0 & z_2 \end{array} \right\rfloor \{f\}$$

or

$$\left(w^{(1)}\right)^2 = \underset{(1\times4)}{\{f\}}^T \underset{(4\times4)}{[\,\overline{D}\,]} \underset{(4\times1)}{\{f\}}, \tag{4.8.3}$$

where

$$[\,\overline{D}\,] = \begin{bmatrix} 1 & 0 & 0 & z_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ z_2 & 0 & 0 & z_2^2 \end{bmatrix}. \tag{4.8.4}$$

Substitution of equation (4.8.3) into equation (4.8.1) yields

$$U_f = \frac{1}{2}\int_0^B\!\!\int_0^L s\,(x,y)\ \underset{(1\times4)}{\{f\}}^T \underset{(4\times4)}{[\,\overline{D}\,]} \underset{(4\times1)}{\{f\}}\ dx\ dy. \tag{4.8.5}$$

4.9 Potential energy of the platform and the cargo in the gravity field

If we set $\varepsilon_{xz}^{(1)} = \varepsilon_{yz}^{(1)} = \varepsilon_{zz}^{(1)} = \varepsilon_{xz}^{(3)} = \varepsilon_{yz}^{(3)} = \varepsilon_{zz}^{(3)} = 0$, equation (3.9.19) for potential energy of the platform and the cargo in the gravity field takes the form

$$\Pi_{platform} + \Pi_{\mathrm{cargo}} = \int_0^B\!\!\int_0^L \underset{(1\times4)}{\{f\}}^T \underset{(4\times1)}{\{\Gamma\}}\ dx\ dy, \tag{4.9.1}$$

where

$$\{f\} = \left\{ \begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{yz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\},$$

$$\underset{(4\times 1)}{\{\Gamma\}} = \left\{ \begin{array}{c} g\left[\rho^{(1)}\left(z_2 - z_1\right) + \rho^{(2)}\left(z_3 - z_2\right) + \rho^{(3)}\left(z_4 - z_3\right) + \mu\ H\left(x,y\right)\right] \\ 0 \\ 0 \\ g\left[\rho^{(1)} z_2\left(z_2 - z_1\right) + \frac{1}{2}\rho^{(2)}\left(z_3^2 - z_2^2\right) + \rho^{(3)}\ z_3\left(z_4 - z_3\right) + \mu\ H\left(x,y\right) z_3\right] \end{array} \right\}. \tag{4.9.2}$$

4.10 Kinetic Energy of the Platform

The kinetic energy of the platform of the k-th sublaminate (k=1, 2, 3), i. e. the kinetic energy of either one of the face sheets or of the core is

$$K^{(k)} = \frac{1}{2}\rho^{(k)} \int_0^B \int_0^L \int_{z_k}^{z_{k+1}} \left\{ \begin{array}{c} \dot{u}^{(k)} \\ \dot{v}^{(k)} \\ \dot{w}^{(k)} \end{array} \right\}^T \left\{ \begin{array}{c} \dot{u}^{(k)} \\ \dot{v}^{(k)} \\ \dot{w}^{(k)} \end{array} \right\} dz\ dx\ dy. \tag{4.10.1}$$

According to equation (4.2.10),

$$\left\{ \begin{array}{c} \dot{u}^{(k)} \\ \dot{v}^{(k)} \\ \dot{w}^{(k)} \end{array} \right\} = \left[\widetilde{Z}^{(k)}\right]\left[\widetilde{\partial}^{(k)}\right]\frac{\partial}{\partial t}\{f\}. \tag{4.10.2}$$

Therefore,

$$K^{(k)} = \frac{1}{2}\rho^{(k)} \int_0^B \int_0^L \left(\left[\widetilde{\partial}^{(k)}\right]\frac{\partial}{\partial t}\{f\}\right)^T \left[\widetilde{D}^{(k)}\right]\left(\left[\widetilde{\partial}^{(k)}\right]\frac{\partial}{\partial t}\{f\}\right)\ dx\ dy, \tag{4.10.3}$$

where

$$\left[\widetilde{D}^{(k)}\right] = \int_{z_k}^{z_{k+1}} \left[\widetilde{Z}^{(k)}\right]^T \left[\widetilde{Z}^{(k)}\right]\ dz \qquad (k = 1, 2, 3). \tag{4.10.4}$$

Substitution of (4.2.12) and (4.2.13) into (4.10.4) yields

$$\left[\widetilde{D}^{(1)}\right] = \int_{z_1}^{z_2} \begin{bmatrix} 1 & 0 & 0 \\ z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & z & 0 & 0 & 0 \\ 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} dz =$$

$$
= \begin{bmatrix} z_2 - z_1 & \frac{1}{2}\left(z_2^2 - z_1^2\right) & 0 & 0 & 0 \\ \frac{1}{2}\left(z_2^2 - z_1^2\right) & \frac{1}{3}\left(z_2^3 - z_1^3\right) & 0 & 0 & 0 \\ 0 & 0 & z_2 - z_1 & \frac{1}{2}\left(z_2^2 - z_1^2\right) & 0 \\ 0 & 0 & \frac{1}{2}\left(z_2^2 - z_1^2\right) & \frac{1}{3}\left(z_2^3 - z_1^3\right) & 0 \\ 0 & 0 & 0 & 0 & z_2 - z_1 \end{bmatrix}. \tag{4.10.5}
$$

$$
\left[\widetilde{D}^{(2)}\right] = \int_{z_2}^{z_3} \begin{bmatrix} z & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & z \end{bmatrix}^T \begin{bmatrix} z & z^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & z^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & z \end{bmatrix} =
$$

$$
= \begin{bmatrix} \frac{1}{3}\left(z_3^3 - z_2^3\right) & \frac{1}{4}\left(z_3^4 - z_2^4\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) \\ \frac{1}{4}\left(z_3^4 - z_2^4\right) & \frac{1}{5}\left(z_3^5 - z_2^5\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) \\ \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{3}\left(z_3^3 - z_2^3\right) & \frac{1}{4}\left(z_3^4 - z_2^4\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) \\ \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{4}\left(z_3^4 - z_2^4\right) & \frac{1}{5}\left(z_3^5 - z_2^5\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) \\ \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & z_3 - z_2 & \frac{1}{2}\left(z_3^2 - z_2^2\right) \\ \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{3}\left(z_3^3 - z_2^3\right) \end{bmatrix}. \tag{4.10.6}
$$

$$
\left[\widetilde{D}^{(3)}\right] = \int_{z_3}^{z_4} \begin{bmatrix} 1 & 0 & 0 \\ z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & z & 0 & 0 & 0 \\ 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} dz =
$$

$$
= \begin{bmatrix} z_4 - z_3 & \frac{1}{2}\left(z_4^2 - z_3^2\right) & 0 & 0 & 0 \\ \frac{1}{2}\left(z_4^2 - z_3^2\right) & \frac{1}{3}\left(z_4^3 - z_3^3\right) & 0 & 0 & 0 \\ 0 & 0 & z_4 - z_3 & \frac{1}{2}\left(z_4^2 - z_3^2\right) & 0 \\ 0 & 0 & \frac{1}{2}\left(z_4^2 - z_3^2\right) & \frac{1}{3}\left(z_4^3 - z_3^3\right) & 0 \\ 0 & 0 & 0 & 0 & z_4 - z_3 \end{bmatrix}. \tag{4.10.7}
$$

So, the kinetic energy of the whole sandwich plate is

$$
K_p = K^{(1)} + K^{(2)} + K^{(3)} =
$$

$$= \frac{1}{2}\rho^{(1)} \int_0^B \int_0^L \left(\underset{(5\times4)}{\left[\widetilde{\partial}^{(1)}\right]} \frac{\partial}{\partial t} \underset{(4\times1)}{\{f\}} \right)^T \underset{(5\times5)}{\left[\widetilde{D}^{(1)}\right]} \left(\underset{(5\times4)}{\left[\widetilde{\partial}^{(1)}\right]} \frac{\partial}{\partial t} \underset{(4\times1)}{\{f\}} \right) dx\ dy +$$

$$+ \frac{1}{2}\rho^{(2)} \int_0^B \int_0^L \left(\underset{(6\times4)}{\left[\widetilde{\partial}^{(2)}\right]} \frac{\partial}{\partial t} \underset{(4\times1)}{\{f\}} \right)^T \underset{(6\times6)}{\left[\widetilde{D}^{(2)}\right]} \left(\underset{(6\times4)}{\left[\widetilde{\partial}^{(2)}\right]} \frac{\partial}{\partial t} \underset{(4\times1)}{\{f\}} \right) dx\ dy +$$

$$+ \frac{1}{2}\rho^{(3)} \int_0^B \int_0^L \left(\underset{(5\times4)}{\left[\widetilde{\partial}^{(3)}\right]} \frac{\partial}{\partial t} \underset{(4\times1)}{\{f\}} \right)^T \underset{(5\times5)}{\left[\widetilde{D}^{(3)}\right]} \left(\underset{(5\times4)}{\left[\widetilde{\partial}^{(3)}\right]} \frac{\partial}{\partial t} \underset{(4\times1)}{\{f\}} \right) dx\ dy. \tag{4.10.8}$$

4.11 Kinetic energy of the cargo

According to equation (3.10.4), kinetic energy of the cargo is

$$K_c = \frac{1}{2} \int_0^B \int_0^L \mu\ H(x,y) \left(\frac{\partial w^{(3)}(x,y,t)}{\partial t} \right)^2 dx\ dy, \tag{4.11.1}$$

where $w^{(3)}(x,y,t)$ is defined by expression (17.6), that can be written in the form

$$w^{(3)} = \left\{ \begin{matrix} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{yz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{matrix} \right\}^T \left\{ \begin{matrix} 1 \\ 0 \\ 0 \\ z_3 \end{matrix} \right\} = \{f\}^T \left\{ \begin{matrix} 1 \\ 0 \\ 0 \\ z_3 \end{matrix} \right\}. \tag{4.11.2}$$

Then

$$\left(\frac{\partial w^{(3)}}{\partial t} \right)^2 = \left(\frac{\partial}{\partial t}\{f\} \right)^T \left\{ \begin{matrix} 1 \\ 0 \\ 0 \\ z_3 \end{matrix} \right\} \lfloor\, 1 \quad 0 \quad 0 \quad z_3 \,\rfloor \left(\frac{\partial}{\partial t}\{f\} \right) =$$

$$= \left(\frac{\partial}{\partial t}\{f\} \right)^T \left[\widetilde{D}_c\right] \left(\frac{\partial}{\partial t}\{f\} \right), \tag{4.11.3}$$

where

$$\left[\widetilde{D}_c\right] = \begin{bmatrix} 1 & 0 & 0 & z_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ z_3 & 0 & 0 & z_3^2 \end{bmatrix}. \tag{4.11.4}$$

Substitution of (4.11.3) into (4.11.1) yields:

$$K_c = \frac{1}{2}\int_0^B\int_0^L \mu\, H(x,y)\left(\frac{\partial}{\partial t}\{f\}\right)^T \left[\widetilde{D}_c\right]\left(\frac{\partial}{\partial t}\{f\}\right)\, dx\, dy. \tag{4.11.5}$$

4.12 Considerations regarding finite element formulation

The Hamilton's principle used for the finite element formulation is discussed in chapter 3, and has the form

$$\delta\int_{t_1}^{t_2} [\text{(strain energy of platform)} + \text{(strain energy of elastic foundation)} +$$

$$+ \text{ (potential energy of platform in gravity field)} + \text{(potential energy of cargo in gravity field)}$$

$$- \text{(kinetic energy of platform)} - \text{(kinetic energy of cargo)}]\ dt$$

$$-\int_{t_1}^{t_2} \text{(virtual work of damping forces)}\ dt = 0. \qquad \text{(eqn 3.11.10)}$$

All the considerations regarding the finite element formulation, presented in chapter 3, are also valid for the simplified model of the chapter 4, except that the simplified model has fewer unknown functions and, therefore, fewer degrees of freedom. The unknown functions of the simplified model are $w_0\,(x,y,t)$, $\varepsilon_{xz}\,(x,y,t)$, $\varepsilon_{yz}\,(x,y,t)$ and $\varepsilon_{zz}(x,y,t)$. In the finite element formulation, the interpolation polynomials for these functions will be the same as those discussed in section 3.12 of chapter 3. The combined finite element for all the unknown functions of the problem will have 40 degrees of freedom: 4 degrees of freedom must be used for interpolation of each of the functions $\varepsilon_{xz}^{(2)}$, $\varepsilon_{yz}^{(2)}$, and 16 degrees of freedom must be used for interpolation of each of the functions w_0 and $\varepsilon_{zz}^{(2)}$. Each node of the combined finite element has 10 degrees of freedom: w_0, $\frac{\partial w_0}{\partial x}$, $\frac{\partial w_0}{\partial y}$, $\frac{\partial^2 w_0}{\partial x \partial y}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{yz}^{(2)}$, $\varepsilon_{zz}^{(2)}$, $\frac{\partial \varepsilon_{zz}^{(2)}}{\partial x}$, $\frac{\partial \varepsilon_{zz}^{(2)}}{\partial y}$, $\frac{\partial^2 \varepsilon_{zz}^{(2)}}{\partial x \partial y}$.

4.13 Post-processing stage of the finite element analysis: expressions for the in-plane stress components and the second form of the transverse stress components in terms of the unknown functions w_0, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{yz}^{(2)}$, $\varepsilon_{zz}^{(2)}$.

To obtain expressions for the in-plane stresses in terms of the unknown functions for this simplified model, one can use the corresponding expressions (3.13.4) of the nonsimplified model and set in them the functions u_0, v_0, $\varepsilon_{xz}^{(1)}$, $\varepsilon_{yz}^{(1)}$, $\varepsilon_{zz}^{(1)}$, $\varepsilon_{xz}^{(3)}$, $\varepsilon_{yz}^{(3)}$, $\varepsilon_{zz}^{(3)}$ equal to zero. Thus, one can receive:

$$\left\{ \begin{array}{c} {}^{H}\sigma_{xx} \\ {}^{H}\sigma_{yy} \\ {}^{H}\sigma_{xy} \end{array} \right\}^{(k)} =$$

$$= \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{16} & \overline{C}_{13} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{26} & \overline{C}_{23} \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{66} & \overline{C}_{36} \end{bmatrix}^{(k)} \left(\left\{ \begin{array}{c} \varphi_{xx0} \\ \varphi_{yy0} \\ \varphi_{xy0} \\ \varepsilon_{zz} \end{array} \right\}^{(k)} + \left\{ \begin{array}{c} \varphi_{xx1} \\ \varphi_{yy1} \\ \varphi_{xy1} \\ 0 \end{array} \right\}^{(k)} z + \left\{ \begin{array}{c} \varphi_{xx2} \\ \varphi_{yy2} \\ \varphi_{xy2} \\ 0 \end{array} \right\}^{(k)} z^2 \right), \tag{4.13.1}$$

where

$$\varphi_{xx0}^{(1)} = u_{0,x} + 2z_2\varepsilon_{xz,x}^{(2)} + \frac{1}{2}z_2^2\varepsilon_{zz,xx}^{(2)} + \underbrace{\frac{1}{2}\left(w_{0,x} + z_2\varepsilon_{zz,x}^{(2)}\right)^2}, \tag{4.13.2}$$

$$\varphi_{xx1}^{(1)} = -w_{0,xx} - z_2\varepsilon_{zz,xx}^{(2)}, \tag{4.13.3}$$

$$\varphi_{xx2}^{(1)} = 0, \tag{4.13.4}$$

$$\varphi_{yy0}^{(1)} = v_{0,y} + 2z_2\varepsilon_{yz,y}^{(2)} + \frac{1}{2}z_2^2\varepsilon_{zz,yy}^{(2)} + \underbrace{\frac{1}{2}\left(w_{0,y} + z_2\varepsilon_{zz,y}^{(2)}\right)^2}, \tag{4.13.5}$$

$$\varphi^{(1)}_{yy1} = -w_{0,yy} - z_2 \varepsilon^{(2)}_{zz,yy} \tag{4.13.6}$$

$$\varphi^{(1)}_{yy2} = 0 \; , \tag{4.13.7}$$

$$\varphi^{(1)}_{xy0} = 2z_2 \left(\varepsilon^{(2)}_{xz,y} + \varepsilon^{(2)}_{yz,x}\right) + z_2^2 \varepsilon^{(2)}_{zz,xy} + \underbrace{\left(w_{0,x} + z_2 \varepsilon^{(2)}_{zz,x}\right)\left(w_{0,y} + z_2 \varepsilon^{(2)}_{zz,y}\right)} \; , \tag{4.13.8}$$

$$\varphi^{(1)}_{xy1} = -2w_{0,xy} - 2z_2 \varepsilon^{(2)}_{zz,xy} \; , \tag{4.13.9}$$

$$\varphi^{(1)}_{xy2} = 0 \; , \tag{4.13.10}$$

$$\varphi^{(2)}_{xx0} = \underbrace{\frac{1}{2}(w_{0,x})^2} \; , \tag{4.13.11}$$

$$\varphi^{(2)}_{xx1} = 2\varepsilon^{(2)}_{xz,x} - w_{0,xx} + \underbrace{w_{0,x}\varepsilon^{(2)}_{zz,x}} \; , \tag{4.13.12}$$

$$\varphi^{(2)}_{xx2} = -\frac{1}{2}\varepsilon^{(2)}_{zz,xx} + \underbrace{\frac{1}{2}\left(\varepsilon^{(2)}_{zz,x}\right)^2} \; , \tag{4.13.13}$$

$$\varphi^{(2)}_{yy0} = \underbrace{\frac{1}{2}(w_{0,y})^2} \; , \tag{4.13.14}$$

$$\varphi^{(2)}_{yy1} = 2\varepsilon^{(2)}_{yz,y} - w_{0,yy} + \underbrace{w_{0,y}\varepsilon^{(2)}_{zz,y}} \; , \tag{4.13.15}$$

$$\varphi^{(2)}_{yy2} = -\frac{1}{2}\varepsilon^{(2)}_{zz,yy} + \underbrace{\frac{1}{2}\left(\varepsilon^{(2)}_{zz,y}\right)^2} \; , \tag{4.13.16}$$

$$\varphi_{xy0}^{(2)} = \underbrace{w_{0,x}w_{0,y}} \ , \tag{4.13.17}$$

$$\varphi_{xy1}^{(2)} = 2\left(\varepsilon_{xz,y}^{(2)} + \varepsilon_{yz,x}^{(2)} - w_{0,xy}\right) + \underbrace{w_{0,y}\varepsilon_{zz,x}^{(2)} + w_{0,x}\varepsilon_{zz,y}^{(2)}} \ , \tag{4.13.18}$$

$$\varphi_{xy2}^{(2)} = -\varepsilon_{zz,xy}^{(2)} + \underbrace{\varepsilon_{zz,x}^{(2)}\varepsilon_{zz,y}^{(2)}} \ , \tag{4.13.19}$$

$$\varphi_{xx0}^{(3)} = 2z_3\varepsilon_{xz,x}^{(2)} + \frac{1}{2}z_3^2\varepsilon_{zz,xx}^{(2)} + \underbrace{\frac{1}{2}\left(w_{0,x} + z_3\varepsilon_{zz,x}^{(2)}\right)^2} \ , \tag{4.13.20}$$

$$\varphi_{xx1}^{(3)} = -w_{0,xx} - z_3\varepsilon_{zz,xx}^{(2)} \ , \tag{4.13.21}$$

$$\varphi_{xx2}^{(3)} = 0 \ , \tag{4.13.22}$$

$$\varphi_{yy0}^{(3)} = 2z_3\varepsilon_{yz,y}^{(2)} + \frac{1}{2}z_3^2\varepsilon_{zz,yy}^{(2)} + \underbrace{\frac{1}{2}\left(w_{0,y} + z_3\varepsilon_{zz,y}^{(2)}\right)^2} \ , \tag{4.13.23}$$

$$\varphi_{yy1}^{(3)} = -w_{0,yy} - z_3\varepsilon_{zz,yy}^{(2)} \ , \tag{4.13.24}$$

$$\varphi_{yy2}^{(3)} = 0 \ , \tag{4.13.25}$$

$$\varphi_{xy0}^{(3)} = 2z_3\left(\varepsilon_{xz,y}^{(2)} + \varepsilon_{yz,x}^{(2)}\right) + z_3^2\varepsilon_{zz,xy}^{(2)} + \underbrace{\left(w_{0,x} + z_3\varepsilon_{zz,x}^{(2)}\right)\left(w_{0,y} + z_3\varepsilon_{zz,y}^{(2)}\right)} \ , \tag{4.13.26}$$

$$\varphi_{xy1}^{(3)} = 2\left(-w_{0,xy} + z_3 - \varepsilon_{zz,xy}^{(2)}\right) \ , \tag{4.13.27}$$

$$\varphi_{xy2}^{(3)} = 0, \tag{4.13.28}$$

where the nonlinear terms are underbraced.

The formulas for the transverse stresses in terms of the in-plane stresses are the same as those presented in section 3.13 of chapter3:

$$\sigma_{\alpha 3}^{(k)} = \sum_{m=1}^{k-1} \int_{z_m}^{z_{m+1}} \left(\rho^{(m)} \ddot{u}_{\alpha}^{(m)} - {}^{H}\sigma_{\alpha\beta,\beta}^{(m)} \right) dz + \int_{z_k}^{z} \left(\rho^{(k)} \ddot{u}_{\alpha}^{(k)} - {}^{H}\sigma_{\alpha\beta,\beta}^{(k)} \right) dz$$

$$(\alpha = 1,2;\;\; \beta = 1,2;\;\; k = 1,2,3) \text{ in the interval } \; z_k \le z \le z_{k+1}, \qquad \text{(eqn 3.13.14)}$$

where the sum is considered to be equal to zero, if the upper value of the summation index m is smaller than the lower value, i.e. if $k = 1$;

$$\sigma_{33}^{(k)} = sw^{(1)}(z_1) + \sum_{m=1}^{k-1} \int_{z_m}^{z_{m+1}} \rho^{(m)} \left(\ddot{u}_3^{(m)} + g \right) dz - \sum_{m=1}^{k-1} \int_{z_m}^{z_{m+1}} \left({}^{H}\sigma_{\alpha\beta}^{(m)}\, u_{3,\alpha}^{(m)} \right)_{,\beta} dz$$

$$-\sum_{m=1}^{k-1} \sum_{n=1}^{m-1} \int_{z_m}^{z_{m+1}} \int_{z_n}^{z_{n+1}} \left(\rho^{(n)} \ddot{u}_{\alpha,\alpha}^{(n)} - {}^{H}\sigma_{\alpha\beta,\alpha\beta}^{(n)} \right) dz\, dz$$

$$-\sum_{m=1}^{k-1} \int_{z_m}^{z_{m+1}} \int_{z_m}^{z} \left(\rho^{(m)} \ddot{u}_{\alpha,\alpha}^{(m)} - {}^{H}\sigma_{\alpha\beta,\alpha\beta}^{(m)} \right) dz\, dz$$

$$+\int_{z_k}^{z} \rho^{(k)} \left(\ddot{u}_3^{(k)} + g \right) dz - \int_{z_k}^{z} \left({}^{H}\sigma_{\alpha\beta}^{(k)}\, u_{3,\alpha}^{(k)} \right)_{,\beta} dz$$

$$-\sum_{n=1}^{k-1} \int_{z_k}^{z} \int_{z_n}^{z_{n+1}} \left(\rho^{(n)}\, \ddot{u}_{\alpha,\alpha}^{(n)} - {}^{H}\sigma_{\alpha\beta,\alpha\beta}^{(n)} \right) dz\, dz$$

$$-\int_{z_k}^{z} \int_{z_k}^{z} \left(\rho^{(k)}\, \ddot{u}_{\alpha,\alpha}^{(k)} - {}^{H}\sigma_{\alpha\beta,\alpha\beta}^{(k)} \right) dz\, dz \qquad (\alpha = 1,2;\;\; \beta = 1,2;\;\; k = 1,2,3) \qquad \text{(eqn 3.13.20)}$$

in the interval $z_k \le z \le z_{k+1}$.

In the next chapter, the computational model of the sandwich plate, presented in this chapter, will be applied for stress and failure analysis of the cargo platform modelled as a wide beam (plate in cylindrical bending), dropped on the ground modelled as elastic Winkler foundation.

Chapter 5

Stress and Failure Analysis of the Sandwich Cargo Platform Modelled as a Plate in Cylindrical Bending

The problem of stress and failure analysis of the cargo sandwich platform dropped on elastic foundation, as formulated in chapters 3 and 4, requires two-dimensional finite element analysis with geometric nonlinearity and the equivalent of material nonlinearity, due to taking account of failure progression. In doing a complex analysis of this type, analysts usually start from simple models and do not attempt a complete solution all at once. A first step toward understanding the response of the composite sandwich platform to the impact against the elastic foundation can be made by solving a simpler problem of cylindrical bending of such a platform. Such a one-dimensional problem has many similar features to the two-dimensional problem of interest, and allows one to discover more easily the inaccuracies that may appear in the finite element formulation and program. The analysis of the cargo platform as a plate in cylindrical bending will become a foundation for the further analysis of the cargo platform with the use of two-dimensional finite elements.

5.1 Some general considerations regarding cylindrical bending

Let us consider an anisotropic plate loaded by surface and body forces, acting in the z-direction, and not varying along the y-direction (Figure 2.1). Let us call the dimension of the plate in x-direction the length, and dimension in y-direction – the width. If the width of the plate is much larger than the length, the displacements do not depend on the y-coordinate:

$$u = u\left(x,z\right), \;\; v = v\left(x,z\right), w = w\left(x,z\right). \tag{5.1.}$$

Such a condition is called **generalized plane strain** (Lekhnitskii, 1981). In this case, the components of the Green's strain tensor, associated with the y-direction, are:

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = 0, \tag{5.1.2}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) = \frac{\partial v}{\partial x}, \tag{5.1.3}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right) = \frac{\partial v}{\partial z}. \tag{5.1.4}$$

If

$$u = u\left(x,z\right), \;\; w = w\left(x,z\right), \;\; v = const \tag{5.1.5}$$

(or $u = u\left(x,z\right), \; w = w\left(x,z\right), \; v = 0$, if rigid body displacements in y-direction are excluded from consideration), then we have a condition of **pure plane strain** or simply **plane strain**. In this case all strain components, associated with y-direction, are equal to zero:

$$\varepsilon_{yy} = 0, \;\; \varepsilon_{xy} = 0, \;\; \varepsilon_{yz} = 0. \tag{5.1.6}$$

The condition of the generalized plane strain reduces to the condition of the pure plane strain if the plate is isotropic, or if the plate is anisotropic and at each point of the plate there is a plane of elastic symmetry parallel to x-z plane (Lekhnitskii, 1981).

If a deformed plate is in the condition of the generalized plane strain, then it is said to be in cylindrical bending. A plate is in cylindrical bending if : 1) its width (dimension in y-direction) is

much larger than its length (dimension in the x-direction), and 2) the load intensity does not vary in the y-direction.

L and B are taken to be dimensions of a rectangular plate relative to the $x-$ and $y-$axes. The aspect ratio $\frac{B}{L}$, required to make the assumption of cylindrical bending for a laminated plate, depends on laminate construction. For unsymmetrical laminates of the class $[0^o/90^o]_n$ it has been shown (Whitney, 1969, 1987) that the maximum deflection under transverse loading rapidly approaches the maximum deflection of cylindrical bending, if the aspect ratio increases. For an aspect ratio $\frac{B}{L} = 3$ the plate center deflection was within 4% of the center deflection of an infinite strip. In the case of angle-ply laminates the convergence to cylindrical bending with increasing aspect ratio is less rapid.

Let us assume the cargo platform satisfies the conditions of cylindrical bending, described above, i.e. the load of the cargo is uniformly distributed in one direction (y-direction), and the face sheets are cross -ply laminates with aspect ratio $\frac{B}{L}$ equal, at least, 3. Then in the platform there is, approximately, the condition of the generalized plane strain, which occurs if the unknown functions of the problem depend only on x-coordinate. If we do the simplified analysis, introduced in chapter 4, the unknown functions in case of cylindrical bending are w_0, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{yz}^{(2)}$, $\varepsilon_{zz}^{(2)}$. As it was mentioned in chapter 4, the middle-surface displacements u_0 and v_0 are considered negligible because, among other reasons, the sublaminates of the sandwich plates are assumed to be either cross-ply, or specially orthotropic, i.e. at each point of the plate there is a plane of elastic symmetry parallel to the x-z plane. Due to the same assumption, the condition of the generalized plane strain reduces to the condition of the pure plane strain (Lekhnitskii, 1981), i.e. $u = u\,(x,z)\,,\ w = w\,(x,z)\,,\ v = 0$ and, therefore, $\varepsilon_{yz} = 0$. So, in the case of cylindrical bending, the unknown functions of the problem are

$$w_0\,(x,t)\,,\ \varepsilon_{xz}^{(2)}\,(x,t)\,,\ \varepsilon_{zz}^{(2)}\,(x,t)\,. \tag{5.1.7}$$

5.2 Displacements in terms of the unknown functions

Equations (4.2.1)-(4.2.9) for a plate in cylindrical bending take the form:

$$w^{(1)}\,(x,t) = w_0\,(x,t) + \varepsilon_{zz}^{(2)}\,(x,t)\,z_2 \qquad (z_1 \le z \le z_2)\,, \tag{5.2.1}$$

$$w^{(2)}\,(x,z,t) = w_0\,(x,t) + \varepsilon_{zz}^{(2)}\,(x,t)\,z \qquad (z_2 \le z \le z_3)\,, \tag{5.2.2}$$

$$w^{(3)}(x,t) = w_0(x,t) + \varepsilon_{zz}^{(2)}(x,t)\, z_3 \qquad (z_3 \le z \le z_4), \tag{5.2.3}$$

$$u^{(1)}(x,z,t) = \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right) z_2 - \frac{1}{2}\varepsilon_{zz,x}^{(2)}\, z_2^2 - \left(\varepsilon_{zz,x}^{(2)}\, z_2 + w_{0,x}\right)(z - z_2) \quad (z_1 \le z \le z_2), \tag{5.2.4}$$

$$u^{(2)}(x,z,t) = \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right) z - \frac{1}{2}\varepsilon_{zz,x}^{(2)}\, z^2 \qquad (z_2 \le z \le z_3), \tag{5.2.5}$$

$$u^{(3)}(x,z,t) = \left(2\varepsilon_{xz}^{(2)} - w_{0,x}\right) z_3 - \frac{1}{2}\varepsilon_{zz,x}^{(2)}\, z_3^2 - \left(w_{0,x} + \varepsilon_{zz,x}^{(2)} z_3\right)(z - z_3) \quad (z_3 \le z \le z_4), \tag{5.2.6}$$

$$v^{(1)} = v^{(2)} = v^{(3)} = 0. \tag{5.2.7}$$

These equations can be written in matrix form as follows:

$$\left\{ \begin{array}{c} u^{(1)} \\ w^{(1)} \end{array} \right\} = \underset{(2\times3)}{\left[\begin{array}{ccc} 1 & z & 0 \\ 0 & 0 & 1 \end{array} \right]} \underset{(3\times3)}{\left[\begin{array}{ccc} 0 & 2z_2 & \frac{1}{2} z_2^2 \frac{d}{dx} \\ -\frac{d}{dx} & 0 & -z_2 \frac{d}{dx} \\ 1 & 0 & z_2 \end{array} \right]} \underset{(3\times1)}{\left\{ \begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\}}, \tag{5.2.8}$$

$$\left\{ \begin{array}{c} u^{(2)} \\ w^{(2)} \end{array} \right\} = \underset{(2\times4)}{\left[\begin{array}{cccc} z & z^2 & 0 & 0 \\ 0 & 0 & 1 & z \end{array} \right]} \underset{(4\times3)}{\left[\begin{array}{ccc} -\frac{d}{dx} & 2 & 0 \\ 0 & 0 & -\frac{1}{2}\frac{d}{dx} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right]} \underset{(3\times1)}{\left\{ \begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\}}, \tag{5.2.9}$$

$$\left\{ \begin{array}{c} u^{(3)} \\ w^{(3)} \end{array} \right\} = \underset{(2\times3)}{\left[\begin{array}{ccc} 1 & z & 0 \\ 0 & 0 & 1 \end{array} \right]} \underset{(3\times3)}{\left[\begin{array}{ccc} 0 & 2z_3 & \frac{1}{2} z_3^2 \frac{d}{dx} \\ -\frac{d}{dx} & 0 & -z_3 \frac{d}{dx} \\ 1 & 0 & z_3 \end{array} \right]} \underset{(3\times1)}{\left\{ \begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\}}. \tag{5.2.10}$$

These equations can also be written in the form

$$\left\{ \begin{array}{c} u^{(k)} \\ w^{(k)} \end{array} \right\} = \left[\widetilde{Z}^{(k)}\right] \left[\widetilde{\partial}^{(k)}\right] \{f\} \qquad (k = 1,2,3), \tag{5.2.11}$$

where

$$\{f\} = \left\{ \begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\}, \tag{5.2.12}$$

$$\underset{(2\times3)}{\left[\widetilde{Z}^{(1)}\right]} = \underset{(2\times3)}{\left[\widetilde{Z}^{(3)}\right]} = \left[\begin{array}{ccc} 1 & z & 0 \\ 0 & 0 & 1 \end{array} \right], \tag{5.2.13}$$

$$\underset{(2\times4)}{\left[\widetilde{Z}^{(2)}\right]} = \underset{(2\times4)}{\left[\begin{array}{cccc} z & z^2 & 0 & 0 \\ 0 & 0 & 1 & z \end{array} \right]}, \tag{5.2.14}$$

$$\underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]} = \left[\begin{array}{ccc} 0 & 2z_2 & \frac{1}{2}z_2^2\frac{d}{dx} \\ -\frac{d}{dx} & 0 & -z_2\frac{d}{dx} \\ 1 & 0 & z_2 \end{array} \right], \tag{5.2.15}$$

$$\underset{(4\times3)}{\left[\widetilde{\partial}^{(2)}\right]} = \left[\begin{array}{ccc} -\frac{d}{dx} & 2 & 0 \\ 0 & 0 & -\frac{1}{2}\frac{d}{dx} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right], \tag{5.2.16}$$

$$\underset{(3\times3)}{\left[\widetilde{\partial}^{(3)}\right]} = \left[\begin{array}{ccc} 0 & 2z_3 & \frac{1}{2}z_3^2\frac{d}{dx} \\ -\frac{d}{dx} & 0 & -z_3\frac{d}{dx} \\ 1 & 0 & z_3 \end{array} \right]. \tag{5.2.17}$$

5.3 Strains in terms of the unknown functions

If we substitute expressions (5.2.1)-(5.2.6) for displacements into the strain-displacement relations

$$\varepsilon_{xx}^{(k)} = u_{,x}^{(k)} + \frac{1}{2}\left(w_{,x}^{(k)}\right)^2,$$

we obtain

$$\varepsilon_{xx}^{(1)} = 2z_2\,\varepsilon_{xz,x}^{(2)} + \tfrac{1}{2}z_2^2\,\varepsilon_{zz,xx}^{(2)} + \underbrace{\frac{1}{2}\left(w_{0,x} + z_2\,\varepsilon_{zz,x}^{(2)}\right)^2} - \left(w_{0,xx} + z_2\,\varepsilon_{zz,xx}^{(2)}\right)z, \tag{5.3.1}$$

$$\varepsilon_{xx}^{(2)} = \tfrac{1}{2}(w_{0,x})^2 + \left(2\varepsilon_{xz,x}^{(2)} - w_{0,xx} + \underbrace{w_{0,x}\varepsilon_{zz,x}^{(2)}}\right) z + \left(-\frac{1}{2}\varepsilon_{zz,xx}^{(2)} + \underbrace{\frac{1}{2}\left(\varepsilon_{zz,x}^{(2)}\right)^2}\right) z^2, \tag{5.3.2}$$

$$\varepsilon_{xx}^{(3)} = 2z_3\,\varepsilon_{xz,x}^{(2)} + \tfrac{1}{2}z_3^2\,\varepsilon_{zz,xx}^{(2)} + \underbrace{\frac{1}{2}\left(w_{0,x} + z_3\,\varepsilon_{zz,x}^{(2)}\right)^2} - \left(w_{0,xx} + z_3\,\varepsilon_{zz,xx}^{(2)}\right) z. \tag{5.3.3}$$

In the last three equations the nonlinear terms are underbraced.

Using expressions (5.3.1)-(5.3.3), we can write strains in terms of the unknown functions as follows:

$$\varepsilon_{xx}^{(1)} = \underset{(1\times 2)}{\left\lfloor Z^{(1)} \right\rfloor} \left(\underset{(2\times 3)}{\left[\partial^{(1)}\right]} \underset{(3\times 1)}{\{f\}} + \underset{(2\times 1)}{\left\{\eta^{(1)}\right\}} \right), \tag{5.3.4}$$

$$\varepsilon_{xx}^{(3)} = \underset{(1\times 2)}{\left\lfloor Z^{(3)} \right\rfloor} \left(\underset{(2\times 3)}{\left[\partial^{(3)}\right]} \underset{(3\times 1)}{\{f\}} + \underset{(2\times 1)}{\left\{\eta^{(3)}\right\}} \right), \tag{5.3.5}$$

$$\left\{ \begin{array}{c} \varepsilon_{xx}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\} = \underset{(2\times 4)}{\left[\widehat{Z}^{(2)}\right]} \left(\underset{(4\times 3)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(3\times 1)}{\{f\}} + \underset{(4\times 1)}{\left\{\widehat{\eta}^{(2)}\right\}} \right), \tag{5.3.6}$$

$$\varepsilon_{xz}^{(2)} = \left\lfloor \begin{array}{ccc} 0 & 1 & 0 \end{array} \right\rfloor \underset{(3\times 1)}{\{f\}}, \tag{5.3.7}$$

where

$$\underset{(3\times 1)}{\{f\}} = \left\{ \begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\}, \tag{5.3.8}$$

$$\left\lfloor Z^{(1)} \right\rfloor = \left\lfloor Z^{(3)} \right\rfloor = \left\lfloor \begin{array}{cc} 1 & z \end{array} \right\rfloor, \qquad \left[\widehat{Z}^{(2)}\right] = \left[\begin{array}{cccc} 1 & z & z^2 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right], \tag{5.3.9}$$

$$\underset{(2\times 3)}{\left[\partial^{(1)}\right]} = \left[\begin{array}{ccc} 0 & 2z_2\frac{d}{dx} & \frac{1}{2}z_2^2\frac{d^2}{dx^2} \\ -\frac{d^2}{dx^2} & 0 & -z_2\frac{d^2}{dx^2} \end{array} \right], \tag{5.3.10}$$

$$\underset{(2\times 3)}{\left[\partial^{(3)}\right]} = \underset{(2\times 3)}{\left[\begin{array}{ccc} 0 & 2z_3\frac{d}{dx} & \frac{1}{2}z_3^2\frac{d^2}{dx^2} \\ -\frac{d^2}{dx^2} & 0 & -z_3\frac{d^2}{dx^2} \end{array} \right]}, \tag{5.3.11}$$

$$\left[\widehat{\partial}^{(2)}\right]_{(4\times3)} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{d^2}{dx^2} & 2\frac{d}{dx} & 0 \\ 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} \\ 0 & 0 & 1 \end{bmatrix}_{(4\times3)}, \tag{5.3.12}$$

$$\left\{\eta^{(1)}\right\}_{(2\times1)} = \left\{ \begin{array}{c} \frac{1}{2}\left(w_{0,x} + z_2\, \varepsilon^{(2)}_{zz,x}\right)^2 \\ 0 \end{array} \right\}, \tag{5.3.13}$$

$$\left\{\eta^{(3)}\right\}_{(2\times1)} = \left\{ \begin{array}{c} \frac{1}{2}\left(w_{0,x} + z_3\, \varepsilon^{(2)}_{zz,x}\right)^2 \\ 0 \end{array} \right\}, \tag{5.3.14}$$

$$\left[\widehat{\eta}^{(2)}\right]_{(4\times1)} = \left\{ \begin{array}{c} \frac{1}{2}(w_{0,x})^2 \\ w_{0,x}\varepsilon^{(2)}_{zz,x} \\ \frac{1}{2}\left(\varepsilon^{(2)}_{zz,x}\right)^2 \\ 0 \end{array} \right\}_{(4\times1)}. \tag{5.3.15}$$

5.4 Constitutive relations

If $\varepsilon^{(1)}_{xz} = \varepsilon^{(1)}_{yz} = \varepsilon^{(1)}_{zz} = \varepsilon^{(3)}_{xz} = \varepsilon^{(3)}_{yz} = \varepsilon^{(3)}_{zz} = 0$, then constitutive equations (3.6.13) for an orthotropic material can be written for the face sheets and the core as follows:

for the upper face sheet:

$$^{H}\sigma^{(1)}_{xx} = \overline{C}^{(1)}_{11}\varepsilon^{(1)}_{xx}\ , \tag{5.4.1}$$

lower face sheet:

$$^{H}\sigma^{(3)}_{xx} = \overline{C}^{(3)}_{11}\varepsilon^{(3)}_{xx}, \tag{5.4.2}$$

core:

$$\left\{ \begin{array}{c} \sigma^{(2)}_{xx} \\ \sigma^{(2)}_{zz} \end{array} \right\} = \begin{bmatrix} \overline{C}^{(2)}_{11} & \overline{C}^{(2)}_{13} \\ C^{(2)}_{13} & \overline{C}^{(2)}_{33} \end{bmatrix} \left\{ \begin{array}{c} \varepsilon^{(2)}_{xx} \\ \varepsilon^{(2)}_{zz} \end{array} \right\}, \tag{5.4.3}$$

$$\sigma^{(2)}_{xz} = \overline{C}^{(2)}_{55}2\varepsilon^{(2)}_{xz}. \tag{5.4.4}$$

5.5 Strain energy of the plate in cylindrical bending

The strain energy of the sandwich plate consists of the strain energies of the face sheets and the core, and it has the form:

$$U = \frac{1}{2}b\int_0^L\int_{z_1}^{z_2} \varepsilon_{xx}^{(1)}\ {}^H\sigma_{xx}^{(1)}\ dz\ dx + \frac{1}{2}b\int_0^L\int_{z_3}^{z_4} \varepsilon_{xx}^{(3)}\ {}^H\sigma_{xx}^{(3)}\ dz\ dx +$$

$$+\frac{1}{2}b\int_0^L\int_{z_2}^{z_3} \begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{Bmatrix}^T \begin{Bmatrix} {}^H\sigma_{xx}^{(2)} \\ {}^H\sigma_{zz}^{(2)} \end{Bmatrix} dz\ dx + \frac{1}{2}b\int_0^L\int_{z_2}^{z_3} 2\varepsilon_{xz}^{(2)}\ {}^H\sigma_{xz}^{(2)} dz\ dx =$$

$$= \frac{1}{2}b\int_0^L\int_{z_1}^{z_2} \varepsilon_{xx}^{(1)}\overline{C}_{11}^{(1)}\varepsilon_{xx}^{(1)}\ dz\ dx + \frac{1}{2}b\int_0^L\int_{z_3}^{z_4} \varepsilon_{xx}^{(3)}\ \overline{C}_{11}^{(3)}\varepsilon_{xx}^{(3)}\ dz\ dx +$$

$$+\frac{1}{2}b\int_0^L\int_{z_2}^{z_3} \begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{Bmatrix}^T \begin{bmatrix} \overline{C}_{11}^{(2)} & \overline{C}_{13}^{(2)} \\ C_{13}^{(2)} & \overline{C}_{33}^{(2)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{Bmatrix} dz\ dx + \frac{1}{2}b\int_0^L\int_{z_2}^{z_3} 2\varepsilon_{xz}^{(2)}\overline{C}_{55}^{(2)}\ 2\varepsilon_{xz}^{(2)} dz\ dx =$$

$$= \frac{1}{2}b\int_0^L \left(\underset{(2\times3)}{\left[\partial^{(1)}\right]}\underset{(3\times1)}{\{f\}} + \underset{(2\times1)}{\left\{\eta^{(1)}\right\}}\right)^T \left(\int_{z_1}^{z_2} \underset{(2\times1)}{\left\lfloor Z^{(1)}\right\rfloor^T} \overline{C}_{11}^{(1)} \underset{(1\times2)}{\left\lfloor Z^{(1)}\right\rfloor}\ dz\right) \left(\underset{(2\times3)}{\left[\partial^{(1)}\right]}\underset{(3\times1)}{\{f\}} + \underset{(2\times1)}{\left\{\eta^{(1)}\right\}}\right) dx +$$

$$+\frac{1}{2}b\int_0^L \left(\underset{(2\times3)}{\left[\partial^{(3)}\right]}\underset{(3\times1)}{\{f\}} + \underset{(2\times1)}{\left\{\eta^{(3)}\right\}}\right)^T \left(\int_{z_3}^{z_4} \underset{(2\times1)}{\left\lfloor Z^{(3)}\right\rfloor^T}\ \overline{C}_{11}^{(3)} \underset{(1\times2)}{\left\lfloor Z^{(3)}\right\rfloor}\ dz\right) \left(\underset{(2\times3)}{\left[\partial^{(3)}\right]}\underset{(3\times1)}{\{f\}} + \underset{(2\times1)}{\left\{\eta^{(3)}\right\}}\right) dx +$$

$$+\frac{1}{2}b\int_0^L \left(\underset{(4\times3)}{\left[\widehat{\partial}^{(2)}\right]}\underset{(3\times1)}{\{f\}} + \underset{(4\times1)}{\left[\widehat{\eta}^{(2)}\right]}\right)^T \left(\int_{z_2}^{z_3} \underset{(4\times2)}{\left[\widehat{Z}^{(2)}\right]^T} \begin{bmatrix} \overline{C}_{11}^{(2)} & \overline{C}_{13}^{(2)} \\ C_{13}^{(2)} & \overline{C}_{33}^{(2)} \end{bmatrix} \underset{(2\times4)}{\left[\widehat{Z}^{(2)}\right]}\right) dz \times$$

$$\times \left(\underset{(4\times3)}{\left[\widehat{\partial}^{(2)}\right]}\underset{(3\times1)}{\{f\}} + \underset{(4\times1)}{\left[\widehat{\eta}^{(2)}\right]}\right) dx +$$

$$+\frac{1}{2}b\int_{0}^{L}\{f\}^T\left(\int_{z_2}^{z_3}2\begin{Bmatrix}0\\1\\0\end{Bmatrix}\overline{C}_{55}^{(2)}\,2\begin{bmatrix}0 & 1 & 0\end{bmatrix}dz\right)\underset{(3\times1)}{\{f\}}\,dx \tag{5.5.1}$$

or

$$U=\frac{1}{2}\int_{0}^{L}\left(\underset{(2\times3)}{\left[\partial^{(1)}\right]}\underset{(3\times1)}{\{f\}}+\underset{(2\times1)}{\left\{\eta^{(1)}\right\}}\right)^T\underset{(2\times2)}{\left[D^{(1)}\right]}\left(\underset{(2\times3)}{\left[\partial^{(1)}\right]}\underset{(3\times1)}{\{f\}}+\underset{(2\times1)}{\left\{\eta^{(1)}\right\}}\right)dx+$$

$$+\frac{1}{2}\int_{0}^{L}\left(\underset{(2\times3)}{\left[\partial^{(3)}\right]}\underset{(3\times1)}{\{f\}}+\underset{(2\times1)}{\left\{\eta^{(3)}\right\}}\right)^T\underset{(2\times2)}{\left[D^{(3)}\right]}\left(\underset{(2\times3)}{\left[\partial^{(3)}\right]}\underset{(3\times1)}{\{f\}}+\underset{(2\times1)}{\left\{\eta^{(3)}\right\}}\right)dx+$$

$$+\frac{1}{2}\int_{0}^{L}\left(\underset{(4\times3)}{\left[\widehat{\partial}^{(2)}\right]}\underset{(3\times1)}{\{f\}}+\underset{(4\times1)}{\left[\widehat{\eta}^{(2)}\right]}\right)^T\underset{(4\times4)}{\left[\widehat{D}^{(2)}\right]}\left(\underset{(4\times3)}{\left[\widehat{\partial}^{(2)}\right]}\underset{(3\times1)}{\{f\}}+\underset{(4\times1)}{\left[\widehat{\eta}^{(2)}\right]}\right)dx+$$

$$+\frac{1}{2}\int_{0}^{L}\{f\}^T\underset{(3\times3)}{\left[\check{D}^{(2)}\right]}\underset{(3\times1)}{\{f\}}\,dx, \tag{5.5.2}$$

where

$$\underset{(4\times4)}{\left[\widehat{D}^{(2)}\right]}=b\int_{z_2}^{z_3}\underset{(4\times2)}{\left[\widehat{Z}^{(2)}\right]}^T\begin{bmatrix}\overline{C}_{11}^{(2)} & \overline{C}_{13}^{(2)}\\ C_{13}^{(2)} & \overline{C}_{33}^{(2)}\end{bmatrix}\underset{(2\times4)}{\left[\widehat{Z}^{(2)}\right]}dz=$$

$$=b\int_{z_2}^{z_3}\begin{bmatrix}\overline{C}_{11}^{(2)} & z\overline{C}_{11}^{(2)} & z^2\overline{C}_{11}^{(2)} & \overline{C}_{13}^{(2)}\\ z\overline{C}_{11}^{(2)} & z^2\overline{C}_{11}^{(2)} & z^3\overline{C}_{11}^{(2)} & z\overline{C}_{13}^{(2)}\\ z^2\overline{C}_{11}^{(2)} & z^3\overline{C}_{11}^{(2)} & z^4\overline{C}_{11}^{(2)} & z^2\overline{C}_{13}^{(2)}\\ \overline{C}_{13}^{(2)} & z\overline{C}_{13}^{(2)} & z^2\overline{C}_{13}^{(2)} & \overline{C}_{33}^{(2)}\end{bmatrix}dz\ , \tag{5.5.3}$$

$$\underset{(3\times3)}{\left[\check{D}^{(2)}\right]}=b\int_{z_2}^{z_3}\overline{C}_{55}^{(2)}2\begin{Bmatrix}0\\1\\0\end{Bmatrix}2\begin{bmatrix}0 & 1 & 0\end{bmatrix}dz\ , \tag{5.5.4}$$

$$\underset{(2\times2)}{\left[D^{(1)}\right]}=b\int_{z_1}^{z_2}\underset{(2\times1)}{\left\lfloor Z^{(1)}\right\rfloor}^T\overline{C}_{11}^{(1)}\underset{(1\times2)}{\left\lfloor Z^{(1)}\right\rfloor}\,dz, \tag{5.5.5}$$

$$\underset{(2\times 2)}{\left[D^{(3)}\right]} = b\int\limits_{z_3}^{z_4} \underset{(2\times 1)}{\left\lfloor Z^{(3)} \right\rfloor^T} \overline{C}_{11}^{(3)} \underset{(1\times 2)}{\left\lfloor Z^{(3)} \right\rfloor} dz \; . \tag{5.5.6}$$

As in chapters 3 and 4, a stiffness coefficient in the Hooke's law for a ply of the lower face sheet, in the laminate coordinate system, will be denoted by ${}^{\alpha}\overline{C}_{ij}^{(1)}$, where the right superscript (1) denotes a stiffness coefficient associated with the 1-st sublaminate (i.e. the lower face sheet), the left superscript α is a number of a ply in the lower face sheet, subscripts i and j denote a position of the stiffness coefficient in the stiffness matrix. Analogously, a stiffness coefficient in the Hooke's law for a ply of the upper face sheet will be denoted by ${}^{\alpha}\overline{C}_{ij}^{(3)}$. Let n be a number of plies in the lower face sheet and let

$$\xi_1 = z_1,\, \xi_2,\, \xi_3,\, ...,\, \xi_n = z_2$$

be z-coordinates of the interfaces between the plies of the lower face sheet (Figure 3.3). Also, let m be a number of plies in the upper face sheet and let

$$\zeta_1 = z_3,\, \zeta_2,\, \zeta_3,\, ...,\, \zeta_m = z_4$$

be z-coordinates of the interfaces between the plies of the upper face sheet. Then

$$\underset{(2\times 2)}{\left[D^{(1)}\right]} = b\int\limits_{z_1}^{z_2} \underset{(2\times 1)}{\left\lfloor Z^{(1)} \right\rfloor^T} \overline{C}_{11}^{(1)} \underset{(1\times 2)}{\left\lfloor Z^{(1)} \right\rfloor} dz = b\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{11}^{(1)} \int\limits_{\xi_\alpha}^{\xi_{\alpha+1}} \underset{(2\times 1)}{\left\lfloor Z^{(1)} \right\rfloor^T} \underset{(1\times 2)}{\left\lfloor Z^{(1)} \right\rfloor} dz =$$

$$= b\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}_{11}^{(1)} \begin{bmatrix} \xi_{\alpha+1} - \xi_\alpha & \frac{1}{2}\left(\xi_{\alpha+1}^2 - \xi_\alpha^2\right) \\ \frac{1}{2}\left(\xi_{\alpha+1}^2 - \xi_\alpha^2\right) & \frac{1}{3}\left(\xi_{\alpha+1}^3 - \xi_\alpha^3\right) \end{bmatrix}, \tag{5.5.7}$$

$$\underset{(2\times 2)}{\left[D^{(3)}\right]} = b\int\limits_{z_3}^{z_4} \underset{(2\times 1)}{\left\lfloor Z^{(3)} \right\rfloor^T} \overline{C}_{11}^{(3)} \underset{(1\times 2)}{\left\lfloor Z^{(3)} \right\rfloor} dz = b\sum_{\alpha=1}^{m} {}^{\alpha}\overline{C}_{11}^{(3)} \int\limits_{\zeta_\alpha}^{\zeta_{\alpha+1}} \underset{(2\times 1)}{\left\lfloor Z^{(3)} \right\rfloor^T} \underset{(1\times 2)}{\left\lfloor Z^{(3)} \right\rfloor} dz =$$

$$= b\sum_{\alpha=1}^{m} {}^{\alpha}\overline{C}_{11}^{(3)} \begin{bmatrix} \zeta_{\alpha+1} - \zeta_\alpha & \frac{1}{2}\left(\zeta_{\alpha+1}^2 - \zeta_\alpha^2\right) \\ \frac{1}{2}\left(\zeta_{\alpha+1}^2 - \zeta_\alpha^2\right) & \frac{1}{3}\left(\zeta_{\alpha+1}^3 - \zeta_\alpha^3\right) \end{bmatrix}. \tag{5.5.8}$$

In absence of damage, the elastic constants of the core do not vary in the thickness direction, i.e. do not depend on z-coordinate. But the damage, that can occur in the core as a result of impact, can be distributed nonuniformly in the thickness direction, and, therefore, the elastic coefficients

$C_{ij}^{(2}$ of the damaged core can depend on z-coordinate. This will be taken into account by dividing the core into a number of nominal layers and by considering the elastic coefficients $C_{ij}^{(2)}$ of the core independent of the z-coordinate within a nominal layer, but varying from layer to layer. A stiffness coefficient in the Hooke's law for a nominal layer in the core, in the laminate coordinate system, will be denoted by ${}^{\alpha}\overline{C}_{ij}^{(2)}$, where the right superscript (2) denotes a stiffness coefficient associated with the second sublaminate (i.e. the core), the left superscript α is an ordinal number of a nominal layer in the core. Let s be a number of nominal layers in the core, and let

$$\eta_1 = z_2,\ \eta_2,\ \eta_3,\ldots,\ \eta_s = z_3$$

be z-coordinates of the interfaces between the nominal layers of the core. Then

$$\underset{(4\times4)}{\left[\widehat{D}^{(2)}\right]} = b\int_{z_2}^{z_3} \underset{(4\times2)}{\left[\widehat{Z}^{(2)}\right]^T} \begin{bmatrix} \overline{C}_{11}^{(2)} & \overline{C}_{13}^{(2)} \\ C_{13}^{(2)} & \overline{C}_{33}^{(2)} \end{bmatrix} \underset{(2\times4)}{\left[\widehat{Z}^{(2)}\right]} dz =$$

$$= b\int_{z_2}^{z_3} \begin{bmatrix} \overline{C}_{11}^{(2)} & z\overline{C}_{11}^{(2)} & z^2\overline{C}_{11}^{(2)} & \overline{C}_{13}^{(2)} \\ z\overline{C}_{11}^{(2)} & z^2\overline{C}_{11}^{(2)} & z^3\overline{C}_{11}^{(2)} & z\overline{C}_{13}^{(2)} \\ z^2\overline{C}_{11}^{(2)} & z^3\overline{C}_{11}^{(2)} & z^4\overline{C}_{11}^{(2)} & z^2\overline{C}_{13}^{(2)} \\ \overline{C}_{13}^{(2)} & z\overline{C}_{13}^{(2)} & z^2\overline{C}_{13}^{(2)} & \overline{C}_{33}^{(2)} \end{bmatrix} dz =$$

$$= b\sum_{\alpha=1}^{s} \int_{\eta_\alpha}^{\eta_{\alpha+1}} \begin{bmatrix} {}^{\alpha}\overline{C}_{11}^{(2)} & z\ {}^{\alpha}\overline{C}_{11}^{(2)} & z^2\ {}^{\alpha}\overline{C}_{11}^{(2)} & {}^{\alpha}\overline{C}_{13}^{(2)} \\ z\ {}^{\alpha}\overline{C}_{11}^{(2)} & z^2\ {}^{\alpha}\overline{C}_{11}^{(2)} & z^3\ {}^{\alpha}\overline{C}_{11}^{(2)} & z\ {}^{\alpha}\overline{C}_{13}^{(2)} \\ z^2\ {}^{\alpha}\overline{C}_{11}^{(2)} & z^3\ {}^{\alpha}\overline{C}_{11}^{(2)} & z^4\ {}^{\alpha}\overline{C}_{11}^{(2)} & z^2\ {}^{\alpha}\overline{C}_{13}^{(2)} \\ {}^{\alpha}\overline{C}_{13}^{(2)} & z\ {}^{\alpha}\overline{C}_{13}^{(2)} & z^2\ {}^{\alpha}\overline{C}_{13}^{(2)} & {}^{\alpha}\overline{C}_{33}^{(2)} \end{bmatrix} dz =$$

$$= b\sum_{\alpha=1}^{s} \begin{bmatrix} {}^{\alpha}\overline{C}_{11}^{(2)}\left(\eta_{\alpha+1}-\eta_\alpha\right) & {}^{\alpha}\overline{C}_{11}^{(2)}\frac{1}{2}\left(\eta_{\alpha+1}^2-\eta_\alpha^2\right) & {}^{\alpha}\overline{C}_{11}^{(2)}\frac{1}{3}\left(\eta_{\alpha+1}^3-\eta_\alpha^3\right) & {}^{\alpha}\overline{C}_{13}^{(2)}\left(\eta_{\alpha+1}-\eta_\alpha\right) \\ {}^{\alpha}\overline{C}_{11}^{(2)}\frac{1}{2}\left(\eta_{\alpha+1}^2-\eta_\alpha^2\right) & {}^{\alpha}\overline{C}_{11}^{(2)}\frac{1}{3}\left(\eta_{\alpha+1}^3-\eta_\alpha^3\right) & {}^{\alpha}\overline{C}_{11}^{(2)}\frac{1}{4}\left(\eta_{\alpha+1}^4-\eta_\alpha^4\right) & {}^{\alpha}\overline{C}_{13}^{(2)}\frac{1}{2}\left(\eta_{\alpha+1}^2-\eta_\alpha^2\right) \\ {}^{\alpha}\overline{C}_{11}^{(2)}\frac{1}{3}\left(\eta_{\alpha+1}^3-\eta_\alpha^3\right) & {}^{\alpha}\overline{C}_{11}^{(2)}\frac{1}{4}\left(\eta_{\alpha+1}^4-\eta_\alpha^4\right) & {}^{\alpha}\overline{C}_{11}^{(2)}\frac{1}{5}\left(\eta_{\alpha+1}^5-\eta_\alpha^5\right) & {}^{\alpha}\overline{C}_{13}^{(2)}\frac{1}{3}\left(\eta_{\alpha+1}^3-\eta_\alpha^3\right) \\ {}^{\alpha}\overline{C}_{13}^{(2)}\left(\eta_{\alpha+1}-\eta_\alpha\right) & {}^{\alpha}\overline{C}_{13}^{(2)}\frac{1}{2}\left(\eta_{\alpha+1}^2-\eta_\alpha^2\right) & {}^{\alpha}\overline{C}_{13}^{(2)}\frac{1}{3}\left(\eta_{\alpha+1}^3-\eta_\alpha^3\right) & {}^{\alpha}\overline{C}_{33}^{(2)}\left(\eta_{\alpha+1}-\eta_\alpha\right) \end{bmatrix} \tag{5.5.9}$$

$$\underset{(3\times3)}{\left[\check{D}^{(2)}\right]} = b\int_{z_2}^{z_3} \begin{Bmatrix} 0 \\ 2 \\ 0 \end{Bmatrix} \overline{C}_{55}^{(2)}(z) \left\lfloor\ 0\quad 2\quad 0\ \right\rfloor dz =$$

$$= b \begin{Bmatrix} 0 \\ 2 \\ 0 \end{Bmatrix} \begin{bmatrix} 0 & 2 & 0 \end{bmatrix} \int_{z_2}^{z_3} \overline{C}_{55}^{(2)}(z)\ dz =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \left(4b \sum_{\alpha=1}^{s} \int_{\eta_\alpha}^{\eta_{\alpha+1}} {}^{\alpha}\overline{C}_{55}^{(2)}\ dz \right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.5.10}$$

5.6 Strain energy of elastic foundation

The strain energy of the elastic foundation is defined by expression

$$U_f = \frac{1}{2} b \int_0^L s(x) \left[w^{(1)}(x,t) \right]^2 dx\ , \tag{5.6.1}$$

where $s(x)$ is a modulus of the foundation. According to equation (5.2.1),

$$w^{(1)} = w_0 + \varepsilon_{zz}^{(2)} z_2 \quad (z_1 \le z \le z_2)$$

or

$$w^{(1)} = \begin{bmatrix} 1 & 0 & z_2 \end{bmatrix} \begin{Bmatrix} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & z_2 \end{bmatrix} \{f\}. \tag{5.6.2}$$

Then

$$\left(w^{(1)} \right)^2 = \{f\}^T \begin{Bmatrix} 1 \\ 0 \\ z_2 \end{Bmatrix} \begin{bmatrix} 1 & 0 & z_2 \end{bmatrix} \{f\} \tag{5.6.3}$$

or

$$\left(w^{(1)} \right)^2 = \underset{(1\times3)}{\{f\}^T} \underset{(3\times3)}{\left[\overline{D} \right]} \underset{(3\times1)}{\{f\}}, \tag{5.6.4}$$

where

$$\left[\overline{D} \right] = \begin{bmatrix} 1 \\ 0 \\ z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & z_2 \\ 0 & 0 & 0 \\ z_2 & 0 & z_2^2 \end{bmatrix}. \tag{5.6.5}$$

Substitution of equation (5.6.4) into equation (5.6.1) yields

$$U_f = \frac{1}{2} b \int_0^L s(x) \underset{(1\times3)}{\{f\}}^T \underset{(3\times3)}{\left[\,\overline{D}\,\right]} \underset{(3\times1)}{\{f\}}\, dx\,. \tag{5.6.6}$$

5.7 Potential energy of the platform and the cargo in the gravity field

In order to obtain the expression for the potential energy Π of a wide beam and the cargo in the gravity field, we need to set in expression (4.9.1) $\varepsilon_{yz}^{(2)} = 0$. Then

$$\Pi = b \int_0^L \underset{(1\times3)}{\{f\}}^T \underset{(3\times1)}{\{\Gamma\}}\, dx\,, \tag{5.7.1}$$

where

$$\{f\} = \begin{Bmatrix} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{Bmatrix},$$

$$\underset{(3\times1)}{\{\Gamma\}} = \begin{Bmatrix} g\left[\rho^{(1)}(z_2 - z_1) + \rho^{(2)}(z_3 - z_2) + \rho^{(3)}(z_4 - z_3) + \mu\, H(x)\right] \\ 0 \\ g\left[\rho^{(1)} z_2 (z_2 - z_1) + \frac{1}{2}\rho^{(2)}\left(z_3^2 - z_2^2\right) + \rho^{(3)}\, z_3 (z_4 - z_3) + \mu\, H(x)\, z_3\right] \end{Bmatrix}. \tag{5.7.2}$$

5.8 Kinetic energy of the platform

The kinetic energy of the platform of the k-th sublaminate (k=1, 2, 3)

$$K^{(k)} = \frac{1}{2}\rho^{(k)} b \int_0^L \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \dot{u}^{(k)} \\ \dot{w}^{(k)} \end{Bmatrix}^T \begin{Bmatrix} \dot{u}^{(k)} \\ \dot{w}^{(k)} \end{Bmatrix} dz\, dx\,. \tag{5.8.1}$$

According to equation (5.2.11),

$$\begin{Bmatrix} \dot{u}^{(k)} \\ \dot{w}^{(k)} \end{Bmatrix} = \left[\widetilde{Z}^{(k)}\right]\left[\widetilde{\partial}^{(k)}\right] \frac{\partial}{\partial t}\{f\}, \tag{5.8.2}$$

where quantities, entering into equation (5.8.2), are defined by equations (5.2.12)-(5.2.17). Therefore,

$$K^{(k)} = \frac{1}{2}\rho^{(k)} b \int_0^L \left(\left[\widetilde{\partial}^{(k)}\right] \frac{\partial}{\partial t}\{f\}\right)^T \left[\widetilde{D}^{(k)}\right] \left(\left[\widetilde{\partial}^{(k)}\right] \frac{\partial}{\partial t}\{f\}\right) dx\,, \tag{5.8.3}$$

where

$$\left[\widetilde{D}^{(k)}\right] = \int_{z_k}^{z_{k+1}} \left[\widetilde{Z}^{(k)}\right]^T \left[\widetilde{Z}^{(k)}\right] dz \qquad (k = 1, 2, 3)\,. \tag{5.8.4}$$

Substitution of (5.2.13) and (5.2.14) into (5.8.4) yields:

$$\underset{(3\times 3)}{\left[\widetilde{D}^{(1)}\right]} = \int_{z_1}^{z_2} \underset{(3\times 2)}{\left[\widetilde{Z}^{(1)}\right]^T} \underset{(2\times 3)}{\left[\widetilde{Z}^{(1)}\right]} dz = \int_{z_1}^{z_2} \begin{bmatrix} 1 & 0 \\ z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & z & 0 \\ 0 & 0 & 1 \end{bmatrix} dz =$$

$$= \begin{bmatrix} z_2 - z_1 & \frac{1}{2}\left(z_2^2 - z_1^2\right) & 0 \\ \frac{1}{2}\left(z_2^2 - z_1^2\right) & \frac{1}{3}\left(z_2^3 - z_1^3\right) & 0 \\ 0 & 0 & z_2 - z_1 \end{bmatrix}, \tag{5.8.5}$$

$$\underset{(4\times 4)}{\left[\widetilde{D}^{(2)}\right]} = \int_{z_2}^{z_3} \underset{(4\times 2)}{\left[\widetilde{Z}^{(2)}\right]^T} \underset{(2\times 4)}{\left[\widetilde{Z}^{(2)}\right]} dz = \int_{z_2}^{z_3} \begin{bmatrix} z & 0 \\ z^2 & 0 \\ 0 & 1 \\ 0 & z \end{bmatrix} \begin{bmatrix} z & z^2 & 0 & 0 \\ 0 & 0 & 1 & z \end{bmatrix} dz =$$

$$= \begin{bmatrix} \frac{1}{3}\left(z_3^3 - z_2^3\right) & \frac{1}{4}\left(z_3^4 - z_2^4\right) & 0 & 0 \\ \frac{1}{4}\left(z_3^4 - z_2^4\right) & \frac{1}{5}\left(z_3^5 - z_2^5\right) & 0 & 0 \\ 0 & 0 & z_3 - z_2 & \frac{1}{2}\left(z_3^2 - z_2^2\right) \\ 0 & 0 & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{3}\left(z_3^3 - z_2^3\right) \end{bmatrix}, \tag{5.8.6}$$

$$\underset{(3\times 3)}{\left[\widetilde{D}^{(3)}\right]} = \int_{z_3}^{z_4} \underset{(3\times 2)}{\left[\widetilde{Z}^{(3)}\right]^T} \underset{(2\times 3)}{\left[\widetilde{Z}^{(3)}\right]} dz = \int_{z_3}^{z_4} \begin{bmatrix} 1 & 0 \\ z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & z & 0 \\ 0 & 0 & 1 \end{bmatrix} dz =$$

$$= \begin{bmatrix} z_4 - z_3 & \frac{1}{2}\left(z_4^2 - z_3^2\right) & 0 \\ \frac{1}{2}\left(z_4^2 - z_3^2\right) & \frac{1}{3}\left(z_4^3 - z_3^3\right) & 0 \\ 0 & 0 & z_4 - z_3 \end{bmatrix}. \tag{5.8.7}$$

So, the kinetic energy of the whole sandwich plate is

$$K_p = K^{(1)} + K^{(2)} + K^{(3)} =$$

$$= \frac{1}{2}\rho^{(1)} b \int_0^L \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \underset{(3\times3)}{\left[\widetilde{D}^{(1)}\right]} \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) dx \; +$$

$$+ \frac{1}{2}\rho^{(2)} b \int_0^L \left(\underset{(4\times3)}{\left[\widetilde{\partial}^{(2)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \underset{(4\times4)}{\left[\widetilde{D}^{(2)}\right]} \left(\underset{(4\times3)}{\left[\widetilde{\partial}^{(2)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) dx \; +$$

$$+ \frac{1}{2}\rho^{(3)} b \int_0^L \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(3)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \underset{(3\times3)}{\left[\widetilde{D}^{(3)}\right]} \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(3)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) dx \; . \tag{5.8.8}$$

5.9 Kinetic energy of the cargo

According to equation (4.11.1), kinetic energy of the cargo is

$$K_c = \frac{1}{2} b \int_0^L \mu \; H(x) \left(\frac{\partial w^{(3)}(x,y,t)}{\partial t} \right)^2 dx \; , \tag{5.9.1}$$

where $w^{(3)}(x,y,t)$ is defined by expression (5.2.3), that can be written in the form

$$w^{(3)} = \left\{ \begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\}^T \left\{ \begin{array}{c} 1 \\ 0 \\ z_3 \end{array} \right\} = \underset{(1\times3)}{\{f\}}^T \left\{ \begin{array}{c} 1 \\ 0 \\ z_3 \end{array} \right\} . \tag{5.9.2}$$

Then

$$\left\{ \frac{\partial w^{(3)}}{\partial t} \right\}^{(2)} = \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \left\{ \begin{array}{c} 1 \\ 0 \\ z_3 \end{array} \right\} \left\lfloor \begin{array}{ccc} 1 & 0 & z_3 \end{array} \right\rfloor \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) =$$

$$= \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \underset{(3\times3)}{\left[\widetilde{D}_c\right]} \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) , \tag{5.9.3}$$

where

$$\underset{(3\times3)}{\left[\widetilde{D}_c\right]} = \left\{ \begin{array}{c} 1 \\ 0 \\ z_3 \end{array} \right\} \left\lfloor \begin{array}{ccc} 1 & 0 & z_3 \end{array} \right\rfloor = \begin{bmatrix} 1 & 0 & z_3 \\ 0 & 0 & 0 \\ z_3 & 0 & z_3^2 \end{bmatrix} . \tag{5.9.4}$$

Substitution of (5.9.4) into (4.11.1) yields:

$$K_c = \frac{1}{2} b \int_0^L \mu \, H(x) \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \underset{(3\times3)}{\left[\widetilde{D}_c\right]} \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) dx \, . \tag{5.9.5}$$

5.10 Finite Element Formulation for the Cargo Platform Modelled as a Plate under Cylindrical Bending

5.10.1 Strain energy in terms of the nodal variables

Strain energy of the finite element that represents a platform is defined by expression (5.5.2), if in this expression the total length of the platform L is substituted by a length l of a finite element, and x implies the local, element coordinate, not a global coordinate as in equation (5.5.2). This expression is

$$U = \frac{1}{2}\int_0^l \left(\underset{(2\times3)}{\left[\partial^{(1)}\right]} \underset{(3\times1)}{\{f\}} + \underset{(2\times1)}{\left\{\eta^{(1)}\right\}} \right)^T \underset{(2\times2)}{\left[D^{(1)}\right]} \left(\underset{(2\times3)}{\left[\partial^{(1)}\right]} \underset{(3\times1)}{\{f\}} + \underset{(2\times1)}{\left\{\eta^{(1)}\right\}} \right) dx +$$

$$+\frac{1}{2}\int_0^l \left(\underset{(2\times3)}{\left[\partial^{(3)}\right]} \underset{(3\times1)}{\{f\}} + \underset{(2\times1)}{\left\{\eta^{(3)}\right\}} \right)^T \underset{(2\times2)}{\left[D^{(3)}\right]} \left(\underset{(2\times3)}{\left[\partial^{(3)}\right]} \underset{(3\times1)}{\{f\}} + \underset{(2\times1)}{\left\{\eta^{(3)}\right\}} \right) dx +$$

$$+\frac{1}{2}\int_0^l \left(\underset{(4\times3)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(3\times1)}{\{f\}} + \underset{(4\times1)}{\left[\widehat{\eta}^{(2)}\right]} \right)^T \underset{(4\times4)}{\left[\widehat{D}^{(2)}\right]} \left(\underset{(4\times3)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(3\times1)}{\{f\}} + \underset{(4\times1)}{\left[\widehat{\eta}^{(2)}\right]} \right) dx +$$

$$+\frac{1}{2}\int_0^l \{f\}^T \underset{(3\times3)}{\left[\check{D}^{(2)}\right]} \underset{(3\times1)}{\{f\}}\, dx, \tag{5.10.1}$$

where

$$\underset{(3\times1)}{\{f\}} = \left\{ \begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\} \tag{5.10.2}$$

is a column-vector of the unknown functions,

$$\underset{(2\times3)}{\left[\partial^{(1)}\right]} = \begin{bmatrix} 0 & 2z_2\frac{d}{dx} & \frac{1}{2}z_2^2\frac{d^2}{dx^2} \\ -\frac{d^2}{dx^2} & 0 & -z_2\frac{d^2}{dx^2} \end{bmatrix}, \qquad \underset{(2\times3)}{\left[\partial^{(3)}\right]} = \begin{bmatrix} 0 & 2z_3\frac{d}{dx} & \frac{1}{2}z_3^2\frac{d^2}{dx^2} \\ -\frac{d^2}{dx^2} & 0 & -z_3\frac{d^2}{dx^2} \end{bmatrix}, \tag{5.10.3}$$

$$\underset{(4\times3)}{\left[\widehat{\partial}^{(2)}\right]} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{d^2}{dx^2} & 2\frac{d}{dx} & 0 \\ 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} \\ 0 & 0 & 1 \end{bmatrix} \tag{5.10.4}$$

are matrices of differential operators, also

$$\underset{(2\times1)}{\left\{\eta^{(1)}\right\}} = \left\{ \begin{array}{c} \frac{1}{2}\left(w_{0,x} + z_2\ \varepsilon^{(2)}_{zz,x}\right)^2 \\ 0 \end{array} \right\}, \quad \underset{(2\times1)}{\left\{\eta^{(3)}\right\}} = \left\{ \begin{array}{c} \frac{1}{2}\left(w_{0,x} + z_3\ \varepsilon^{(2)}_{zz,x}\right)^2 \\ 0 \end{array} \right\}, \tag{5.10.5}$$

$$\underset{(4\times1)}{\left[\widehat{\eta}^{(2)}\right]} = \left\{ \begin{array}{c} \frac{1}{2}(w_{0,x})^2 \\ w_{0,x}\varepsilon^{(2)}_{zz,x} \\ \frac{1}{2}\left(\varepsilon^{(2)}_{zz,x}\right)^2 \\ 0 \end{array} \right\} \tag{5.10.6}$$

are the column-matrices of non-linear combinations of the unknown functions of the problem. In addition,

$$\underset{(2\times2)}{\left[D^{(1)}\right]} = b\sum_{\alpha=1}^{n} {}^{\alpha}\overline{C}^{(1)}_{11} \begin{bmatrix} \xi_{\alpha+1} - \xi_\alpha & \frac{1}{2}\left(\xi^2_{\alpha+1} - \xi^2_\alpha\right) \\ \frac{1}{2}\left(\xi^2_{\alpha+1} - \xi^2_\alpha\right) & \frac{1}{3}\left(\xi^3_{\alpha+1} - \xi^3_\alpha\right) \end{bmatrix}, \tag{5.10.7}$$

$$\underset{(2\times2)}{\left[D^{(3)}\right]} = b\sum_{\alpha=1}^{m} {}^{\alpha}\overline{C}^{(3)}_{11} \begin{bmatrix} \zeta_{\alpha+1} - \zeta_\alpha & \frac{1}{2}\left(\zeta^2_{\alpha+1} - \zeta^2_\alpha\right) \\ \frac{1}{2}\left(\zeta^2_{\alpha+1} - \zeta^2_\alpha\right) & \frac{1}{3}\left(\zeta^3_{\alpha+1} - \zeta^3_\alpha\right) \end{bmatrix}, \tag{5.10.8}$$

$$\underset{(4\times4)}{\left[\widehat{D}^{(2)}\right]} = b\sum_{\alpha=1}^{s} \begin{bmatrix} {}^{\alpha}\overline{C}^{(2)}_{11}\left(\eta_{\alpha+1} - \eta_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{11}\frac{1}{2}\left(\eta^2_{\alpha+1} - \eta^2_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{11}\frac{1}{3}\left(\eta^3_{\alpha+1} - \eta^3_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{13}\left(\eta_{\alpha+1} - \eta_\alpha\right) \\ {}^{\alpha}\overline{C}^{(2)}_{11}\frac{1}{2}\left(\eta^2_{\alpha+1} - \eta^2_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{11}\frac{1}{3}\left(\eta^3_{\alpha+1} - \eta^3_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{11}\frac{1}{4}\left(\eta^4_{\alpha+1} - \eta^4_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{13}\frac{1}{2}\left(\eta^2_{\alpha+1} - \eta^2_\alpha\right) \\ {}^{\alpha}\overline{C}^{(2)}_{11}\frac{1}{3}\left(\eta^3_{\alpha+1} - \eta^3_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{11}\frac{1}{4}\left(\eta^4_{\alpha+1} - \eta^4_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{11}\frac{1}{5}\left(\eta^5_{\alpha+1} - \eta^5_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{13}\frac{1}{3}\left(\eta^3_{\alpha+1} - \eta^3_\alpha\right) \\ {}^{\alpha}\overline{C}^{(2)}_{13}\left(\eta_{\alpha+1} - \eta_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{13}\frac{1}{2}\left(\eta^2_{\alpha+1} - \eta^2_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{13}\frac{1}{3}\left(\eta^3_{\alpha+1} - \eta^3_\alpha\right) & {}^{\alpha}\overline{C}^{(2)}_{33}\left(\eta_{\alpha+1} - \eta_\alpha\right) \end{bmatrix}. \tag{5.10.9}$$

$$\underset{(3\times3)}{\left[\breve{D}^{(2)}\right]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \left(4b\sum_{\alpha=1}^{s}\int_{\eta_\alpha}^{\eta_{\alpha+1}} {}^{\alpha}\overline{C}^{(2)}_{55}\ dz\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.10.10}$$

are the matrices of material constants, averaged over the thickness of a sublaminate.

Let us represent the unknown functions w_0, $\varepsilon^{(2)}_{xz}$, $\varepsilon^{(2)}_{zz}$ by **interpolation polynomials**. The general rules for choosing the interpolation polynomials are the following: if the Hamilton's principle contains derivatives of a field variable through order m, then an interpolation polynomial must satisfy

the following requirements (Cook, Malkus, Plesha, 1989):

1) it must be a complete polynomial of degree m;

2) across boundaries between elements, there must be continuity of the field variable and its derivative through order $m-1$, therefore these derivatives must be carried as nodal variables.

The first requirement ensures that the m-th derivative of the interpolation polynomial does not vanish in the Hamilton's principle. The completeness of the interpolation polynomial is necessary in order to make an element capable to represent a constant value of any of the m derivatives of the field variable. The second requirement is due to the fact that if the Hamilton's principle contains derivatives of a field variable ϕ through order m, then the primary variables associated with this field variable, are ϕ, $\frac{\partial\phi}{\partial x}, \ldots, \frac{\partial^{m-1}\phi}{\partial x^{m-1}}$, and the primary variables must be continuous at the interelement boundaries (Reddy, 1993).

In the problem under consideration, the interpolation polynomials will be chosen to satisfy the minimal requirements of general rules, presented above. In other words, the simplest allowable elements will be used, which is a general practice in solving the transient and nonlinear problems (Cook, Malkus, Plesha, 1989).

The maximum order of derivatives of $\varepsilon_{xz}^{(2)}$, entering into the Hamilton's principle, is 1. Therefore, an interpolation polynomial for $\varepsilon_{xz}^{(2)}$ must be of at least first degree, and across boundaries between elements there must be continuity of, at least, $\varepsilon_{xz}^{(2)}$ (continuity of derivatives of $\varepsilon_{xz}^{(2)}$ is not required). Therefore, we choose the first degree Lagrange polynomials to interpolate $\varepsilon_{xz}^{(2)}$ $(k=1,2,3)$ as functions of x[1]:

$$\varepsilon_{xz}^{(2)} = \lfloor M \rfloor \{\bar{\varepsilon}\} = \lfloor M_1 \; M_2 \rfloor \{\bar{\varepsilon}\} \ , \tag{5.10.11}$$

where

$$M_1 = 1 - \frac{x}{l}, \; M_2 = \frac{x}{l}, \tag{5.10.12}$$

$$\left\{\bar{\varepsilon}^{(2)}\right\} = \left\{ \begin{array}{c} \varepsilon_{xz}^{(2)}(0) \\ \varepsilon_{xz}^{(2)}(l) \end{array} \right\}. \tag{5.10.13}$$

The maximum order of the derivatives of w_0 and $\varepsilon_{zz}^{(2)}$ is 2. Therefore, interpolation polynomials for w_0 and $\varepsilon_{zz}^{(2)}$ must be of at least second degree and must have derivatives, continuous at the element boundaries up to the first order (i.e. w_0, $\frac{dw_0}{dx}$, $\varepsilon_{zz}^{(2)}$ and $\frac{d\varepsilon_{zz}^{(2)}}{dx}$ must be continuous). Therefore, we

[1]Here and further in this section devoted to the FE formulation, it is implied for simplicity of notations, that x is a coordinate in the element (local) coordinate system.

choose the Hermit polynomial of the third degree to interpolate w_0 and $\varepsilon_{zz}^{(2)}$ (the lowest degree of the Hermit polynomials is three):

$$w_0 = \lfloor N \rfloor \{\bar{w}\} = \lfloor N_1 \ N_2 \ N_3 \ N_4 \rfloor \{\bar{w}\}, \tag{5.10.14}$$

$$\varepsilon_{zz}^{(2)} = \lfloor N \rfloor \{\bar{\varepsilon}\} = \lfloor N_1 \ N_2 \ N_3 \ N_4 \rfloor \{\bar{\varepsilon}\}, \tag{5.10.15}$$

where

$$N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}, \ N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}, \ N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3}, \ N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2}, \tag{5.10.16}$$

$$\{\bar{w}\} = \left\{ \begin{array}{c} w_0(0) \\ \frac{dw_0}{dx}(0) \\ w_0(l) \\ \frac{dw_0}{dx}(l) \end{array} \right\}, \tag{5.10.17}$$

$$\{\bar{\varepsilon}\} = \left\{ \begin{array}{c} \varepsilon_{zz}^{(2)}(0) \\ \frac{d\varepsilon_{zz}^{(2)}}{dx}(0) \\ \varepsilon_{zz}^{(2)}(l) \\ \frac{d\varepsilon_{zz}^{(2)}}{dx}(l) \end{array} \right\}. \tag{5.10.18}$$

So, the combined finite element has 10 degrees of freedom. At each node there are **5 nodal variables**: w_0, $\frac{dw_0}{dx}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{zz}^{(2)}$, $\frac{d\varepsilon_{zz}^{(2)}}{dx}$.

Let us write expression (5.10.1) for the strain energy in terms of the nodal variables. First, we will obtain an expression for $\left[\partial^{(1)}\right]\{f\}$ in terms of the nodal variables:

$$\underset{(2\times3)}{\left[\partial^{(1)}\right]} \underset{(3\times1)}{\{f\}} = \left[\partial^{(1)}\right] \left\{ \begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\} =$$

$$= \left[\partial^{(1)}\right] \left\{ \begin{array}{c} \underset{(1\times4)(4\times1)}{\lfloor N \rfloor \{\bar{w}\}} \\ \underset{(1\times2)(2\times1)}{\lfloor M \rfloor \{\bar{\varepsilon}\}} \\ \underset{(1\times4)(4\times1)}{\lfloor N \rfloor \{\bar{\varepsilon}\}} \end{array} \right\} = \underset{(2\times3)}{\left[\partial^{(1)}\right]} \left[\begin{array}{ccc} \underset{(1\times4)}{\lfloor N \rfloor} & \underset{(1\times2)}{\lfloor 0 \rfloor} & \underset{(1\times4)}{\lfloor 0 \rfloor} \\ \underset{(1\times4)}{\lfloor 0 \rfloor} & \underset{(1\times2)}{\lfloor M \rfloor} & \underset{(1\times4)}{\lfloor 0 \rfloor} \\ \underset{(1\times4)}{\lfloor 0 \rfloor} & \underset{(1\times2)}{\lfloor 0 \rfloor} & \underset{(1\times4)}{\lfloor N \rfloor} \end{array} \right] \left\{ \begin{array}{c} \underset{(4\times1)}{\{\overline{w}\}} \\ \underset{(2\times1)}{\{\bar{\varepsilon}\}} \\ \underset{(4\times1)}{\{\bar{\varepsilon}\}} \end{array} \right\} =$$

$$= \underset{(2\times 10)}{\left[B^{(1)}\right]} \underset{(10\times 1)}{\{d\}} , \tag{5.10.19}$$

where

$$\underset{(2\times 10)}{\left[B^{(1)}\right]} \equiv \underset{(2\times 3)}{\left[\partial^{(1)}\right]} \underset{(3\times 10)}{\begin{bmatrix} \underset{(1\times 4)}{\lfloor N \rfloor} & \underset{(1\times 2)}{\lfloor 0 \rfloor} & \underset{(1\times 4)}{\lfloor 0 \rfloor} \\ \underset{(1\times 4)}{\lfloor 0 \rfloor} & \underset{(1\times 2)}{\lfloor M \rfloor} & \underset{(1\times 4)}{\lfloor 0 \rfloor} \\ \underset{(1\times 4)}{\lfloor 0 \rfloor} & \underset{(1\times 2)}{\lfloor 0 \rfloor} & \underset{(1\times 4)}{\lfloor N \rfloor} \end{bmatrix}} =$$

$$= \begin{bmatrix} 0 & 2z_2\frac{d}{dx} & \frac{1}{2}z_2^2\frac{d^2}{dx^2} \\ -\frac{d^2}{dx^2} & 0 & -z_2\frac{d^2}{dx^2} \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_1 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 2z_2\frac{dM_1}{dx} & 2z_2\frac{dM_2}{dx} & \frac{1}{2}z_2^2\frac{d^2N_1}{dx^2} & \frac{1}{2}z_2^2\frac{d^2N_2}{dx^2} & \frac{1}{2}z_2^2\frac{d^2N_3}{dx^2} & \frac{1}{2}z_2^2\frac{d^2N_4}{dx^2} \\ -\frac{d^2N_1}{dx^2} & -\frac{d^2N_2}{dx^2} & -\frac{d^2N_3}{dx^2} & -\frac{d^2N_4}{dx^2} & 0 & 0 & -z_2\frac{d^2N_1}{dx^2} & -z_2\frac{d^2N_2}{dx^2} & -z_2\frac{d^2N_3}{dx^2} & -z_2\frac{d^2N_4}{dx^2} \end{bmatrix}, \tag{5.10.20}$$

$$\{d\} \equiv \left\{ \begin{array}{c} \underset{(4\times 1)}{\{\overline{w}\}} \\ \underset{(2\times 1)}{\{\overline{\epsilon}\}} \\ \underset{(4\times 1)}{\{\overline{\varepsilon}\}} \end{array} \right\} = \left\{ \begin{array}{c} d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_{10} \end{array} \right\}. \tag{5.10.21}$$

In equation (5.10.21)

$$d_1 = w_0(0),\ d_2 = w_0'(0),\ d_3 = w_0(l),\ d_4 = w_0'(l),\ d_5 = \varepsilon_{xz}^{(2)}(0),\ d_6 = \varepsilon_{xz}^{(2)}(l),$$

$$d_7 = \varepsilon_{zz}^{(2)}(0),\ d_8 = \frac{d\varepsilon_{zz}^{(2)}}{dx}(0),\ d_9 = \varepsilon_{zz}^{(2)}(l),\ d_{10} = \frac{d\varepsilon_{zz}^{(2)}}{dx}(l). \tag{5.10.22}$$

These are the nodal variables of a finite element.

Now, let us obtain expression for $\left\{\eta^{(1)}\right\}$, defined by expression (5.10.5), in terms of the nodal variables.

$$\underset{(2\times 1)}{\left\{\eta^{(1)}\right\}} = \left\{ \begin{array}{c} \frac{1}{2}\left(w_{0,x} + z_2\, \varepsilon_{zz,x}^{(2)}\right)^2 \\ 0 \end{array} \right\} = \left\{ \begin{array}{c} \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 + z_2\frac{dw_0}{dx}\frac{d\varepsilon_{zz}^{(2)}}{dx} + \frac{1}{2}z_2^2\left(\frac{d\varepsilon_{zz}^{(2)}}{dx}\right)^2 \\ 0 \end{array} \right\}. \tag{5.10.23}$$

Using representation of the unknown functions w_0 and $\varepsilon_{zz}^{(2)}$ in terms of the nodal variables (equations (5.10.14) and (5.10.15)), one can obtain

$$\frac{dw_0}{dx} = \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \underset{(4\times1)}{\{\overline{w}\}} = \underset{(1\times4)}{\{\overline{w}\}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}}, \qquad \frac{d\varepsilon_{zz}^{(2)}}{dx} = \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \underset{(4\times1)}{\{\overline{\varepsilon}\}} = \underset{(1\times4)}{\{\overline{\varepsilon}\}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \tag{5.10.24}$$

Therefore,

$$\left(\frac{dw_0}{dx}\right)^2 = \underset{(1\times4)}{\{\overline{w}\}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \underset{(4\times1)}{\{\overline{w}\}}, \tag{5.10.25}$$

$$\left(\frac{d\varepsilon_{zz}^{(2)}}{dx}\right)^2 = \underset{(1\times4)}{\{\overline{\varepsilon}\}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \underset{(4\times1)}{\{\overline{\varepsilon}\}}, \tag{5.10.26}$$

$$\frac{dw_0}{dx}\frac{d\varepsilon_{zz}^{(2)}}{dx} = \underset{(1\times4)}{\{\overline{w}\}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \underset{(4\times1)}{\{\overline{\varepsilon}\}}. \tag{5.10.27}$$

The substitution of expressions (5.10.24)-(5.10.26) into the expression $\frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 + z_2\frac{dw_0}{dx}\frac{d\varepsilon_{zz}^{(2)}}{dx} + \frac{1}{2}z_2^2\left(\frac{d\varepsilon_{zz}^{(2)}}{dx}\right)^2$ yields

$$\frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 + z_2\frac{dw_0}{dx}\frac{d\varepsilon_{zz}^{(2)}}{dx} + \frac{1}{2}z_2^2\left(\frac{d\varepsilon_{zz}^{(2)}}{dx}\right)^2 =$$

$$= \frac{1}{2}\underset{(1\times4)}{\{\overline{w}\}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \underset{(4\times1)}{\{\overline{w}\}} + z_2\underset{(1\times4)}{\{\overline{w}\}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \underset{(4\times1)}{\{\overline{\varepsilon}\}} + \frac{1}{2}z_2^2\underset{(1\times4)}{\{\overline{\varepsilon}\}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \underset{(4\times1)}{\{\overline{\varepsilon}\}} =$$

$$= \frac{1}{2}\underset{(1\times4)}{\{\overline{w}\}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \underset{(4\times1)}{\{\overline{w}\}} + \left(z_2\underset{(1\times4)}{\{\overline{w}\}}^T + \frac{1}{2}z_2^2\underset{(1\times4)}{\{\overline{\varepsilon}\}}^T\right) \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \underset{(4\times1)}{\{\overline{\varepsilon}\}} =$$

$$\left\lfloor \frac{1}{2}\underset{(1\times4)}{\{\overline{w}\}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \quad 0 \quad 0 \quad \left(z_2\underset{(1\times4)}{\{\overline{w}\}}^T + \frac{1}{2}z_2^2\underset{(1\times4)}{\{\overline{\varepsilon}\}}^T\right) \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \right\rfloor \left\{ \begin{array}{c} \underset{(4\times1)}{\{\overline{w}\}} \\ \underset{(2\times1)}{\{\overline{\varepsilon}\}} \\ \underset{(4\times1)}{\{\overline{\varepsilon}\}} \end{array} \right\} =$$

$$\left\lfloor \frac{1}{2}\underset{(1\times4)}{\{\overline{w}\}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \quad 0 \quad 0 \quad \left(z_2\underset{(1\times4)}{\{\overline{w}\}}^T + \frac{1}{2}z_2^2\underset{(1\times4)}{\{\overline{\varepsilon}\}}^T\right) \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \right\rfloor \underset{(10\times1)}{\{d\}}. \tag{5.10.28}$$

Therefore,

$$\underset{(2\times1)}{\left\{\eta^{(1)}\right\}} \equiv \left\{\begin{array}{c} \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2 + z_2\frac{dw_0}{dx}\frac{d\varepsilon_{zz}^{(2)}}{dx} + \frac{1}{2}z_2^2\left(\frac{d\varepsilon_{zz}^{(2)}}{dx}\right)^2 \\ 0 \end{array}\right\} = \underset{(2\times10)}{\left[\beta^{(1)}\right]}\underset{(10\times1)}{\{d\}}, \tag{5.10.29}$$

where

$$\underset{(2\times10)}{\left[\beta^{(1)}\right]} \equiv \left[\begin{array}{ccc} \frac{1}{2}\underset{(1\times4)}{\{\overline{w}\}}^T\underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} & 0\quad 0 & \left(z_2\underset{(1\times4)}{\{\overline{w}\}}^T + \frac{1}{2}z_2^2\underset{(1\times4)}{\{\overline{\varepsilon}\}}^T\right)\underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}} \\ & \underset{(1\times10)}{\lfloor 0\rfloor} & \end{array}\right]. \tag{5.10.30}$$

Column-matrices $\underset{(1\times4)}{\{\overline{w}\}}$ and $\underset{(1\times4)}{\{\overline{\varepsilon}\}}$ can be written in the form

$$\underset{(4\times1)}{\{\overline{w}\}} = \underset{(4\times10)}{\left[\begin{array}{cc}\underset{(4\times4)}{[I]} & \underset{(4\times6)}{[0]}\end{array}\right]}\underset{(10\times1)}{\left\{\begin{array}{c}\underset{(4\times1)}{\{\overline{w}\}} \\ \underset{(2\times1)}{\{\overline{\epsilon}\}} \\ \underset{(4\times1)}{\{\overline{\varepsilon}\}}\end{array}\right\}} = \left[\begin{array}{cc}\underset{(4\times4)}{[I]} & \underset{(4\times6)}{[0]}\end{array}\right]\underset{(10\times1)}{\{d\}},$$

$$\underset{(4\times1)}{\{\overline{\varepsilon}\}} = \left[\begin{array}{cc}\underset{(4\times6)}{[0]} & \underset{(4\times4)}{[I]}\end{array}\right]\left\{\begin{array}{c}\underset{(4\times1)}{\{\overline{w}\}} \\ \underset{(2\times1)}{\{\overline{\epsilon}\}} \\ \underset{(4\times1)}{\{\overline{\varepsilon}\}}\end{array}\right\} = \left[\begin{array}{cc}\underset{(4\times6)}{[0]} & \underset{(4\times4)}{[I]}\end{array}\right]\underset{(10\times1)}{\{d\}},$$

where

$$\underset{(4\times4)}{[I]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\underset{(2\times10)}{\left[\beta^{(1)}\right]} \equiv \left[\begin{array}{ccc} \left(\underset{(1\times10)}{\{d\}}^T\underset{(10\times4)}{[\Phi]}\right) & 0\quad 0 & \left(\underset{(1\times10)}{\{d\}}^T\underset{(10\times4)}{[\Psi]}\right) \\ & \underset{(1\times10)}{\lfloor 0\rfloor} & \end{array}\right] =$$

$$= \underset{(2\times1)}{\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}} \underset{(1\times10)}{\{d\}}^T \underset{(10\times10)}{\begin{bmatrix} \underset{(10\times4)}{[\Phi]} & \underset{(10\times2)}{[0]} & \underset{(10\times4)}{\left[\Psi^{(1)}\right]} \end{bmatrix}}, \tag{5.10.31}$$

where

$$\underset{(10\times4)}{[\Phi]} = \frac{1}{2} \underset{(10\times4)}{\begin{bmatrix} \underset{(4\times4)}{[I]} & \underset{(4\times6)}{[0]} \end{bmatrix}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}}, \tag{5.10.32}$$

$$\underset{(10\times4)}{\left[\Psi^{(1)}\right]} = z_2 \left(\underset{(10\times4)}{\begin{bmatrix} \underset{(4\times4)}{[I]} & \underset{(4\times6)}{[0]} \end{bmatrix}}^T + \frac{1}{2} z_2 \underset{(10\times4)}{\begin{bmatrix} \underset{(4\times6)}{[0]} & \underset{(4\times4)}{[I]} \end{bmatrix}}^T \right) \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}}. \tag{5.10.33}$$

So,

$$\underset{(2\times3)}{\left[\partial^{(1)}\right]} \underset{(3\times1)}{\{f\}} + \underset{(2\times1)}{\left\{\eta^{(1)}\right\}} = \left(\underset{(2\times10)}{\left[B^{(1)}\right]} + \underset{(2\times10)}{\left[\beta^{(1)}\right]} \right) \underset{(10\times1)}{\{d\}}, \tag{5.10.34}$$

where

$$\underset{(2\times10)}{\left[B^{(1)}\right]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 2z_2\frac{dM_1}{dx} & 2z_2\frac{dM_2}{dx} & \frac{1}{2}z_2^2\frac{d^2N_1}{dx^2} & \frac{1}{2}z_2^2\frac{d^2N_2}{dx^2} & \frac{1}{2}z_2^2\frac{d^2N_3}{dx^2} & \frac{1}{2}z_2^2\frac{d^2N_4}{dx^2} \\ -\frac{d^2N_1}{dx^2} & -\frac{d^2N_2}{dx^2} & -\frac{d^2N_3}{dx^2} & -\frac{d^2N_4}{dx^2} & 0 & 0 & -z_2\frac{d^2N_1}{dx^2} & -z_2\frac{d^2N_2}{dx^2} & -z_2\frac{d^2N_3}{dx^2} & -z_2\frac{d^2N_4}{dx^2} \end{bmatrix},$$

$$\underset{(2\times10)}{\left[\beta^{(1)}\right]} = \underset{(2\times1)}{\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}} \underset{(1\times10)}{\{d\}}^T \underset{(10\times10)}{\begin{bmatrix} \underset{(10\times4)}{[\Phi]} & \underset{(10\times2)}{[0]} & \underset{(10\times4)}{\left[\Psi^{(1)}\right]} \end{bmatrix}},$$

$$\underset{(10\times4)}{[\Phi]} = \frac{1}{2} \underset{(10\times4)}{\begin{bmatrix} \underset{(4\times4)}{[I]} & \underset{(4\times6)}{[0]} \end{bmatrix}}^T \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}},$$

$$\underset{(10\times4)}{\left[\Psi^{(1)}\right]} = z_2 \left(\underset{(10\times4)}{\begin{bmatrix} \underset{(4\times4)}{[I]} & \underset{(4\times6)}{[0]} \end{bmatrix}}^T + \frac{1}{2} z_2 \underset{(10\times4)}{\begin{bmatrix} \underset{(4\times6)}{[0]} & \underset{(4\times4)}{[I]} \end{bmatrix}}^T \right) \underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}} \underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}},$$

$$\underset{(1\times4)}{\lfloor N\rfloor} = \lfloor N_1 \; N_2 \; N_3 \; N_4 \rfloor,$$

$$\lfloor M\rfloor = \lfloor M_1 \; M_2 \rfloor,$$

$$N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3},\ N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2},\ N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3},\ N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2},$$

$$M_1 = 1 - \frac{x}{l},\ M_2 = \frac{x}{l},$$

$\underset{(10\times1)}{\{d\}} = \left\lfloor\, w_0(0)\quad w_0'(0)\quad w_0(l)\quad w_0'(l)\quad \varepsilon_{xz}^{(k)}(0)\quad \varepsilon_{xz}^{(k)}(l)\quad \varepsilon_{zz}^{(2)}(0)\quad \frac{d\varepsilon_{zz}^{(2)}}{dx}(0)\quad \varepsilon_{zz}^{(2)}(l)\quad \frac{d\varepsilon_{zz}^{(2)}}{dx}(l)\,\right\rfloor^T$ is a column-matrix of the nodal variables.

So, the **first term in the expression (5.10.1) for the strain energy of the finite element of the plate** is

$$\frac{1}{2}\int_0^l \left(\underset{(2\times3)}{\left[\partial^{(1)}\right]} \underset{(3\times1)}{\{f\}} + \underset{(2\times1)}{\left\{\eta^{(1)}\right\}} \right)^T \underset{(2\times2)}{\left[D^{(1)}\right]} \left(\underset{(2\times3)}{\left[\partial^{(1)}\right]} \underset{(3\times1)}{\{f\}} + \underset{(2\times1)}{\left\{\eta^{(1)}\right\}} \right) dx =$$

$$= \frac{1}{2} \underset{(1\times10)}{\{d\}}^T \int_0^l \underset{(10\times2)}{\left(\underset{(2\times10)}{\left[B^{(1)}\right]} + \underset{(2\times10)}{\left[\beta^{(1)}\right]} \right)^T} \underset{(2\times2)}{\left[D^{(1)}\right]} \left(\underset{(2\times10)}{\left[B^{(1)}\right]} + \underset{(2\times10)}{\left[\beta^{(1)}\right]} \right) dx \underset{(10\times1)}{\{d\}} =$$

$$= \frac{1}{2} \underset{(1\times10)}{\{d\}}^T \underbrace{\int_0^l \underset{(10\times2)}{\left[B^{(1)}\right]^T} \underset{(2\times2)}{\left[D^{(1)}\right]} \underset{(2\times10)}{\left[B^{(1)}\right]} dx}_{\left[k^{(1)}\right]} \underset{(10\times1)}{\{d\}} +$$

$$+ \frac{1}{2} \underset{(1\times10)}{\{d\}}^T \int_0^l \underset{(10\times2)}{\left[\beta^{(1)}\right]^T} \underset{(2\times2)}{\left[D^{(1)}\right]} \underset{(2\times10)}{\left[B^{(1)}\right]} dx \underset{(10\times1)}{\{d\}} +$$

$$+ \frac{1}{2} \underset{(1\times10)}{\{d\}}^T \int_0^l \underset{(10\times2)}{\left[B^{(1)}\right]^T} \underset{(2\times2)}{\left[D^{(1)}\right]} \underset{(2\times10)}{\left[\beta^{(1)}\right]} dx \underset{(10\times1)}{\{d\}} +$$

$$+ \frac{1}{2} \underset{(1\times10)}{\{d\}}^T \int_0^l \underset{(10\times2)}{\left[\beta^{(1)}\right]^T} \underset{(2\times2)}{\left[D^{(1)}\right]} \underset{(2\times10)}{\left[\beta^{(1)}\right]} dx \underset{(10\times1)}{\{d\}} . \tag{5.10.35}$$

Matrix $\left[k^{(1)}\right]$ in the first term of expression (5.10.35) is part of the stiffness matrix of the linearly formulated problem. Its components are shown in Appendix 5-A. The last three terms in the expression (5.10.35) are not quadratic with respect to the nodal variables. They lead to the part of

the internal force vector, that is nonlinear with respect to the nodal variables. The components of the nonlinear part of the internal force vector were derived with the use of MAPLE, a program for symbolic computation. As an illustration, the first component of the nonlinear part of the internal force vector is shown in Appendix 5-C[2].

Now, let us write the **second term in the expression (5.10.1) for the strain energy in terms of the nodal variables**. Analogously to equation (5.10.34), we obtain

$$\underset{(2\times3)}{\left[\partial^{(3)}\right]}\underset{(3\times1)}{\{f\}}+\underset{(2\times1)}{\left\{\eta^{(3)}\right\}}=\left(\underset{(2\times10)}{\left[B^{(3)}\right]}+\underset{(2\times10)}{\left[\beta^{(3)}\right]}\right)\underset{(10\times1)}{\{d\}}, \tag{5.10.36}$$

where

$$\underset{(2\times10)}{\left[B^{(3)}\right]}=$$

$$=\begin{bmatrix} 0 & 0 & 0 & 0 & 2z_3\frac{dM_1}{dx} & 2z_3\frac{dM_2}{dx} & \frac{1}{2}z_3^2\frac{d^2N_1}{dx^2} & \frac{1}{2}z_3^2\frac{d^2N_2}{dx^2} & \frac{1}{2}z_3^2\frac{d^2N_3}{dx^2} & \frac{1}{2}z_3^2\frac{d^2N_4}{dx^2} \\ -\frac{d^2N_1}{dx^2} & -\frac{d^2N_2}{dx^2} & -\frac{d^2N_3}{dx^2} & -\frac{d^2N_4}{dx^2} & 0 & 0 & -z_3\frac{d^2N_1}{dx^2} & -z_3\frac{d^2N_2}{dx^2} & -z_3\frac{d^2N_3}{dx^2} & -z_3\frac{d^2N_4}{dx^2} \end{bmatrix},$$

$$\underset{(2\times10)}{\left[\beta^{(3)}\right]}=\underset{(2\times1)}{\left\{\begin{matrix}1\\0\end{matrix}\right\}}\underset{(1\times10)}{\{d\}}^T\underset{(10\times10)}{\left[\begin{matrix}\underset{(10\times4)}{[\Phi]} & \underset{(10\times2)}{[0]} & \underset{(10\times4)}{\left[\Psi^{(3)}\right]}\end{matrix}\right]},$$

$$\underset{(10\times4)}{[\Phi]}=\frac{1}{2}\underset{(10\times4)}{\left[\begin{matrix}\underset{(4\times4)}{[I]} & \underset{(4\times6)}{[0]}\end{matrix}\right]}^T\underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}},$$

$$\underset{(10\times4)}{\left[\Psi^{(3)}\right]}=z_3\left(\underset{(10\times4)}{\left[\begin{matrix}\underset{(4\times4)}{[I]} & \underset{(4\times6)}{[0]}\end{matrix}\right]}^T+\frac{1}{2}z_3\underset{(10\times4)}{\left[\begin{matrix}\underset{(4\times6)}{[0]} & \underset{(4\times4)}{[I]}\end{matrix}\right]}^T\right)\underset{(4\times1)}{\frac{d\lfloor N\rfloor^T}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}}.$$

So, the **second term in the expression (5.10.1) for the strain energy of the beam** is

$$\frac{1}{2}\int_0^l\left(\underset{(2\times3)}{\left[\partial^{(3)}\right]}\underset{(3\times1)}{\{f\}}+\underset{(2\times1)}{\left\{\eta^{(3)}\right\}}\right)^T\underset{(2\times2)}{\left[D^{(3)}\right]}\left(\underset{(2\times3)}{\left[\partial^{(3)}\right]}\underset{(3\times1)}{\{f\}}+\underset{(2\times1)}{\left\{\eta^{(3)}\right\}}\right)dx=$$

[2] In Appendix 5-C, the first component of the nonlinear part of the internal force vector is written in terms of the nodal variables θ_i, numbered, for convenience, in a different way than the nodal variables d_i, as described in subsequent text

$$= \frac{1}{2} \underset{(1\times10)}{\{d\}}^T \int_0^l \underset{(10\times2)}{\left(\underset{(2\times10)}{\left[B^{(3)}\right]} + \underset{(2\times10)}{\left[\beta^{(3)}\right]} \right)^T} \underset{(2\times2)}{\left[D^{(3)}\right]} \left(\underset{(2\times10)}{\left[B^{(3)}\right]} + \underset{(2\times10)}{\left[\beta^{(3)}\right]} \right) dx \underset{(10\times1)}{\{d\}} =$$

$$= \frac{1}{2} \underset{(1\times10)}{\{d\}}^T \underbrace{\int_0^l \underset{(10\times2)}{\left[B^{(3)}\right]^T} \underset{(2\times2)}{\left[D^{(3)}\right]} \underset{(2\times10)}{\left[B^{(3)}\right]} dx}_{\left[k^{(3)}\right]} \underset{(10\times1)}{\{d\}} +$$

$$+ \frac{1}{2} \underset{(1\times10)}{\{d\}}^T \int_0^l \underset{(10\times2)}{\left[\beta^{(3)}\right]^T} \underset{(2\times2)}{\left[D^{(3)}\right]} \underset{(2\times10)}{\left[B^{(3)}\right]} dx \underset{(10\times1)}{\{d\}} +$$

$$+ \frac{1}{2} \underset{(1\times10)}{\{d\}}^T \int_0^l \underset{(10\times2)}{\left[B^{(3)}\right]^T} \underset{(2\times2)}{\left[D^{(3)}\right]} \underset{(2\times10)}{\left[\beta^{(3)}\right]} dx \underset{(10\times1)}{\{d\}} +$$

$$+ \frac{1}{2} \underset{(1\times10)}{\{d\}}^T \int_0^l \underset{(10\times2)}{\left[\beta^{(3)}\right]^T} \underset{(2\times2)}{\left[D^{(3)}\right]} \underset{(2\times10)}{\left[\beta^{(3)}\right]} dx \underset{(10\times1)}{\{d\}} . \tag{5.10.37}$$

Matrix $\left[k^{(3)}\right]$ in the first term of expression (5.10.37) is part of the stiffness matrix of the linearly formulated problem. Its components are shown in Appendix 5-A. The last three terms in the expression (5.10.37) are not quadratic with respect to the nodal variables. They lead to the part of the internal force vector, that is nonlinear with respect to the nodal variables. As an illustration, the first component of the nonlinear part of the internal force vector is shown in Appendix 5-C.

Let us write the **third term in the expression (5.10.1) for the strain energy in terms of the nodal variables.** First, we will obtain an expression for $\underset{(4\times3)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(3\times1)}{\{f\}}$ in terms of the nodal variables:

$$\underset{(4\times3)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(3\times1)}{\{f\}} = \underset{(4\times3)}{\left[\widehat{\partial}^{(2)}\right]} \left\{ \begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array} \right\} = \underset{(4\times3)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(3\times10)}{\left[\begin{array}{ccc} \underset{(1\times4)}{\lfloor N \rfloor} & \underset{(1\times2)}{\lfloor 0 \rfloor} & \underset{(1\times4)}{\lfloor 0 \rfloor} \\ \underset{(1\times4)}{\lfloor 0 \rfloor} & \underset{(1\times2)}{\lfloor M \rfloor} & \underset{(1\times4)}{\lfloor 0 \rfloor} \\ \underset{(1\times4)}{\lfloor 0 \rfloor} & \underset{(1\times2)}{\lfloor 0 \rfloor} & \underset{(1\times4)}{\lfloor N \rfloor} \end{array} \right]} \underset{(10\times1)}{\left\{ \begin{array}{c} \underset{(4\times1)}{\{\overline{w}\}} \\ \underset{(2\times1)}{\{\overline{\epsilon}\}} \\ \underset{(4\times1)}{\{\overline{\varepsilon}\}} \end{array} \right\}} =$$

$$= \underset{(4\times 10)}{\left[\widehat{B}^{(2)}\right]} \underset{(10\times 1)}{\left\{ \begin{array}{c} \underset{(4\times 1)}{\{\overline{w}\}} \\ \underset{(2\times 1)}{\{\overline{\epsilon}\}} \\ \underset{(4\times 1)}{\{\overline{\varepsilon}\}} \end{array} \right\}} = \underset{(4\times 10)}{\left[\widehat{B}^{(2)}\right]} \underset{(10\times 1)}{\{d\}}, \tag{5.10.38}$$

where

$$\underset{(4\times 10)}{\left[\widehat{B}^{(2)}\right]} = \underset{(4\times 3)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(3\times 10)}{\begin{bmatrix} \underset{(1\times 4)}{\lfloor N \rfloor} & \underset{(1\times 2)}{\lfloor 0 \rfloor} & \underset{(1\times 4)}{\lfloor 0 \rfloor} \\ \underset{(1\times 4)}{\lfloor 0 \rfloor} & \underset{(1\times 2)}{\lfloor M \rfloor} & \underset{(1\times 4)}{\lfloor 0 \rfloor} \\ \underset{(1\times 4)}{\lfloor 0 \rfloor} & \underset{(1\times 2)}{\lfloor 0 \rfloor} & \underset{(1\times 4)}{\lfloor N \rfloor} \end{bmatrix}} =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ -\frac{d^2}{dx^2} & 2\frac{d}{dx} & 0 \\ 0 & 0 & -\frac{1}{2}\frac{d^2}{dx^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_1 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{d^2N_1}{dx^2} & -\frac{d^2N_2}{dx^2} & -\frac{d^2N_3}{dx^2} & -\frac{d^2N_4}{dx^2} & 2\frac{dM_1}{dx} & 2\frac{dM_2}{dx} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{d^2N_1}{dx^2} & -\frac{1}{2}\frac{d^2N_2}{dx^2} & -\frac{1}{2}\frac{d^2N_3}{dx^2} & -\frac{1}{2}\frac{d^2N_4}{dx^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix}. \tag{5.10.39}$$

Now, let us obtain an expression for $\left\{\widehat{\eta}^{(2)}\right\}$, defined by expression (5.10.6), in terms of the nodal variables:

$$\underset{(4\times 1)}{\left[\widehat{\eta}^{(2)}\right]} = \underset{(4\times 1)}{\left\{ \begin{array}{c} \frac{1}{2}(w_{0,x})^2 \\ w_{0,x}\varepsilon^{(2)}_{zz,x} \\ \frac{1}{2}\left(\varepsilon^{(2)}_{zz,x}\right)^2 \\ 0 \end{array} \right\}} = \left\{ \begin{array}{c} \frac{1}{2} \underset{(1\times 4)}{\{\overline{w}\}^T} \underset{(4\times 1)}{\frac{d\lfloor N \rfloor^T}{dx}} \underset{(1\times 4)}{\frac{d\lfloor N \rfloor}{dx}} \underset{(4\times 1)}{\{\overline{w}\}} \\ \underset{(1\times 4)}{\{\overline{w}\}^T} \underset{(4\times 1)}{\frac{d\lfloor N \rfloor^T}{dx}} \underset{(1\times 4)}{\frac{d\lfloor N \rfloor}{dx}} \underset{(4\times 1)}{\{\overline{\varepsilon}\}} \\ \frac{1}{2} \underset{(1\times 4)}{\{\overline{\varepsilon}\}^T} \underset{(4\times 1)}{\frac{d\lfloor N \rfloor^T}{dx}} \underset{(1\times 4)}{\frac{d\lfloor N \rfloor}{dx}} \underset{(4\times 1)}{\{\overline{\varepsilon}\}} \\ 0 \end{array} \right\} =$$

$$
= \underset{(4\times 10)}{\begin{bmatrix}
\left(\frac{1}{2}\underset{(1\times4)}{\{\overline{w}\}}^{T}\underset{(4\times1)}{\frac{d\lfloor N\rfloor^{T}}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}}\right) & \underset{(1\times2)}{\lfloor 0\rfloor} & \underset{(1\times4)}{\lfloor 0\rfloor} \\
\underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \left(\underset{(1\times4)}{\{\overline{w}\}}^{T}\underset{(4\times1)}{\frac{d\lfloor N\rfloor^{T}}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}}\right) \\
\underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \left(\frac{1}{2}\underset{(1\times4)}{\{\overline{\varepsilon}\}}^{T}\underset{(4\times1)}{\frac{d\lfloor N\rfloor^{T}}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}}\right) \\
\underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \underset{(1\times4)}{\lfloor 0\rfloor}
\end{bmatrix}}
\begin{Bmatrix}
\underset{(4\times1)}{\{\overline{w}\}} \\
\underset{(2\times1)}{\{\overline{\varepsilon}\}} \\
\underset{(4\times1)}{\{\overline{\varepsilon}\}}
\end{Bmatrix} =
$$

$$
= \underset{(4\times10)}{\left[\widehat{\beta}^{(2)}\right]}\underset{(10\times1)}{\{d\}},
$$

where

$$
\underset{(4\times10)}{\left[\widehat{\beta}^{(2)}\right]} = \underset{(4\times 10)}{\begin{bmatrix}
\left(\frac{1}{2}\underset{(1\times4)}{\{\overline{w}\}}^{T}\underset{(4\times1)}{\frac{d\lfloor N\rfloor^{T}}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}}\right) & \underset{(1\times2)}{\lfloor 0\rfloor} & \underset{(1\times4)}{\lfloor 0\rfloor} \\
\underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \left(\underset{(1\times4)}{\{\overline{w}\}}^{T}\underset{(4\times1)}{\frac{d\lfloor N\rfloor^{T}}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}}\right) \\
\underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \left(\frac{1}{2}\underset{(1\times4)}{\{\overline{\varepsilon}\}}^{T}\underset{(4\times1)}{\frac{d\lfloor N\rfloor^{T}}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}}\right) \\
\underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \underset{(1\times4)}{\lfloor 0\rfloor}
\end{bmatrix}} =
$$

$$
= \underset{(4\times 10)}{\begin{bmatrix}
\left(\underset{(1\times10)}{\{d\}}^{T}\underset{(10\times4)}{[\Phi]}\right) & \underset{(1\times2)}{\lfloor 0\rfloor} & \underset{(1\times4)}{\lfloor 0\rfloor} \\
\underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \left(2\ \underset{(1\times10)}{\{d\}}^{T}\underset{(10\times4)}{[\Phi]}\right) \\
\underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \left(\underset{(1\times10)}{\{d\}}^{T}\underset{(10\times4)}{\left[\Psi^{(2)}\right]}\right) \\
\underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \underset{(1\times4)}{\lfloor 0\rfloor}
\end{bmatrix}}, \qquad (5.10.40)
$$

where

$$
\underset{(10\times4)}{[\Phi]} = \frac{1}{2}\underset{(10\times4)}{\begin{bmatrix} \underset{(4\times4)}{[I]} & \underset{(4\times6)}{[0]} \end{bmatrix}^{T}}\underset{(4\times1)}{\frac{d\lfloor N\rfloor^{T}}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}},
$$

$$
\underset{(10\times4)}{\left[\Psi^{(2)}\right]} = \frac{1}{2}\underset{(10\times4)}{\begin{bmatrix} \underset{(4\times6)}{[0]} & \underset{(4\times4)}{[I]} \end{bmatrix}^{T}}\underset{(4\times1)}{\frac{d\lfloor N\rfloor^{T}}{dx}}\underset{(1\times4)}{\frac{d\lfloor N\rfloor}{dx}}, \qquad (5.10.41)
$$

and $\underset{(4\times 4)}{[I]}$ is a unit matrix:

$$\underset{(4\times 4)}{[I]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So, the **third term in the expression (5.10.1) for the strain energy of the beam** is

$$\frac{1}{2}\int_0^l \left(\underset{(4\times 3)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(3\times 1)}{\{f\}} + \underset{(4\times 1)}{\left[\widehat{\eta}^{(2)}\right]} \right)^T \underset{(4\times 4)}{\left[\widehat{D}^{(2)}\right]} \left(\underset{(4\times 3)}{\left[\widehat{\partial}^{(2)}\right]} \underset{(3\times 1)}{\{f\}} + \underset{(4\times 1)}{\left[\widehat{\eta}^{(2)}\right]} \right) dx =$$

$$= \frac{1}{2}\int_0^l \left(\underset{(4\times 10)}{\left[\widehat{B}^{(2)}\right]} \underset{(10\times 1)}{\{d\}} + \underset{(4\times 10)}{\left[\widehat{\beta}^{(2)}\right]} \underset{(10\times 1)}{\{d\}} \right)^T \underset{(4\times 4)}{\left[\widehat{D}^{(2)}\right]} \underset{(4\times 4)}{\left(\underset{(4\times 10)}{\left[\widehat{B}^{(2)}\right]} \underset{(10\times 1)}{\{d\}} + \underset{(4\times 10)}{\left[\widehat{\beta}^{(2)}\right]} \underset{(10\times 1)}{\{d\}} \right)} dx =$$

$$= \frac{1}{2} \underset{(1\times 10)}{\{d\}}^T \int_0^l \underset{(10\times 4)}{\left(\underset{(4\times 10)}{\left[\widehat{B}^{(2)}\right]} + \underset{(4\times 10)}{\left[\widehat{\beta}^{(2)}\right]} \right)^T} \underset{(4\times 4)}{\left[\widehat{D}^{(2)}\right]} \left(\underset{(4\times 10)}{\left[\widehat{B}^{(2)}\right]} + \underset{(4\times 10)}{\left[\widehat{\beta}^{(2)}\right]} \right) dx \underset{(10\times 1)}{\{d\}} =$$

$$= \frac{1}{2} \underset{(1\times 10)}{\{d\}}^T \underbrace{\int_0^l \underset{(10\times 4)}{\left[\widehat{B}^{(2)}\right]^T} \underset{(4\times 4)}{\left[\widehat{D}^{(2)}\right]} \underset{(4\times 10)}{\left[\widehat{B}^{(2)}\right]} dx}_{\left[\widehat{k}^{(2)}\right]} \underset{(10\times 1)}{\{d\}} +$$

$$+ \frac{1}{2} \underset{(1\times 10)}{\{d\}}^T \int_0^l \underset{(10\times 4)}{\left[\widehat{\beta}^{(2)}\right]^T} \underset{(4\times 4)}{\left[\widehat{D}^{(2)}\right]} \underset{(4\times 10)}{\left[\widehat{B}^{(2)}\right]} dx \underset{(10\times 1)}{\{d\}} +$$

$$+ \frac{1}{2} \underset{(1\times 10)}{\{d\}}^T \int_0^l \underset{(10\times 4)}{\left[\widehat{B}^{(2)}\right]^T} \underset{(4\times 4)}{\left[\widehat{D}^{(2)}\right]} \underset{(4\times 10)}{\left[\widehat{\beta}^{(2)}\right]} dx \underset{(10\times 1)}{\{d\}} +$$

$$+ \frac{1}{2} \underset{(1\times 10)}{\{d\}}^T \int_0^l \underset{(10\times 4)}{\left[\widehat{\beta}^{(2)}\right]^T} \underset{(4\times 4)}{\left[\widehat{D}^{(2)}\right]} \underset{(4\times 10)}{\left[\widehat{\beta}^{(2)}\right]} dx \underset{(10\times 1)}{\{d\}}. \qquad (5.10.42)$$

Matrix $\left[\widehat{k}^{(2)}\right]$ in the first term of expression (5.10.42) is a part of the stiffness matrix of the linearly formulated problem. Its components are shown in Appendix 5-A. The last three terms in the expression (5.10.42) are not quadratic with respect to the nodal variables. They lead to the part of the internal force vector, that is nonlinear with respect to the nodal variables. As an illustration, the first component of the nonlinear part of the internal force vector is shown in Appendix 5-C.

Let us write the **fourth term in the expression (5.10.1) for the strain energy of the wide beam in terms of the nodal variables**. This term is

$$\frac{1}{2}\int_0^l \underset{(1\times3)}{\{f\}}^T \underset{(3\times3)}{\left[\check{D}^{(2)}\right]} \underset{(3\times1)}{\{f\}}\, dx,$$

where

$$\underset{(3\times1)}{\{f\}} = \left\{\begin{array}{c} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{array}\right\} = \left\{\begin{array}{c} \underset{(1\times4)(4\times1)}{\lfloor N\rfloor\{\bar{w}\}} \\ \underset{(1\times2)(2\times1)}{\lfloor M\rfloor\{\bar{\epsilon}\}} \\ \underset{(1\times4)(4\times1)}{\lfloor N\rfloor\{\bar{\epsilon}\}} \end{array}\right\} = \begin{bmatrix} \underset{(1\times4)}{\lfloor N\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \underset{(1\times4)}{\lfloor 0\rfloor} \\ \underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor M\rfloor} & \underset{(1\times4)}{\lfloor 0\rfloor} \\ \underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \underset{(1\times4)}{\lfloor N\rfloor} \end{bmatrix} \left\{\begin{array}{c} \underset{(4\times1)}{\{\overline{w}\}} \\ \underset{(2\times1)}{\{\bar{\epsilon}\}} \\ \underset{(4\times1)}{\{\bar{\epsilon}\}} \end{array}\right\} =$$

$$= \begin{bmatrix} \underset{(1\times4)}{\lfloor N\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \underset{(1\times4)}{\lfloor 0\rfloor} \\ \underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor M\rfloor} & \underset{(1\times4)}{\lfloor 0\rfloor} \\ \underset{(1\times4)}{\lfloor 0\rfloor} & \underset{(1\times2)}{\lfloor 0\rfloor} & \underset{(1\times4)}{\lfloor N\rfloor} \end{bmatrix} \underset{(10\times1)}{\{d\}} = \underset{(3\times10)(10\times1)}{[Q]\;\{d\}} \tag{5.10.43}$$

and $\left[\check{D}^{(2)}\right]$ is defined by equation (5.10.10):

$$\left[\check{D}^{(2)}\right] = 4b\overline{C}_{55}^{(2)}\left(z_3 - z_2\right)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \check{D}_{22}^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.10.44}$$

Therefore,

$$\frac{1}{2}\int_0^l \underset{(1\times3)}{\{f\}}^T \underset{(3\times3)}{\left[\check{D}^{(2)}\right]} \underset{(3\times1)}{\{f\}}\, dx = \frac{1}{2}\underset{(1\times10)}{\{d\}}^T \underbrace{\int_0^l \underset{(10\times3)}{[Q]}^T \underset{(3\times3)}{\left[\check{D}^{(2)}\right]} \underset{(3\times10)}{[Q]}\, dx}_{[\check{k}^{(2)}]} \underset{(10\times1)}{\{d\}} =$$

$$= \frac{1}{2}\underset{(1\times10)}{\{d\}}^T \underset{(10\times10)}{[\check{k}^{(2)}]} \underset{(10\times1)}{\{d\}}, \tag{5.10.45}$$

where

$$\underset{(10\times10)}{[\check{k}^{(2)}]} = \int_0^l \underset{(10\times3)}{[Q]}^T \underset{(3\times3)}{\left[\check{D}^{(2)}\right]} \underset{(3\times10)}{[Q]}\, dx =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3}l\check{D}_{22}^{(2)} & \frac{1}{6}l\check{D}_{22}^{(2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{6}l\check{D}_{22}^{(2)} & \frac{1}{3}l\check{D}_{22}^{(2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (5.10.46)$$

is a part of the stiffness matrix.

Now, let us write the **strain energy of the elastic foundation in terms of the nodal variables**. According to equation (5.6.6), the strain energy of the elastic foundation is

$$U_f = \frac{1}{2}b\int_0^l s(x)\, \underset{(1\times3)}{\{f\}}^T \underset{(3\times3)}{[\,\overline{D}\,]} \underset{(3\times1)}{\{f\}}\, dx,$$

where, according to equation (5.6.5),

$$[\,\overline{D}\,] = \begin{bmatrix} 1 & 0 & z_2 \\ 0 & 0 & 0 \\ z_2 & 0 & z_2^2 \end{bmatrix}$$

and, according to equation (5.10.43),

$$\underset{(3\times1)}{\{f\}} = \underset{(3\times10)}{[Q]} \underset{(10\times1)}{\{d\}},$$

where

$$\underset{(3\times10)}{[Q]} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_1 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix},$$

and $\{d\}$ is a column-matrix of the nodal variables. So,

$$U_f = \frac{1}{2} b \int_0^l s(x) \underset{(1\times 3)}{\{f\}}^T \underset{(3\times 3)}{\left[\overline{D}\right]} \underset{(3\times 1)}{\{f\}} \, dx = \frac{1}{2} \underset{(1\times 10)}{\{d\}}^T \underset{(10\times 10)}{\left[k^{(f)}\right]} \underset{(10\times 1)}{\{d\}} , \tag{5.10.47}$$

where

$$\underset{(10\times 10)}{\left[k^{(f)}\right]} = b \int_0^l s(x) \underset{(10\times 3)}{[Q]}^T \underset{(3\times 3)}{\left[\overline{D}\right]} \underset{(3\times 10)}{[Q]} \, dx \tag{5.10.48}$$

is part of the stiffness matrix of the system.

If $s(x)$ =const, then

$$\left[k^{(f)}\right] = bs \int_0^l [Q]^T \begin{bmatrix} 1 & 0 & z_2 \\ 0 & 0 & 0 \\ z_2 & 0 & z_2^2 \end{bmatrix} [Q] \, dx =$$

$$= bs \begin{bmatrix}
\frac{13}{35} l & \frac{11}{210} l^2 & \frac{9}{70} l & -\frac{13}{420} l^2 & 0 & 0 & \frac{13}{35} l z_2 & \frac{11}{210} l^2 z_2 & \frac{9}{70} l z_2 & -\frac{13}{420} l^2 z_2 \\
\frac{11}{210} l^2 & \frac{1}{105} l^3 & \frac{13}{420} l^2 & -\frac{1}{140} l^3 & 0 & 0 & \frac{11}{210} l^2 z_2 & \frac{1}{105} l^3 z_2 & \frac{13}{420} l^2 z_2 & -\frac{1}{140} l^3 z_2 \\
\frac{9}{70} l & \frac{13}{420} l^2 & \frac{13}{35} l & -\frac{11}{210} l^2 & 0 & 0 & \frac{9}{70} l z_2 & \frac{13}{420} l^2 z_2 & \frac{13}{35} l z_2 & -\frac{11}{210} l^2 z_2 \\
-\frac{13}{420} l^2 & -\frac{1}{140} l^3 & -\frac{11}{210} l^2 & \frac{1}{105} l^3 & 0 & 0 & -\frac{13}{420} l^2 z_2 & -\frac{1}{140} l^3 z_2 & -\frac{11}{210} l^2 z_2 & \frac{1}{105} l^3 z_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{13}{35} l z_2 & \frac{11}{210} l^2 z_2 & \frac{9}{70} l z_2 & -\frac{13}{420} l^2 z_2 & 0 & 0 & \frac{13}{35} l z_2^2 & \frac{11}{210} l^2 z_2^2 & \frac{9}{70} l z_2^2 & -\frac{13}{420} l^2 z_2^2 \\
\frac{11}{210} l^2 z_2 & \frac{1}{105} l^3 z_2 & \frac{13}{420} l^2 z_2 & -\frac{1}{140} l^3 z_2 & 0 & 0 & \frac{11}{210} l^2 z_2^2 & \frac{1}{105} l^3 z_2^2 & \frac{13}{420} l^2 z_2^2 & -\frac{1}{140} l^3 z_2^2 \\
\frac{9}{70} l z_2 & \frac{13}{420} l^2 z_2 & \frac{13}{35} l z_2 & -\frac{11}{210} l^2 z_2 & 0 & 0 & \frac{9}{70} l z_2^2 & \frac{13}{420} l^2 z_2^2 & \frac{13}{35} l z_2^2 & -\frac{11}{210} l^2 z_2^2 \\
-\frac{13}{420} l^2 z_2 & -\frac{1}{140} l^3 z_2 & -\frac{11}{210} l^2 z_2 & \frac{1}{105} l^3 z_2 & 0 & 0 & -\frac{13}{420} l^2 z_2^2 & -\frac{1}{140} l^3 z_2^2 & -\frac{11}{210} l^2 z_2^2 & \frac{1}{105} l^3 z_2^2
\end{bmatrix} \tag{19.49}$$

The strain energy of the mechanical system under consideration is the sum of the strain energies of the lower face sheet, the upper face sheet, the core and the elastic foundation. Therefore, according to the equations (5.10.35), (5.10.37), (5.10.42), (5.10.45) and(5.10.47), the **part of the strain energy of the system that is quadratic with respect to the nodal variables**[3], is

$$U_l = \frac{1}{2} \underset{(1\times 10)}{\{d\}}^T \underset{(10\times 10)}{\left[k^{(l)}\right]} \underset{(1\times 10)}{\{d\}} , \tag{5.10.50}$$

[3]i.e. the strain energy of the linearly formulated problem

where

$$\left[k^{(l)}\right] = \left[k^{(1)}\right] + \left[k^{(3)}\right] + \left[\widehat{k}^{(2)}\right] + \left[\check{k}^{(2)}\right] + \left[k^{(f)}\right], \tag{5.10.51}$$

is the stiffness matrix of the linearly formulated problem. The part of the strain energy of the system that is not quadratic with respect to the nodal variables[4]

$$\begin{aligned}
U_{nl} = {} & \frac{1}{2} \underset{(1\times10)}{\{d\}}^{T} \int_0^l \underset{(10\times2)}{\left[\beta^{(1)}\right]^T} \underset{(2\times2)}{\left[D^{(1)}\right]} \underset{(2\times10)}{\left[B^{(1)}\right]} dx \underset{(10\times1)}{\{d\}} + \\
& + \frac{1}{2} \underset{(1\times10)}{\{d\}}^{T} \int_0^l \underset{(10\times2)}{\left[B^{(1)}\right]^T} \underset{(2\times2)}{\left[D^{(1)}\right]} \underset{(2\times10)}{\left[\beta^{(1)}\right]} dx \underset{(10\times1)}{\{d\}} + \\
& + \frac{1}{2} \underset{(1\times10)}{\{d\}}^{T} \int_0^l \underset{(10\times2)}{\left[\beta^{(1)}\right]^T} \underset{(2\times2)}{\left[D^{(1)}\right]} \underset{(2\times10)}{\left[\beta^{(1)}\right]} dx \underset{(10\times1)}{\{d\}} + \\
& + \frac{1}{2} \underset{(1\times10)}{\{d\}}^{T} \int_0^l \underset{(10\times2)}{\left[\beta^{(3)}\right]^T} \underset{(2\times2)}{\left[D^{(3)}\right]} \underset{(2\times10)}{\left[B^{(3)}\right]} dx \underset{(10\times1)}{\{d\}} + \\
& + \frac{1}{2} \underset{(1\times10)}{\{d\}}^{T} \int_0^l \underset{(10\times2)}{\left[B^{(3)}\right]^T} \underset{(2\times2)}{\left[D^{(3)}\right]} \underset{(2\times10)}{\left[\beta^{(3)}\right]} dx \underset{(10\times1)}{\{d\}} + \\
& + \frac{1}{2} \underset{(1\times10)}{\{d\}}^{T} \int_0^l \underset{(10\times2)}{\left[\beta^{(3)}\right]^T} \underset{(2\times2)}{\left[D^{(3)}\right]} \underset{(2\times10)}{\left[\beta^{(3)}\right]} dx \underset{(10\times1)}{\{d\}} + \\
& + \frac{1}{2} \underset{(1\times10)}{\{d\}}^{T} \int_0^l \underset{(10\times4)}{\left[\widehat{\beta}^{(2)}\right]^T} \underset{(4\times4)}{\left[\widehat{D}^{(2)}\right]} \underset{(4\times10)}{\left[\widehat{B}^{(2)}\right]} dx \underset{(10\times1)}{\{d\}} + \\
& + \frac{1}{2} \underset{(1\times10)}{\{d\}}^{T} \int_0^l \underset{(10\times4)}{\left[\widehat{B}^{(2)}\right]^T} \underset{(4\times4)}{\left[\widehat{D}^{(2)}\right]} \underset{(4\times10)}{\left[\widehat{\beta}^{(2)}\right]} dx \underset{(10\times1)}{\{d\}} + \\
& + \frac{1}{2} \underset{(1\times10)}{\{d\}}^{T} \int_0^l \underset{(10\times4)}{\left[\widehat{\beta}^{(2)}\right]^T} \underset{(4\times4)}{\left[\widehat{D}^{(2)}\right]} \underset{(4\times10)}{\left[\widehat{\beta}^{(2)}\right]} dx \underset{(10\times1)}{\{d\}} .
\end{aligned} \tag{5.10.52}$$

[4] i.e. the part of the strain energy that appears due to nonlinear terms in the strain-displacement relations

5.10.2 Kinetic energy in terms of the nodal variables

Now, we need to derive a matrix of inertia of a finite element. For this we need to derive an expression for a kinetic energy of the system in terms of the time derivatives of the nodal variables. The kinetic energy of the system is a sum of kinetic energies of the sandwich plate and the cargo. According to equations (4.10.8) and (4.11.5), the kinetic energy of the sandwich plate and the cargo, over a finite element, is

$$K = \frac{1}{2}\rho^{(1)} b \int_0^l \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \underset{(3\times3)}{\left[\widetilde{D}^{(1)}\right]} \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) dx \; +$$

$$+\frac{1}{2}\rho^{(2)} b \int_0^l \left(\underset{(4\times3)}{\left[\widetilde{\partial}^{(2)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \underset{(4\times4)}{\left[\widetilde{D}^{(2)}\right]} \left(\underset{(4\times3)}{\left[\widetilde{\partial}^{(2)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) dx \; +$$

$$+\frac{1}{2}\rho^{(3)} b \int_0^l \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(3)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \underset{(3\times3)}{\left[\widetilde{D}^{(3)}\right]} \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(3)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) dx +$$

$$+\frac{1}{2} b \int_0^l \mu \; H(x) \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \underset{(3\times3)}{\left[\widetilde{D}_c\right]} \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) dx \;, \tag{5.10.54}$$

where

$$\underset{(3\times3)}{\left[\widetilde{D}^{(1)}\right]} = \begin{bmatrix} z_2 - z_1 & \frac{1}{2}\left(z_2^2 - z_1^2\right) & 0 \\ \frac{1}{2}\left(z_2^2 - z_1^2\right) & \frac{1}{3}\left(z_2^3 - z_1^3\right) & 0 \\ 0 & 0 & z_2 - z_1 \end{bmatrix},$$

$$\underset{(4\times4)}{\left[\widetilde{D}^{(2)}\right]} = \begin{bmatrix} \frac{1}{3}\left(z_3^3 - z_2^3\right) & \frac{1}{4}\left(z_3^4 - z_2^4\right) & 0 & 0 \\ \frac{1}{4}\left(z_3^4 - z_2^4\right) & \frac{1}{5}\left(z_3^5 - z_2^5\right) & 0 & 0 \\ 0 & 0 & z_3 - z_2 & \frac{1}{2}\left(z_3^2 - z_2^2\right) \\ 0 & 0 & \frac{1}{2}\left(z_3^2 - z_2^2\right) & \frac{1}{3}\left(z_3^3 - z_2^3\right) \end{bmatrix},$$

$$\underset{(3\times3)}{\left[\widetilde{D}^{(3)}\right]} = \begin{bmatrix} z_4 - z_3 & \frac{1}{2}\left(z_4^2 - z_3^2\right) & 0 \\ \frac{1}{2}\left(z_4^2 - z_3^2\right) & \frac{1}{3}\left(z_4^3 - z_3^3\right) & 0 \\ 0 & 0 & z_4 - z_3 \end{bmatrix},$$

$$\underset{(3\times3)}{\left[\widetilde{D}_c\right]} = \begin{bmatrix} 1 & 0 & z_3 \\ 0 & 0 & 0 \\ z_3 & 0 & z_3^2 \end{bmatrix},$$

$$\underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]} = \begin{bmatrix} 0 & 2z_2 & \frac{1}{2}z_2^2\frac{d}{dx} \\ -\frac{d}{dx} & 0 & -z_2\frac{d}{dx} \\ 1 & 0 & z_2 \end{bmatrix},$$

$$\underset{(4\times3)}{\left[\widetilde{\partial}^{(2)}\right]} = \begin{bmatrix} -\frac{d}{dx} & 2 & 0 \\ 0 & 0 & -\frac{1}{2}\frac{d}{dx} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\underset{(3\times3)}{\left[\widetilde{\partial}^{(3)}\right]} = \begin{bmatrix} 0 & 2z_3 & \frac{1}{2}z_3^2\frac{d}{dx} \\ -\frac{d}{dx} & 0 & -z_3\frac{d}{dx} \\ 1 & 0 & z_3 \end{bmatrix},$$

μ is a mass of the cargo per unit area of contact with the platform; $H(x)$ is a function, defined as follows:

$$H(x) = \begin{cases} 1 \text{ in region of the upper surface, occupied by the cargo} \\ 0 \text{ in region of the upper surface, not occupied by the cargo} \end{cases},$$

$$\underset{(3\times1)}{\{f\}} = \begin{Bmatrix} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{Bmatrix}.$$

According to equation (5.10.43),

$$\underset{(3\times1)}{\{f\}} = \underset{(3\times10)}{[Q]}\underset{(10\times1)}{\{d\}},$$

where

$$\underset{(3\times10)}{[Q]} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_1 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix},$$

and $\{d\}$ is a column-matrix of the nodal variables.

Let us write the **first term of the expression (5.10.54) for the kinetic energy in terms of the time derivatives of the nodal variables**. This term is

$$\frac{1}{2}\rho^{(1)}b\int_0^l \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right)^T \underset{(3\times3)}{\left[\widetilde{D}^{(1)}\right]}\left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right)\,dx.$$

In this expression

$$\underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}} = \underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]}\frac{\partial}{\partial t}\underset{(3\times10)}{[Q]}\underset{(10\times1)}{\{d\}} =$$

$$= \underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]}\underset{(3\times10)}{[Q]}\frac{\partial}{\partial t}\underset{(10\times1)}{\{d\}} = \underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]}\underset{(3\times10)}{[Q]}\underset{(10\times1)}{\left\{\dot{d}\right\}} =$$

$$= \underset{(3\times3)}{\begin{bmatrix} 0 & 2z_2 & \frac{1}{2}z_2^2\frac{d}{dx} \\ -\frac{d}{dx} & 0 & -z_2\frac{d}{dx} \\ 1 & 0 & z_2 \end{bmatrix}} \underset{(3\times10)}{\begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_1 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix}} \underset{(10\times1)}{\left\{\dot{d}\right\}} =$$

$$= \underset{(3\times10)}{\begin{bmatrix} 0 & 0 & 0 & 0 & 2z_2M_1 & 2z_2M_2 & \frac{1}{2}z_2^2\frac{dN_1}{dx} & \frac{1}{2}z_2^2\frac{dN_2}{dx} & \frac{1}{2}z_2^2\frac{dN_3}{dx} & \frac{1}{2}z_2^2\frac{dN_4}{dx} \\ -\frac{dN_1}{dx} & -\frac{dN_2}{dx} & -\frac{dN_3}{dx} & -\frac{dN_4}{dx} & 0 & 0 & -z_2\frac{dN_1}{dx} & -z_2\frac{dN_2}{dx} & -z_2\frac{dN_3}{dx} & -z_2\frac{dN_4}{dx} \\ N_1 & N_2 & N_3 & N_4 & 0 & 0 & z_2N_1 & z_2N_2 & z_2N_3 & z_2N_4 \end{bmatrix}} \underset{(10\times1)}{\left\{\dot{d}\right\}} =$$

$$= \underset{(3\times10)}{\left[G^{(1)}\right]}\underset{(10\times1)}{\left\{\dot{d}\right\}}, \tag{5.10.55}$$

where

$$\underset{(3\times10)}{\left[G^{(1)}\right]} = \underset{(3\times10)}{\begin{bmatrix} 0 & 0 & 0 & 0 & 2z_2M_1 & 2z_2M_2 & \frac{1}{2}z_2^2\frac{dN_1}{dx} & \frac{1}{2}z_2^2\frac{dN_2}{dx} & \frac{1}{2}z_2^2\frac{dN_3}{dx} & \frac{1}{2}z_2^2\frac{dN_4}{dx} \\ -\frac{dN_1}{dx} & -\frac{dN_2}{dx} & -\frac{dN_3}{dx} & -\frac{dN_4}{dx} & 0 & 0 & -z_2\frac{dN_1}{dx} & -z_2\frac{dN_2}{dx} & -z_2\frac{dN_3}{dx} & -z_2\frac{dN_4}{dx} \\ N_1 & N_2 & N_3 & N_4 & 0 & 0 & z_2N_1 & z_2N_2 & z_2N_3 & z_2N_4 \end{bmatrix}} =$$

$$
= \begin{bmatrix}
0 & 6x\frac{l-x}{l^3} & 1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3} \\
0 & -\frac{l^2-4xl+3x^2}{l^2} & x-2\frac{x^2}{l}+\frac{x^3}{l^2} \\
0 & -6x\frac{l-x}{l^3} & 3\frac{x^2}{l^2}-2\frac{x^3}{l^3} \\
0 & x\frac{2l-3x}{l^2} & -\frac{x^2}{l}+\frac{x^3}{l^2} \\
2z_2\left(1-\frac{x}{l}\right) & 0 & 0 \\
2z_2\frac{x}{l} & 0 & 0 \\
-3z_2^2 x\frac{l-x}{l^3} & 6z_2 x\frac{l-x}{l^3} & z_2\left(1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3}\right) \\
\frac{1}{2}z_2^2\frac{l^2-4xl+3x^2}{l^2} & -z_2\frac{l^2-4xl+3x^2}{l^2} & z_2\left(x-2\frac{x^2}{l}+\frac{x^3}{l^2}\right) \\
3z_2^2 x\frac{l-x}{l^3} & -6z_2 x\frac{l-x}{l^3} & z_2\left(3\frac{x^2}{l^2}-2\frac{x^3}{l^3}\right) \\
-\frac{1}{2}z_2^2 x\frac{2l-3x}{l^2} & z_2 x\frac{2l-3x}{l^2} & z_2\left(-\frac{x^2}{l}+\frac{x^3}{l^2}\right)
\end{bmatrix}^T \qquad (5.10.56)
$$

Substitution of (5.10.55) into the first term of expression (5.10.54) yields

$$
\frac{1}{2}\rho^{(1)}b\int_0^l \left(\underset{(3\times3)}{\left[\tilde{\partial}^{(1)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right)^T \underset{(3\times3)}{\left[\tilde{D}^{(1)}\right]}\left(\underset{(3\times3)}{\left[\tilde{\partial}^{(1)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right) dx =
$$

$$
= \frac{1}{2}\underset{(1\times10)}{\left\{\dot{d}\right\}}^T \underset{(10\times10)}{\left[m^{(1)}\right]}\underset{(10\times1)}{\left\{\dot{d}\right\}}, \qquad (5.10.57)
$$

where

$$
\underset{(10\times10)}{\left[m^{(1)}\right]} = \rho^{(1)}b\int_0^l \underset{(10\times3)}{\left[G^{(1)}\right]}^T \underset{(3\times3)}{\left[\tilde{D}^{(1)}\right]}\underset{(3\times10)}{\left[G^{(1)}\right]} dx. \qquad (5.10.58)
$$

The components of the matrix $\underset{(10\times10)}{\left[m^{(1)}\right]}$ are written in Appendix 5-B.

Now, let us write the **second term of the expression (5.10.54) for the kinetic energy in terms of the time derivatives of the nodal variables.** This term is

$$
\frac{1}{2}\rho^{(2)}b\int_0^l \left(\underset{(4\times3)}{\left[\tilde{\partial}^{(2)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right)^T \underset{(4\times4)}{\left[\tilde{D}^{(2)}\right]}\left(\underset{(4\times3)}{\left[\tilde{\partial}^{(2)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right) dx\ .
$$

In this expression

$$
\underset{(4\times3)}{\left[\tilde{\partial}^{(2)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}} = \underset{(4\times3)}{\left[\tilde{\partial}^{(2)}\right]}\frac{\partial}{\partial t}\underset{(3\times10)}{[Q]}\underset{(10\times1)}{\{d\}} = \underset{(4\times3)}{\left[\tilde{\partial}^{(2)}\right]}\underset{(3\times10)}{[Q]}\underset{(10\times1)}{\left\{\dot{d}\right\}} =
$$

$$= \begin{bmatrix} -\frac{d}{dx} & 2 & 0 \\ 0 & 0 & -\frac{1}{2}\frac{d}{dx} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_1 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \underset{(10\times 1)}{\left\{\dot{d}\right\}} =$$

$$\begin{bmatrix} -\frac{dN_1}{dx} & -\frac{dN_2}{dx} & -\frac{dN_3}{dx} & -\frac{dN_4}{dx} & 2M_1 & 2M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{dN_1}{dx} & -\frac{1}{2}\frac{dN_2}{dx} & -\frac{1}{2}\frac{dN_3}{dx} & -\frac{1}{2}\frac{dN_4}{dx} \\ N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \underset{(10\times 1)}{\left\{\dot{d}\right\}} =$$

$$= \underset{(4\times 10)}{\left[G^{(2)}\right]} \underset{(10\times 1)}{\left\{\dot{d}\right\}}, \tag{5.10.59}$$

where

$$\underset{(4\times 10)}{\left[G^{(2)}\right]} = \begin{bmatrix} -\frac{dN_1}{dx} & -\frac{dN_2}{dx} & -\frac{dN_3}{dx} & -\frac{dN_4}{dx} & 2M_1 & 2M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}\frac{dN_1}{dx} & -\frac{1}{2}\frac{dN_2}{dx} & -\frac{1}{2}\frac{dN_3}{dx} & -\frac{1}{2}\frac{dN_4}{dx} \\ N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} =$$

$$= \begin{bmatrix} -6x\frac{-l+x}{l^3} & 0 & 1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3} & 0 \\ -\frac{l^2-4xl+3x^2}{l^2} & 0 & x-2\frac{x^2}{l}+\frac{x^3}{l^2} & 0 \\ 6x\frac{-l+x}{l^3} & 0 & 3\frac{x^2}{l^2}-2\frac{x^3}{l^3} & 0 \\ -x\frac{-2l+3x}{l^2} & 0 & -\frac{x^2}{l}+\frac{x^3}{l^2} & 0 \\ 2-2\frac{x}{l} & 0 & 0 & 0 \\ 2\frac{x}{l} & 0 & 0 & 0 \\ 0 & -3x\frac{-l+x}{l^3} & 0 & 1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3} \\ 0 & -\frac{1}{2}\frac{l^2-4xl+3x^2}{l^2} & 0 & x-2\frac{x^2}{l}+\frac{x^3}{l^2} \\ 0 & 3x\frac{-l+x}{l^3} & 0 & 3\frac{x^2}{l^2}-2\frac{x^3}{l^3} \\ 0 & -\frac{1}{2}x\frac{-2l+3x}{l^2} & 0 & -\frac{x^2}{l}+\frac{x^3}{l^2} \end{bmatrix}^T. \tag{5.10.60}$$

Substitution of expression (5.10.59) into the second term of expression (5.10.54) for the kinetic

energy yields

$$\frac{1}{2}\rho^{(2)}b\int_0^l \left(\underset{(4\times3)}{\left[\tilde{\partial}^{(2)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right)^T \underset{(4\times4)}{\left[\tilde{D}^{(2)}\right]}\left(\underset{(4\times3)}{\left[\tilde{\partial}^{(2)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right)\,dx =$$

$$=\frac{1}{2}\underset{(1\times10)}{\left\{\dot{d}\right\}}^T\underset{(10\times10)}{\left[m^{(2)}\right]}\underset{(10\times1)}{\left\{\dot{d}\right\}}, \tag{5.10.61}$$

where

$$\underset{(10\times10)}{\left[m^{(2)}\right]}=\rho^{(1)}b\int_0^l \underset{(10\times4)}{\left[G^{(2)}\right]^T}\underset{(4\times4)}{\left[\tilde{D}^{(2)}\right]}\underset{(4\times10)}{\left[G^{(2)}\right]}\,dx\,. \tag{5.10.62}$$

The components of the matrix $\underset{(10\times10)}{\left[m^{(2)}\right]}$ are written in Appendix 5-B.

Now, let us write the **third term of the expression (5.10.54) for the kinetic energy in terms of the time derivatives of the nodal variables.** This term is

$$\frac{1}{2}\rho^{(3)}b\int_0^l \left(\underset{(3\times3)}{\left[\tilde{\partial}^{(3)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right)^T \underset{(3\times3)}{\left[\tilde{D}^{(3)}\right]}\left(\underset{(3\times3)}{\left[\tilde{\partial}^{(3)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right)\,dx.$$

In this expression

$$\underset{(3\times3)}{\left[\tilde{\partial}^{(3)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}=\underset{(3\times3)}{\left[\tilde{\partial}^{(3)}\right]}\frac{\partial}{\partial t}\underset{(3\times10)}{[Q]}\underset{(10\times1)}{\{d\}}=$$

$$=\underset{(3\times3)}{\left[\tilde{\partial}^{(3)}\right]}\underset{(3\times10)}{[Q]}\underset{(10\times1)}{\left\{\dot{d}\right\}}=$$

$$=\underset{(3\times3)}{\begin{bmatrix} 0 & 2z_3 & \frac{1}{2}z_3^2\frac{d}{dx} \\ -\frac{d}{dx} & 0 & -z_3\frac{d}{dx} \\ 1 & 0 & z_3 \end{bmatrix}}\underset{(3\times10)}{\begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_1 & M_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix}}\underset{(10\times1)}{\left\{\dot{d}\right\}}=$$

$$=\underset{(3\times10)}{\begin{bmatrix} 0 & 0 & 0 & 0 & 2z_3M_1 & 2z_3M_2 & \frac{1}{2}z_3^2\frac{dN_1}{dx} & \frac{1}{2}z_3^2\frac{dN_2}{dx} & \frac{1}{2}z_3^2\frac{dN_3}{dx} & \frac{1}{2}z_3^2\frac{dN_4}{dx} \\ -\frac{dN_1}{dx} & -\frac{dN_2}{dx} & -\frac{dN_3}{dx} & -\frac{dN_4}{dx} & 0 & 0 & -z_3\frac{dN_1}{dx} & -z_3\frac{dN_2}{dx} & -z_3\frac{dN_3}{dx} & -z_3\frac{dN_4}{dx} \\ N_1 & N_2 & N_3 & N_4 & 0 & 0 & z_3N_1 & z_3N_2 & z_3N_3 & z_3N_4 \end{bmatrix}}\underset{(10\times1)}{\left\{\dot{d}\right\}}=$$

$$= \underset{(3\times 10)}{\left[G^{(3)}\right]} \underset{(10\times 1)}{\left\{\dot{d}\right\}}, \qquad (5.10.63)$$

where

$$\underset{(3\times 10)}{\left[G^{(3)}\right]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 2z_3M_1 & 2z_3M_2 & \frac{1}{2}z_3^2\frac{dN_1}{dx} & \frac{1}{2}z_3^2\frac{dN_2}{dx} & \frac{1}{2}z_3^2\frac{dN_3}{dx} & \frac{1}{2}z_3^2\frac{dN_4}{dx} \\ -\frac{dN_1}{dx} & -\frac{dN_2}{dx} & -\frac{dN_3}{dx} & -\frac{dN_4}{dx} & 0 & 0 & -z_3\frac{dN_1}{dx} & -z_3\frac{dN_2}{dx} & -z_3\frac{dN_3}{dx} & -z_3\frac{dN_4}{dx} \\ N_1 & N_2 & N_3 & N_4 & 0 & 0 & z_3N_1 & z_3N_2 & z_3N_3 & z_3N_4 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -6x\frac{-l+x}{l^3} & 1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3} \\ 0 & -\frac{l^2-4xl+3x^2}{l^2} & x-2\frac{x^2}{l}+\frac{x^3}{l^2} \\ 0 & 6x\frac{-l+x}{l^3} & 3\frac{x^2}{l^2}-2\frac{x^3}{l^3} \\ 0 & -x\frac{-2l+3x}{l^2} & -\frac{x^2}{l}+\frac{x^3}{l^2} \\ 2z_3\left(1-\frac{x}{l}\right) & 0 & 0 \\ 2z_3\frac{x}{l} & 0 & 0 \\ 3z_3^2x\frac{-l+x}{l^3} & -6z_3x\frac{-l+x}{l^3} & z_3\left(1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3}\right) \\ \frac{1}{2}z_3^2\frac{l^2-4xl+3x^2}{l^2} & -z_3\frac{l^2-4xl+3x^2}{l^2} & z_3\left(x-2\frac{x^2}{l}+\frac{x^3}{l^2}\right) \\ -3z_3^2x\frac{-l+x}{l^3} & 6z_3x\frac{-l+x}{l^3} & z_3\left(3\frac{x^2}{l^2}-2\frac{x^3}{l^3}\right) \\ \frac{1}{2}z_3^2x\frac{-2l+3x}{l^2} & -z_3x\frac{-2l+3x}{l^2} & z_3\left(-\frac{x^2}{l}+\frac{x^3}{l^2}\right) \end{bmatrix}^T . \qquad (5.10.64)$$

Substitution of (5.10.63) into the third term of expression (5.10.54) yields

$$\frac{1}{2}\rho^{(3)}b\int_0^l \left(\underset{(3\times 3)}{\left[\tilde{\partial}^{(3)}\right]}\frac{\partial}{\partial t}\underset{(3\times 1)}{\{f\}}\right)^T \underset{(3\times 3)}{\left[\tilde{D}^{(3)}\right]} \left(\underset{(3\times 3)}{\left[\tilde{\partial}^{(3)}\right]}\frac{\partial}{\partial t}\underset{(3\times 1)}{\{f\}}\right) dx =$$

$$= \frac{1}{2}\underset{(1\times 10)}{\left\{\dot{d}\right\}^T} \underset{(10\times 10)}{\left[m^{(3)}\right]} \underset{(10\times 1)}{\left\{\dot{d}\right\}}, \qquad (5.10.65)$$

where

$$\underset{(10\times 10)}{\left[m^{(3)}\right]} = \rho^{(3)}b\int_0^l \underset{(10\times 3)}{\left[G^{(3)}\right]^T} \underset{(3\times 3)}{\left[\tilde{D}^{(3)}\right]} \underset{(3\times 10)}{\left[G^{(3)}\right]} dx. \qquad (5.10.66)$$

The components of the matrix $\underset{(10\times 10)}{\left[m^{(3)}\right]}$ are written in Appendix 5-B.

Now, let us write the **fourth term of the expression (5.10.54) for the kinetic energy in terms of the time derivatives of the nodal variables.** This term is

$$\frac{1}{2}b\int_0^l \mu\, H(x) \left(\frac{\partial}{\partial t}\underset{(3\times 1)}{\{f\}}\right)^T \underset{(3\times 3)}{\left[\tilde{D}_c\right]} \left(\frac{\partial}{\partial t}\underset{(3\times 1)}{\{f\}}\right) dx,$$

where μ is a mass of the cargo per unit area of contact with the platform; $H(x)$ is a function, defined as follows:

$$H(x) = \begin{cases} 1 \text{ in region of the upper surface, occupied by the cargo} \\ 0 \text{ in region of the upper surface, not occupied by the cargo} \end{cases} .$$

Let us consider **a finite element, the upper surface of which is fully occupied by the cargo**. Then $H(x) \equiv 1$ within this finite element. In this case the fourth term of the expression (19.54) for the kinetic energy is

$$\frac{1}{2} b \int_0^l \mu \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \underset{(3\times3)}{\left[\widetilde{D}_c\right]} \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) dx =$$

$$= \frac{1}{2} b \int_0^l \mu \left(\underset{(3\times10)}{[Q]} \underset{(10\times1)}{\left\{\dot{d}\right\}} \right)^T \underset{(3\times3)}{\left[\widetilde{D}_c\right]} \underset{(3\times10)}{[Q]} \underset{(10\times1)}{\left\{\dot{d}\right\}} dx =$$

$$= \frac{1}{2} \underset{(1\times10)}{\left\{\dot{d}\right\}}^T \left(b \int_0^l \mu \underset{(10\times3)}{[Q]}^T \underset{(3\times3)}{\left[\widetilde{D}_c\right]} \underset{(3\times10)}{[Q]} dx \right) \underset{(10\times1)}{\left\{\dot{d}\right\}} =$$

$$= \frac{1}{2} \underset{(1\times10)}{\left\{\dot{d}\right\}}^T \underset{(10\times10)}{\left[m^{(c)}\right]} \underset{(10\times1)}{\left\{\dot{d}\right\}}, \tag{5.10.67}$$

where

$$\underset{(10\times10)}{\left[m^{(c)}\right]} = b \int_0^l \mu \underset{(10\times3)}{[Q]}^T \underset{(3\times3)}{\left[\widetilde{D}_c\right]} \underset{(3\times10)}{[Q]} dx . \tag{5.10.68}$$

The components of the matrix $\left[m^{(c)}\right]$ are written in Appendix 5-B.

If the upper surface of a finite element does not have a cargo on it, then the fourth term of the expression (5.10.54) for the kinetic energy is equal to zero.

So, kinetic energy of the system is

$$K = \frac{1}{2} \left\{\dot{d}\right\}^T [m] \left\{\dot{d}\right\}, \tag{5.10.69}$$

where

$$[m] = \left[m^{(1)}\right] + \left[m^{(2)}\right] + \left[m^{(3)}\right] + \left[m^{(c)}\right]. \tag{5.10.70}$$

5.10.3 Potential energy of the platform and the cargo in the gravity field

According to equation (5.7.1), potential energy of the platform and the cargo in the gravity field is

$$\Pi = b\int_0^L \underset{(1\times3)}{\{f\}}^T \underset{(3\times1)}{\{\Gamma\}}\, dx\,, \tag{5.10.71}$$

where

$$\{f\} = \begin{Bmatrix} w_0 \\ \varepsilon_{xz}^{(2)} \\ \varepsilon_{zz}^{(2)} \end{Bmatrix},$$

$$\underset{(3\times1)}{\{\Gamma\}} = \begin{Bmatrix} g\left[\rho^{(1)}(z_2 - z_1) + \rho^{(2)}(z_3 - z_2) + \rho^{(3)}(z_4 - z_3) + \mu\, H(x)\right] \\ 0 \\ g\left[\rho^{(1)} z_2 (z_2 - z_1) + \frac{1}{2}\rho^{(2)}\left(z_3^2 - z_2^2\right) + \rho^{(3)}\, z_3 (z_4 - z_3) + \mu\, H(x)\, z_3\right] \end{Bmatrix}. \tag{5.10.72}$$

Substitution of relation $\underset{(3\times1)}{\{f\}} = \underset{(3\times10)}{[Q]}\underset{(10\times1)}{\{d\}}$ (equation (5.10.43)) into (5.10.71) yields

$$\Pi = \underset{(1\times10)}{\{d\}}^T b\int_0^l \underset{(10\times3)}{[Q]}^T \underset{(3\times1)}{\{\Gamma\}}\, dx =$$

$$= \underset{(1\times10)}{\{d\}}^T b\int_0^l \begin{bmatrix} N_1 & 0 & 0 \\ N_2 & 0 & 0 \\ N_3 & 0 & 0 \\ N_4 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & N_1 \\ 0 & 0 & N_2 \\ 0 & 0 & N_3 \\ 0 & 0 & N_4 \end{bmatrix} \underset{(3\times1)}{\{\Gamma\}}\, dx\,.$$

Calculations give the following result

$$\Pi = -\underset{(1\times10)}{\{d\}}^T \underset{(10\times1)}{\{r\}}\,, \tag{5.10.73}$$

where [5]

$$\underset{(10\times 1)}{\{r\}} = -bg \begin{bmatrix} \frac{1}{2}l\rho^{(1)}(z_2-z_1)+\frac{1}{2}l\rho^{(2)}(z_3-z_2)+\frac{1}{2}l\rho^{(3)}(z_4-z_3)+\frac{1}{2}l\mu \\ \frac{1}{12}l^2\rho^{(1)}(z_2-z_1)+\frac{1}{12}l^2\rho^{(2)}(z_3-z_2)+\frac{1}{12}l^2\rho^{(3)}(z_4-z_3)+\frac{1}{12}l^2\mu \\ \frac{1}{2}l\rho^{(1)}(z_2-z_1)+\frac{1}{2}l\rho^{(2)}(z_3-z_2)+\frac{1}{2}l\rho^{(3)}(z_4-z_3)+\frac{1}{2}l\mu \\ \frac{1}{12}l^2\rho^{(1)}(z_1-z_2)+\frac{1}{12}l^2\rho^{(2)}(z_2-z_3)+\frac{1}{12}l^2\rho^{(3)}(z_3-z_4)-\frac{1}{12}l^2\mu \\ 0 \\ 0 \\ \frac{1}{2}l\rho^{(1)}z_2(z_2-z_1)+\frac{1}{4}l\rho^{(2)}\left(z_3^2-z_2^2\right)+\frac{1}{2}l\rho^{(3)}z_3(z_4-z_3)+\frac{1}{2}l\mu z_3 \\ \frac{1}{12}l^2\rho^{(1)}z_2(z_2-z_1)+\frac{1}{24}l^2\rho^{(2)}\left(z_3^2-z_2^2\right)+\frac{1}{12}l^2\rho^{(3)}z_3(z_4-z_3)+\frac{1}{12}l^2\mu z_3 \\ \frac{1}{2}l\rho^{(1)}z_2(z_2-z_1)+\frac{1}{4}l\rho^{(2)}\left(z_3^2-z_2^2\right)+\frac{1}{2}l\rho^{(3)}z_3(z_4-z_3)+\frac{1}{2}l\mu z_3 \\ \frac{1}{12}l^2\rho^{(1)}z_2(z_1-z_2)+\frac{1}{24}l^2\rho^{(2)}\left(z_2^2-z_3^2\right)+\frac{1}{12}l^2\rho^{(3)}z_3(z_3-z_4)-\frac{1}{12}l^2\mu z_3 \end{bmatrix} \tag{5.10.74}$$

if the upper surface of the finite element is fully covered by the cargo, and

$$\underset{(10\times 1)}{\{r\}} = -bg \begin{bmatrix} \frac{1}{2}l\rho^{(1)}(z_2-z_1)+\frac{1}{2}l\rho^{(2)}(z_3-z_2)+\frac{1}{2}l\rho^{(3)}(z_4-z_3) \\ \frac{1}{12}l^2\rho^{(1)}(z_2-z_1)+\frac{1}{12}l^2\rho^{(2)}(z_3-z_2)+\frac{1}{12}l^2\rho^{(3)}(z_4-z_3) \\ \frac{1}{2}l\rho^{(1)}(z_2-z_1)+\frac{1}{2}l\rho^{(2)}(z_3-z_2)+\frac{1}{2}l\rho^{(3)}(z_4-z_3) \\ \frac{1}{12}l^2\rho^{(1)}(z_1-z_2)+\frac{1}{12}l^2\rho^{(2)}(z_2-z_3)+\frac{1}{12}l^2\rho^{(3)}(z_3-z_4) \\ 0 \\ 0 \\ \frac{1}{2}l\rho^{(1)}z_2(z_2-z_1)+\frac{1}{4}l\rho^{(2)}\left(z_3^2-z_2^2\right)+\frac{1}{2}l\rho^{(3)}z_3(z_4-z_3) \\ \frac{1}{12}l^2\rho^{(1)}z_2(z_2-z_1)+\frac{1}{24}l^2\rho^{(2)}\left(z_3^2-z_2^2\right)+\frac{1}{12}l^2\rho^{(3)}z_3(z_4-z_3) \\ \frac{1}{2}l\rho^{(1)}z_2(z_2-z_1)+\frac{1}{4}l\rho^{(2)}\left(z_3^2-z_2^2\right)+\frac{1}{2}l\rho^{(3)}z_3(z_4-z_3) \\ \frac{1}{12}l^2\rho^{(1)}z_2(z_1-z_2)+\frac{1}{24}l^2\rho^{(2)}\left(z_2^2-z_3^2\right)+\frac{1}{12}l^2\rho^{(3)}z_3(z_3-z_4) \end{bmatrix} \tag{5.10.75}$$

if on the upper surface of the finite element the cargo is totally absent. We do not consider the case of the cargo occupying a part of the upper surface of the finite element, because a finite element mesh can be created in such a way that some of the finite elements are totally covered by the cargo, and the rest of the finite elements have totally free upper surfaces.

[5] vector $\{d\}$ is shown in equation (5.10.22)

5.10.4 Equations of motion in terms of the nodal variables

The Hamilton's principle for the system, that consists of the platform, the cargo on the upper surface of the platform and the elastic foundation, written in terms of the nodal variables, has the form:

$$\int_{t_1}^{t_2}\left(\delta\left(\frac{1}{2}\{d\}^T[k]\{d\}\right)+\delta U_{nl}-\delta\left(\{d\}^T\{r\}\right)-\delta\left(\frac{1}{2}\left\{\dot{d}\right\}^T[m]\left\{\dot{d}\right\}\right)+\{\delta d\}[c]\left\{\dot{d}\right\}\right)dt=0, \tag{5.10.76}$$

where $[c]$ is an element damping matrix. It is difficult to determine the element damping matrix experimentally because the damping characteristics of the plate depend on the properties of the whole plate. For this reason, the global damping matrix is in general not assembled from element damping matrices, but is constructed from the mass and stiffness matrix of the complete element assemblage together with experimental results on the amount of damping in the whole plate. We will use the Rayleigh damping model, in which the global damping matrix $[C]$ is presented as a linear combination of the global mass matrix [M] and the global stiffness matrix [K]

$$[C]=\alpha\,[\mathrm{K}]+\beta\,[\mathrm{M}], \tag{5.10.77}$$

where α and β are constants to be determined from two given logarithmic decrements δ_1 and δ_2 that correspond to two unequal frequencies of vibrations ω_1 and ω_2 by the formulas

$$\alpha=\frac{\delta_1\omega_1-\delta_2\omega_2}{\pi\left(\omega_1^2-\omega_2^2\right)}, \tag{5.10.78}$$

$$\beta=\frac{\omega_1\omega_2\left(\delta_2\omega_1-\delta_1\omega_2\right)}{\pi\left(\omega_1^2-\omega_2^2\right)}. \tag{5.10.79}$$

The Lagrange equations of motion in terms of the nodal variables, that follow from the Hamilton's principle (5.10.76), are

$$\frac{\partial}{\partial d_i}\left(\frac{1}{2}\{d\}^T[k]\{d\}\right)+\frac{\partial U_{nl}}{\partial d_i}-\frac{\partial}{\partial d_i}\left(\{d\}^T\{r\}\right)+\frac{d}{dt}\frac{\partial}{\partial\dot{d}_i}\left(\frac{1}{2}\left\{\dot{d}\right\}^T[m]\left\{\dot{d}\right\}\right)+$$

$$+[c]\left\{\dot{d}\right\}=0 \qquad (i=1,2,...,10) \tag{5.10.80}$$

or in matrix notations

$$\frac{\partial}{\partial\{d\}}\left(\frac{1}{2}\{d\}^T[k]\{d\}\right)+\frac{\partial U_{nl}}{\partial\{d\}}-\frac{\partial}{\partial\{d\}}\left(\{d\}^T\{r\}\right)+\frac{d}{dt}\frac{\partial}{\partial\left\{\dot{d}\right\}}\left(\frac{1}{2}\left\{\dot{d}\right\}^T[m]\left\{\dot{d}\right\}\right)+$$

$$+[c]\left\{\dot{d}\right\} = \{0\}. \tag{5.10.81}$$

From equation (5.10.81) we obtain

$$[k]\{d\} + \frac{\partial U_{nl}}{\partial\{d\}} + [m]\left\{\ddot{d}\right\} + [c]\left\{\dot{d}\right\} = \{r\}. \tag{5.10.82}$$

Part of the strain energy U_{nl} is due to the nonlinear terms in the strain-displacement relations (geometric non-linearity of the von-Karman type). U_{nl} is not a quadratic form of the nodal variables, therefore vector $\frac{\partial U_{nl}}{\partial\{d\}}$ is not linear with respect to the nodal variables. All the quantities that enter into the equation (5.10.82), except the element damping matrix $[c]$, are defined in this chapter. But, as it was written before, the element damping matrix is not required because the global damping matrix will be constructed from the global mass matrix and the global stiffness matrix.

5.10.5 The more convenient numbering scheme for the local degrees of freedom

So far, the nodal variables of an element were defined as follows, i.e. were given the following local numbers (equation (5.10.22)):

$$d_1 = w_0(0),\ d_2 = w_0'(0),\ d_3 = w_0(l),\ d_4 = w_0'(l),\ d_5 = \varepsilon_{xz}^{(2)}(0),$$

$$d_6 = \varepsilon_{xz}^{(2)}(l),\ d_7 = \varepsilon_{zz}^{(2)}(0),\ d_8 = \frac{d\varepsilon_{zz}^{(2)}}{dx}(0),\ d_9 = \varepsilon_{zz}^{(2)}(l),\ d_{10} = \frac{d\varepsilon_{zz}^{(2)}}{dx}(l),$$

where, for example, $w_0(0) \equiv w_0|_{\overline{x}=0}$ is displacement at the left node of an element, or, which is the same, at point $\overline{x} = 0$, where $\overline{x}$ is local (element) x-coordinate; $w_0(l) \equiv w_0|_{\overline{x}=l}$ is displacement at the right node of an element, or, which is the same, at $\overline{x} = l$, where l is length of an element.

For the sake of convenience of assembling the global matrices, we will introduce a different local numbering scheme of the nodal variables:

$$\theta_1 = w_0(0) = d_1,$$

$$\theta_2 = \frac{dw_0}{dx}(0) = d_2,$$

$$\theta_3 = \varepsilon_{xz}^{(2)}(0) = d_5,$$

$$\theta_4 = \varepsilon_{zz}^{(2)}(0) = d_7,$$

$$\theta_5 = \frac{d\varepsilon_{zz}^{(2)}}{dx}(0) = d_8,$$

$$\theta_6 = w_0(l) = d_3,$$

$$\theta_7 = \frac{dw_0}{dx}(l) = d_4,$$

$$\theta_8 = \varepsilon_{xz}^{(2)}(l) = d_6,$$

$$\theta_9 = \varepsilon_{zz}^{(2)}(l) = d_9,$$

$$\theta_{10} = \frac{d\varepsilon_{zz}^{(2)}}{dx}(l) = d_{10}. \tag{5.10.84}$$

These new nodal variables θ_i are more convenient for assembling of global matrices, because the numbering order of θ_i is such that:

the first nodal variable of the left node (θ_1) is w_0 and the first nodal variable of the right node (θ_6) is also w_0;

the second nodal variable of the left node (θ_2) is $\frac{dw_0}{dx}$ and the second nodal variable of the right node (θ_7) is also $\frac{dw_0}{dx}$;

the third nodal variable of the left node (θ_3) is $\varepsilon_{xz}^{(2)}$ and the third nodal variable of the right node (θ_8) is also $\varepsilon_{xz}^{(2)}$;

the fourth nodal variable of the left node (θ_4) is $\varepsilon_{zz}^{(2)}$ and the fourth nodal variable of the right node (θ_9) is also $\varepsilon_{zz}^{(2)}$;

the fifth nodal variable of the left node (θ_5) is $\frac{d\varepsilon_{zz}^{(2)}}{dx}$ and the fifth nodal variable of the right node (θ_{10}) is also $\frac{d\varepsilon_{zz}^{(2)}}{dx}$.

Such numbering scheme allows to establish the correspondence between the local and global notations of the nodal variables by a simple formula

$$\theta_{\underset{\substack{\downarrow \\ \text{local \#} \\ \text{of d.o.f.}}}{i}}^{\overset{\text{elem.\#}}{\overset{\uparrow}{(iel)}}} = \Theta_{\underbrace{5(iel-1)+i}_{\substack{\text{global \#} \\ \text{of d.o.f.}}}}. \tag{5.10.85}$$

Let $A_1, A_2, ..., A_{nel+1}$ be the notations of the nodal points. Then, from relations (5.10.84), which establish correspondence between the meaning of nodal variables and their local numbering, and

from relations (5.10.85) (which establish correspondence between the local and global numbering of the nodal variables) we obtain, for example,

for the first element:

$$w_0\,(A_1) = \theta_1^{(1)} = \Theta_1,$$

$$\frac{dw_0}{dx}(A_1) = \theta_2^{(1)} = \Theta_2,$$

$$\varepsilon_{xz}^{(2)}(A_1) = \theta_3^{(1)} = \Theta_3,$$

$$\varepsilon_{zz}^{(2)}(A_1) = \theta_4^{(1)} = \Theta_4,$$

$$\frac{d\varepsilon_{zz}^{(2)}}{dx}(A_1) = \theta_5^{(1)} = \Theta_5,$$

$$w_0(A_2) = \theta_6^{(1)} = \Theta_6,$$

$$\frac{dw_0}{dx}(A_2) = \theta_7^{(1)} = \Theta_7,$$

$$\varepsilon_{xz}^{(2)}(A_2) = \theta_8^{(1)} = \Theta_8,$$

$$\varepsilon_{zz}^{(2)}(A_2) = \theta_9^{(1)} = \Theta_9,$$

$$\frac{d\varepsilon_{zz}^{(2)}}{dx}(A_2) = \theta_{10}^{(1)} = \Theta_{10},$$

for the second element:

$$w_0\,(A_2) = \theta_1^{(2)} = \Theta_6,$$

$$\frac{dw_0}{dx}\,(A_2) = \theta_2^{(2)} = \Theta_7,$$

$$\varepsilon_{xz}^{(2)}(A_2) = \theta_3^{(2)} = \Theta_8,$$

$$\varepsilon_{zz}^{(2)}(A_2) = \theta_4^{(2)} = \Theta_9,$$

$$\frac{d\varepsilon_{zz}^{(2)}}{dx}(A_2) = \theta_5^{(2)} = \Theta_{10},$$

$$w_0\,(A_3) = \theta_6^{(2)} = \Theta_{11},$$

$$\frac{dw_0}{dx}\,(A_3) = \theta_7^{(2)} = \Theta_{12},$$

$$\varepsilon_{xz}^{(2)}(A_3) = \theta_8^{(2)} = \Theta_{13},$$

$$\varepsilon_{zz}^{(2)}(A_3) = \theta_9^{(2)} = \Theta_{14},$$

$$\frac{d\varepsilon_{zz}^{(2)}}{dx}(A_3) = \theta_{10}^{(2)} = \Theta_{15}.$$

So, when we pass from local θ_i to global Θ_i notations of nodal variables by formula (5.10.85), each nodal variable, that belongs to both adjacent elements, is denoted as one and the same in global notations, that is required for providing continuity of the nodal variables at the interelement boundaries. For example, the nodal variable $\theta_6^{(1)} \equiv w_0\,(A_2)$, that belongs to the first element, and the nodal variable $\theta_1^{(2)} \equiv w_0(A_2)$, that belongs to the second element, are both denoted as Θ_6. The numbering of nodal variables, can be presented as follows:

$$w_0(\underset{\substack{\downarrow\\\text{global}\\\text{node \#}}}{A_{iel}}) = \theta\,\substack{\text{elem.\#}\\\uparrow\\(iel)\\1\\\downarrow\\\text{local \#}\\\text{of d.o.f.}} = \underbrace{\Theta_{5(iel-1)+1}}_{\substack{\text{global \#}\\\text{of d.o.f.}}} \quad \text{or} \quad w_0\left(\underbrace{A_{iel+1}}_{\substack{\downarrow\\\text{global}\\\text{node \#}}}\right) = \theta\,\substack{\text{elem.\#}\\\uparrow\\(iel)\\6\\\downarrow\\\text{local \#}\\\text{of d.o.f.}} = \underbrace{\Theta_{5(iel-1)+6}}_{\substack{\text{global \#}\\\text{of d.o.f.}}}$$

$$\frac{dw_0}{dx}(\underset{\substack{\downarrow\\\text{global}\\\text{node \#}}}{A_{iel}}) = \theta\,\substack{\text{elem.\#}\\\uparrow\\(iel)\\2\\\downarrow\\\text{local \#}\\\text{of d.o.f.}} = \underbrace{\Theta_{5(iel-1)+2}}_{\substack{\text{global \#}\\\text{of d.o.f.}}} \quad \text{or} \quad \frac{dw_0}{dx}\left(\underbrace{A_{iel+1}}_{\substack{\downarrow\\\text{global}\\\text{node \#}}}\right) = \theta\,\substack{\text{elem.\#}\\\uparrow\\(iel)\\7\\\downarrow\\\text{local \#}\\\text{of d.o.f.}} = \underbrace{\Theta_{5(iel-1)+7}}_{\substack{\text{global \#}\\\text{of d.o.f.}}}$$

$$\varepsilon_{xz}^{(2)}(\underset{\substack{\downarrow\\\text{global}\\\text{node \#}}}{A_{iel}}) = \theta\,\substack{\text{elem.\#}\\\uparrow\\(iel)\\3\\\downarrow\\\text{local \#}\\\text{of d.o.f.}} = \underbrace{\Theta_{5(iel-1)+3}}_{\substack{\text{global \#}\\\text{of d.o.f.}}} \quad \text{or} \quad \varepsilon_{xz}^{(2)}\left(\underbrace{A_{iel+1}}_{\substack{\downarrow\\\text{global}\\\text{node \#}}}\right) = \theta\,\substack{\text{elem.\#}\\\uparrow\\(iel)\\8\\\downarrow\\\text{local \#}\\\text{of d.o.f.}} = \underbrace{\Theta_{5(iel-1)+8}}_{\substack{\text{global \#}\\\text{of d.o.f.}}}$$

$$\varepsilon_{zz}^{(2)}(\underset{\substack{\downarrow\\\text{global}\\\text{node \#}}}{A_{iel}}) = \theta\,\substack{\text{elem.\#}\\\uparrow\\(iel)\\4\\\downarrow\\\text{local \#}\\\text{of d.o.f.}} = \underbrace{\Theta_{5(iel-1)+4}}_{\substack{\text{global \#}\\\text{of d.o.f.}}} \quad \text{or} \quad \varepsilon_{zz}^{(2)}\left(\underbrace{A_{iel+1}}_{\substack{\downarrow\\\text{global}\\\text{node \#}}}\right) = \theta\,\substack{\text{elem.\#}\\\uparrow\\(iel)\\9\\\downarrow\\\text{local \#}\\\text{of d.o.f.}} = \underbrace{\Theta_{5(iel-1)+9}}_{\substack{\text{global \#}\\\text{of d.o.f.}}}$$

$$\frac{d\varepsilon_{zz}^{(2)}}{dx}(\underset{\substack{\downarrow\\ \text{global}\\ \text{node }\#}}{A_{iel}}) = \theta\,\overset{\substack{\text{elem.}\#\\ \uparrow}}{\underset{\substack{\downarrow\\ \text{local }\#\\ \text{of d.o.f.}}}{{}^{(iel)}_{5}}} = \Theta_{\underbrace{5(iel-1)+5}_{\substack{\text{global }\#\\ \text{of d.o.f.}}}} \quad \text{or} \quad \frac{d\varepsilon_{zz}^{(2)}}{dx}\left(A_{\underbrace{iel+1}_{\substack{\downarrow\\ \text{global}\\ \text{node }\#}}}\right) = \theta\,\overset{\substack{\text{elem.}\#\\ \uparrow}}{\underset{\substack{\downarrow\\ \text{local }\#\\ \text{of d.o.f.}}}{{}^{(iel)}_{10}}} = \Theta_{\underbrace{5(iel-1)+10}_{\substack{\text{global }\#\\ \text{of d.o.f.}}}}$$

The element stiffness matrix, mass matrix, damping matrix and force vector, corresponding to the newly defined nodal variables θ_i, will be denoted, respectively, as $[\kappa]$, $[mass]$, $[\varrho]$ and $\{p\}$.

The correspondence between the components of the element force vector in old notations, r_i, and the components of the element force vector in new notations, p_i, is the following:

$$p_1 = r_1,\ p_2 = r_2,\ p_3 = r_5,\ p_4 = r_7,\ p_5 = r_8,$$

$$p_6 = r_3,\ p_7 = r_4,\ p_8 = r_6,\ p_9 = r_9,\ p_{10} = r_{10}.$$

The correspondence between the components of the element stiffness matrix in old notations, k_{ij}, and the components of the element stiffness matrix in new notations, κ_{ij}, is given below:

$\boxed{d_1 = \theta_1}$	$\boxed{d_2 = \theta_2}$	$\boxed{d_3 = \theta_6}$	$\boxed{d_4 = \theta_7}$	$\boxed{d_5 = \theta_3}$	$\boxed{d_6 = \theta_8}$	$\boxed{d_7 = \theta_4}$	$\boxed{d_8 = \theta_5}$	$\boxed{d_9 = \theta_9}$	$\boxed{d_{10} = \theta_{10}}$	
$k_{11} = \kappa_{11}$	$k_{12} = \kappa_{12}$	$k_{13} = \kappa_{16}$	$k_{14} = \kappa_{17}$	$k_{15} = \kappa_{13}$	$k_{16} = \kappa_{18}$	$k_{17} = \kappa_{14}$	$k_{18} = \kappa_{15}$	$k_{19} = \kappa_{19}$	$k_{1,10} = \kappa_{1,10}$	$\boxed{d_1 = \theta_1}$
	$k_{22} = \kappa_{22}$	$k_{23} = \kappa_{26}$	$k_{24} = \kappa_{27}$	$k_{25} = \kappa_{23}$	$k_{26} = \kappa_{28}$	$k_{27} = \kappa_{24}$	$k_{28} = \kappa_{25}$	$k_{29} = \kappa_{29}$	$k_{2,10} = \kappa_{2,10}$	$\boxed{d_2 = \theta_2}$
		$k_{33} = \kappa_{66}$	$k_{34} = \kappa_{67}$	$k_{35} = \kappa_{63}$	$k_{36} = \kappa_{68}$	$k_{37} = \kappa_{64}$	$k_{38} = \kappa_{65}$	$k_{39} = \kappa_{69}$	$k_{3,10} = \kappa_{6,10}$	$\boxed{d_3 = \theta_6}$
			$k_{44} = \kappa_{77}$	$k_{45} = \kappa_{73}$	$k_{46} = \kappa_{78}$	$k_{47} = \kappa_{74}$	$k_{48} = \kappa_{75}$	$k_{49} = \kappa_{79}$	$k_{4,10} = \kappa_{7,10}$	$\boxed{d_4 = \theta_7}$
				$k_{55} = \kappa_{33}$	$k_{56} = \kappa_{38}$	$k_{57} = \kappa_{34}$	$k_{58} = \kappa_{35}$	$k_{59} = \kappa_{39}$	$k_{5,10} = \kappa_{3,10}$	$\boxed{d_5 = \theta_3}$
					$k_{66} = \kappa_{88}$	$k_{67} = \kappa_{84}$	$k_{68} = \kappa_{85}$	$k_{69} = \kappa_{89}$	$k_{6,10} = \kappa_{8,10}$	$\boxed{d_6 = \theta_8}$
						$k_{77} = \kappa_{44}$	$k_{78} = \kappa_{45}$	$k_{79} = \kappa_{49}$	$k_{7,10} = \kappa_{4,10}$	$\boxed{d_7 = \theta_4}$
							$k_{88} = \kappa_{55}$	$k_{89} = \kappa_{59}$	$k_{8,10} = \kappa_{5,10}$	$\boxed{d_8 = \theta_5}$
								$k_{99} = \kappa_{99}$	$k_{9,10} = \kappa_{9,10}$	$\boxed{d_9 = \theta_9}$
									$k_{10,10} = \kappa_{10,10}$	$\boxed{d_{10} = \theta_{10}}$

The correspondence between the components of the element mass matrix in old notations, m_{ij} and the components of the element mass matrix in new notations, $(mass)_{ij}$, is established in the same manner.

So, the equation of motion of a finite element (5.10.82), written with the use of the new nodal variables θ_i, defined by equation (5.10.84), takes the form

$$[-]\{\theta\} + \frac{\partial U_{nl}}{\partial \{\theta\}} + [mass]\left\{\ddot{\theta}\right\} + [\varrho]\left\{\dot{\theta}\right\} = \{p\}. \tag{5.10.86}$$

The first component of the nonlinear part of the internal force vector $\frac{\partial U_{nl}}{\partial\{\theta\}}$ is written explicitly in Appendix 5-C. The other nine components are not written in Appendix 5-C due to the limitation on the size of the dissertation.

5.11 Post-Processing of Results of the Finite Element Analysis of the Cargo Platform, Modelled as a Wide Beam

5.11.1 Formulas for stresses in terms of the field variables

After computation of the nodal variables, the stresses need to be computed in the post-processing procedure, in order to substitute them into the failure criteria. As it was written in the previous chapters, the in-plane stresses σ_{xx} will be computed from the constitutive relations, and the transverse stresses σ_{xz} and σ_{zz} - from the equations of motion in terms of the second Piola-Kirchhoff stress tensor, equations (3.1.21)-(3.1.23). These equations, written here again, are

$$\sigma^{(k)}_{xx,x} + \sigma^{(k)}_{xy,y} + \sigma^{(k)}_{xz,z} = \rho^{(k)}\ddot{u}^{(k)} \ , \tag{5.11.1}$$

$$\sigma^{(k)}_{yx,x} + \sigma^{(k)}_{yy,y} + \sigma^{k}_{yz,z} = \rho^{(k)}\ddot{v}^{(k)} \ , \tag{5.11.2}$$

$$\sigma^{(k)}_{zx,x} + \sigma^{(k)}_{zy,y} + \sigma^{(k)}_{zz,z} + \frac{\partial}{\partial x}\left(\sigma^{(k)}_{xx} w^{(k)}_{,x} + \sigma^{(k)}_{yx} w^{(k)}_{,y}\right) +$$

$$+\frac{\partial}{\partial y}\left(\sigma^{(k)}_{xy} \ w^{(k)}_{,x} + \sigma^{(k)}_{yy} \ w^{(k)}_{,y}\right) - \rho^{(k)} g = \rho^{(k)}\ddot{w}^{(k)} \tag{5.11.3}$$

$$(k = 1, 2, 3) \, .$$

In case of cylindrical bending these equations of motion take the form:

$$\sigma^{(k)}_{xx,x} + \sigma^{(k)}_{xz,z} = \rho^{(k)}\ddot{u}^{(k)} \ , \tag{5.11.4}$$

$$\sigma^{(k)}_{zx,x} + \sigma^{(k)}_{zz,z} + \frac{\partial}{\partial x}\left(\sigma^{(k)}_{xx} w^{(k)}_{,x}\right) - \rho^{(k)} g = \rho^{(k)}\ddot{w}^{(k)} \tag{5.11.5}$$

$$(k = 1, 2, 3) \, .$$

From the constitutive equations (3.6.13) and formulas (5.3.1)–(5.3.3) for the strains in terms of the field variables $w_0(x,t)$, $\varepsilon_{xz}^{(2)}(x,t)$, $\varepsilon_{zz}^{(2)}(x,t)$, we find the following expressions for the in-plane stresses in terms of the field variables:

$$^H\sigma_{xx}^{(1)} = \overline{C}_{11}^{(1)}\varepsilon_{xx}^{(1)} + \overline{C}_{12}^{(1)}\underbrace{\varepsilon_{yy}^{(1)}}_{0} + \overline{C}_{13}^{(1)}\underbrace{\varepsilon_{zz}^{(1)}}_{0} + \overline{C}_{16}^{(1)} 2\underbrace{\varepsilon_{xy}^{(1)}}_{0} =$$

$$= \overline{C}_{11}^{(1)}\left[\ 2z_2\ \varepsilon_{xz,x}^{(2)} + \tfrac{1}{2}z_2^2\ \varepsilon_{zz,xx}^{(2)} + \tfrac{1}{2}\left(w_{0,x} + z_2\ \varepsilon_{zz,x}^{(2)}\right)^2 - \left(w_{0,xx} + z_2\ \varepsilon_{zz,xx}^{(2)}\right) z\right], \qquad (5.11.6)$$

$$^H\sigma_{xx}^{(2)} = \overline{C}_{11}^{(2)}\varepsilon_{xx}^{(2)} + \overline{C}_{12}^{(2)}\underbrace{\varepsilon_{yy}^{(2)}}_{0} + \overline{C}_{13}^{(2)}\varepsilon_{zz}^{(2)} + \overline{C}_{16}^{(2)} 2\underbrace{\varepsilon_{xy}^{(2)}}_{0} =$$

$$= \overline{C}_{11}^{(2)}\left[\ \tfrac{1}{2}(w_{0,x})^2 + \left(2\varepsilon_{xz,x}^{(2)} - w_{0,xx} + w_{0,x}\varepsilon_{zz,x}^{(2)}\right) z + \left(-\frac{1}{2}\varepsilon_{zz,xx}^{(2)} + \frac{1}{2}\left(\varepsilon_{zz,x}^{(2)}\right)^2\right) z^2\right] +$$

$$+ \overline{C}_{13}^{(2)}\varepsilon_{zz}^{(2)}, \qquad (5.11.7)$$

$$^H\sigma_{xx}^{(3)} = \overline{C}_{11}^{(3)}\varepsilon_{xx}^{(3)} + \overline{C}_{12}^{(3)}\underbrace{\varepsilon_{yy}^{(3)}}_{0} + \overline{C}_{13}^{(3)}\underbrace{\varepsilon_{zz}^{(3)}}_{0} + \overline{C}_{16}^{(3)} 2\underbrace{\varepsilon_{xy}^{(3)}}_{0} =$$

$$= \overline{C}_{11}^{(3)}\left[\ 2z_3\ \varepsilon_{xz,x}^{(2)} + \tfrac{1}{2}z_3^2\ \varepsilon_{zz,xx}^{(2)} + \tfrac{1}{2}\left(w_{0,x} + z_3\ \varepsilon_{zz,x}^{(2)}\right)^2 - \left(w_{0,xx} + z_3\ \varepsilon_{zz,xx}^{(2)}\right) z\right]. \qquad (5.11.8)$$

$$^H\sigma_{yy}^{(1)} = \overline{C}_{12}^{(1)}\varepsilon_{xx}^{(1)} + \overline{C}_{22}^{(1)}\underbrace{\varepsilon_{yy}^{(1)}}_{0} + \overline{C}_{23}^{(1)}\underbrace{\varepsilon_{zz}^{(1)}}_{0} + \overline{C}_{26}^{(1)} 2\underbrace{\varepsilon_{xy}^{(1)}}_{0} =$$

$$= \overline{C}_{12}^{(1)}\left[\ 2z_2\ \varepsilon_{xz,x}^{(2)} + \tfrac{1}{2}z_2^2\ \varepsilon_{zz,xx}^{(2)} + \tfrac{1}{2}\left(w_{0,x} + z_2\ \varepsilon_{zz,x}^{(2)}\right)^2 - \left(w_{0,xx} + z_2\ \varepsilon_{zz,xx}^{(2)}\right) z\right], \qquad (5.11.9)$$

$$^H\sigma_{yy}^{(2)} = \overline{C}_{12}^{(2)}\varepsilon_{xx}^{(2)} + \overline{C}_{22}^{(2)}\underbrace{\varepsilon_{yy}^{(2)}}_{0} + \overline{C}_{23}^{(2)}\varepsilon_{zz}^{(2)} + \overline{C}_{26}^{(2)} 2\underbrace{\varepsilon_{xy}^{(2)}}_{0} =$$

$$= \overline{C}_{12}^{(2)} \left[\tfrac{1}{2}(w_{0,x})^2 + \left(2\varepsilon_{xz,x}^{(2)} - w_{0,xx} + w_{0,x}\varepsilon_{zz,x}^{(2)}\right) z + \left(-\frac{1}{2}\varepsilon_{zz,xx}^{(2)} + \frac{1}{2}\left(\varepsilon_{zz,x}^{(2)}\right)^2\right) z^2 \right] +$$

$$+\overline{C}_{23}^{(2)}\varepsilon_{zz}^{(2)}, \tag{5.11.10}$$

$${}^H\sigma_{yy}^{(3)} = \overline{C}_{12}^{(3)}\varepsilon_{xx}^{(3)} + \overline{C}_{22}^{(3)}\underbrace{\varepsilon_{yy}^{(3)}}_{0} + \overline{C}_{23}^{(2)}\underbrace{\varepsilon_{zz}^{(3)}}_{0} + \overline{C}_{26}^{(3)}2\underbrace{\varepsilon_{xy}^{(3)}}_{0} =$$

$$= \overline{C}_{12}^{(3)} \left[2z_3\ \varepsilon_{xz,x}^{(2)} + \tfrac{1}{2}z_3^2\ \varepsilon_{zz,xx}^{(2)} + \underbrace{\frac{1}{2}\left(w_{0,x} + z_3\ \varepsilon_{zz,x}^{(2)}\right)^2} - \left(w_{0,xx} + z_3\ \varepsilon_{zz,xx}^{(2)}\right) z \right], \tag{5.11.11}$$

$${}^H\sigma_{xy}^{(1)} = \overline{C}_{16}^{(1)}\varepsilon_{xx}^{(1)} + \overline{C}_{26}^{(1)}\underbrace{\varepsilon_{yy}^{(1)}}_{0} + \overline{C}_{36}^{(1)}\underbrace{\varepsilon_{zz}^{(1)}}_{0} + \overline{C}_{66}^{(1)}2\underbrace{\varepsilon_{xy}^{(1)}}_{0} =$$

$$= \overline{C}_{16}^{(1)} \left[2z_2\ \varepsilon_{xz,x}^{(2)} + \tfrac{1}{2}z_2^2\ \varepsilon_{zz,xx}^{(2)} + \tfrac{1}{2}\left(w_{0,x} + z_2\ \varepsilon_{zz,x}^{(2)}\right)^2 - \left(w_{0,xx} + z_2\ \varepsilon_{zz,xx}^{(2)}\right) z \right], \tag{5.11.12}$$

$${}^H\sigma_{xy}^{(2)} = \overline{C}_{16}^{(2)}\varepsilon_{xx}^{(2)} + \overline{C}_{26}^{(2)}\underbrace{\varepsilon_{yy}^{(2)}}_{0} + \overline{C}_{36}^{(2)}\varepsilon_{zz}^{(2)} + \overline{C}_{66}^{(2)}2\underbrace{\varepsilon_{xy}^{(2)}}_{0} =$$

$$= \overline{C}_{16}^{(2)} \left[\underbrace{\frac{1}{2}(w_{0,x})^2} + \left(2\varepsilon_{xz,x}^{(2)} - w_{0,xx} + \underbrace{w_{0,x}\varepsilon_{zz,x}^{(2)}}\right) z + \left(-\frac{1}{2}\varepsilon_{zz,xx}^{(2)} + \underbrace{\frac{1}{2}\left(\varepsilon_{zz,x}^{(2)}\right)^2}\right) z^2 \right] +$$

$$+\overline{C}_{36}^{(2)}\varepsilon_{zz}^{(2)}, \tag{5.11.13}$$

$${}^H\sigma_{xy}^{(3)} = \overline{C}_{16}^{(3)}\varepsilon_{xx}^{(3)} + \overline{C}_{26}^{(3)}\underbrace{\varepsilon_{yy}^{(3)}}_{0} + \overline{C}_{36}^{(3)}\underbrace{\varepsilon_{zz}^{(3)}}_{0} + \overline{C}_{66}^{(3)}2\underbrace{\varepsilon_{xy}^{(3)}}_{0} =$$

$$= \overline{C}_{16}^{(3)} \left[2z_3\ \varepsilon_{xz,x}^{(2)} + \tfrac{1}{2}z_3^2\ \varepsilon_{zz,xx}^{(2)} + \underbrace{\frac{1}{2}\left(w_{0,x} + z_3\ \varepsilon_{zz,x}^{(2)}\right)^2} - \left(w_{0,xx} + z_3\ \varepsilon_{zz,xx}^{(2)}\right) z \right]. \tag{5.11.14}$$

Coefficients $\overline{C}_{11}^{(1)}$, $\overline{C}_{11}^{(2)}$, $\overline{C}_{13}^{(2)}$ and $\overline{C}_{11}^{(3)}$ depend on z-coordinate because they vary from ply to ply.

Now, let us find expressions for the **transverse stresses** σ_{xz} and σ_{zz} by integrating equations of motion. These expressions will be called the second forms of expressions for the transverse stresses, in contrast with the expressions for the transverse stresses that can be found from the constitutive equations.

In section 3.13 of chapter 3, the second form of expressions for the transverse stresses was found by integrating the equations of motion (3.16)–(3.18). Now, let us use these formulas to express the second form of the transverse stresses in terms of the nodal variables of a finite element, for the case of cylindrical bending of the platform.

From formulas (3.13.5), (3.13.9), (3.13.10), (3.13.15), (3.13.16) and (3.13.17), one can receive the formulas for the transverse stresses in the cylindrically bent plate by setting $\sigma_{xy} = 0$ and $\sigma_{yz} = 0$ (the nonlinear terms are underbraced):

$$\sigma_{xz}^{(1)} = \int_{z_1}^{z} \left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma_{xx,x}^{(1)}\right) dz \quad (z_1 \le z \le z_2), \tag{5.11.15}$$

$$\sigma_{xz}^{(2)} = \int_{z_1}^{z_2} \left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma_{xx,x}^{(1)}\right) dz + \int_{z_2}^{z} \left(\rho^{(2)}\ddot{u}^{(2)} - {}^{H}\sigma_{xx,x}^{(2)}\right) dz \quad (z_2 \le z \le z_3) \tag{5.11.16}$$

$$\sigma_{xz}^{(3)} = \int_{z_1}^{z_2} \left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma_{xx,x}^{(1)}\right) dz + \int_{z_2}^{z_3} \left(\rho^{(2)}\ddot{u}^{(2)} - {}^{H}\sigma_{xx,x}^{(2)}\right) dz$$
$$+ \int_{z_3}^{z} \left(\rho^{(3)}\ddot{u}^{(3)} - {}^{H}\sigma_{xx,x}^{(3)}\right) dz \quad (z_3 \le z \le z_4) \tag{5.11.17}$$

$$\sigma_{zz}^{(1)} = \underbrace{\sigma_{zz}^{(1)}(z_1)}_{sw^{(1)}(z_1)} + \int_{z_1}^{z} \left[\rho^{(1)}\left(\ddot{w}^{(1)} + g\right) - \frac{\partial}{\partial x}\left(\sigma_{xx}^{(1)} w_{,x}^{(1)}\right) - \sigma_{zx,x}^{(1)}\right] dz, \tag{5.11.18}$$

where s is modulus of elastic foundation.

$$\sigma_{zz}^{(2)} = sw^{(1)}(z_1) + \int_{z_1}^{z_2} \left[\rho^{(1)}\left(\ddot{w}^{(1)} + g\right) - \frac{\partial}{\partial x}\left(\sigma_{xx}^{(1)} w_{,x}^{(1)}\right) - \sigma_{zx,x}^{(1)}\right] dz +$$
$$+ \int_{z_2}^{z} \left[\rho^{(2)}\left(\ddot{w}^{(2)} + g\right) - \frac{\partial}{\partial x}\left(\sigma_{xx}^{(2)} w_{,x}^{(2)}\right) - \sigma_{zx,x}^{(2)}\right] dz , \tag{5.11.9}$$

$$\sigma_{zz}^{(3)} = sw^{(1)}(z_1) + \int_{z_1}^{z_2} \left[\rho^{(1)}\left(\ddot{w}^{(1)} + g\right) - \frac{\partial}{\partial x}\left(\sigma_{xx}^{(1)} w_{,x}^{(1)}\right) - \sigma_{zx,x}^{(1)}\right] dz$$

$$+ \int_{z_2}^{z_3} \left[\rho^{(2)}\left(\ddot{w}^{(2)} + g\right) - \frac{\partial}{\partial x}\left(\sigma_{xx}^{(2)} w_{,x}^{(2)}\right) - \sigma_{zx,x}^{(2)}\right] dz$$

$$+ \int_{z_3}^{z} \left[\rho^{(3)}\left(\ddot{w}^{(3)} + g\right) - \frac{\partial}{\partial x}\left(\sigma_{xx}^{(3)} w_{,x}^{(3)}\right) - \sigma_{zx,x}^{(3)}\right] dz \,. \tag{5.11.20}$$

The substitution of expression (5.2.4) for $u^{(1)}(x,z,t)$ and expression (5.11.6) for ${}^{H}\sigma_{xx}^{(1)}$ into expression (5.11.15) for $\sigma_{xz}^{(1)}$ yields:

$$\sigma_{xz}^{(1)}(x,z,t) =$$

$$= \rho^{(1)}(z - z_1)\left[\left(2\,\ddot{\varepsilon}_{xz}^{(2)} - \ddot{w}_{0,x}\right) z_2 - \frac{1}{2}\ddot{\varepsilon}_{zz,x}^{(2)}\, z_2^2 + \left(\ddot{\varepsilon}_{zz,x}^{(2)}\, z_2 + \ddot{w}_{0,x}\right)\frac{1}{2}\left(2z_2 - z_1 - z\right)\right]$$

$$- \left[\, 2z_2\, \varepsilon_{xz,xx}^{(2)} + \tfrac{1}{2} z_2^2\, \varepsilon_{zz,xxx}^{(2)} \;+ \underbrace{\left(w_{0,x} + z_2\, \varepsilon_{zz,x}^{(2)}\right)\left(w_{0,xx} + z_2\, \varepsilon_{zz,xx}^{(2)}\right)}\right] \int_{z_1}^{z} \overline{C}_{11}^{(1)}(z)\; dz +$$

$$+ \left(w_{0,xxx} + z_2\, \varepsilon_{zz,xxx}^{(2)}\right) \int_{z_1}^{z} \overline{C}_{11}^{(1)}(z)\; z\; dz. \tag{5.11.21}$$

If one substitutes equation (5.2.5) for $u^{(2)}$ and equation (5.11.7) for ${}^{H}\sigma_{xx}^{(2)}$ into equation (5.11.16) for $\sigma_{xz}^{(2)}$, one obtains:

$$\sigma_{xz}^{(2)} = I_2(x,t) + \rho^{(2)}\left[\left(2\ddot{\varepsilon}_{xz}^{(2)} - \ddot{w}_{0,x}\right)\frac{1}{2}\left(z^2 - z_2^2\right) - \frac{1}{6}\ddot{\varepsilon}_{zz,x}^{(2)}\left(z^3 - z_2^3\right)\right] - \underbrace{w_{0,x} w_{0,xx} \int_{z_2}^{z} \overline{C}_{11}^{(2)}(z)\; dz}$$

$$- \left(2\varepsilon_{xz,xx}^{(2)} - w_{0,xxx} + \underbrace{w_{0,xx}\varepsilon_{zz,x}^{(2)} + w_{0,x}\varepsilon_{zz,xx}^{(2)}}\right) \int_{z_2}^{z} \overline{C}_{11}^{(2)}(z)\; z\; dz$$

$$- \left(-\frac{1}{2}\varepsilon_{zz,xxx}^{(2)} + \underbrace{\varepsilon_{zz,x}^{(2)}\varepsilon_{zz,xx}^{(2)}}\right) \int_{z_2}^{z} \overline{C}_{11}^{(2)}(z)\; z^2\; dz - \varepsilon_{zz,x}^{(2)} \int_{z_2}^{z} \overline{C}_{13}^{(2)}(z)\; dz \,, \tag{5.11.22}$$

where

$$I_2(x,t) \equiv \left.\sigma_{xz}^{(1)}\right|_{z=z_2} =$$

$$= \rho^{(1)}(z_2 - z_1)\left[\left(2\,\ddot{\varepsilon}_{xz}^{(2)} - \ddot{w}_{0,x}\right) z_2 - \frac{1}{2}\ddot{\varepsilon}_{zz,x}^{(2)}\, z_2^2 + \left(\ddot{\varepsilon}_{zz,x}^{(2)}\, z_2 + \ddot{w}_{0,x}\right)\frac{1}{2}(z_2 - z_1)\right]$$

$$-\left[\; 2z_2\,\varepsilon_{xz,xx}^{(2)} + \tfrac{1}{2}z_2^2\,\varepsilon_{zz,xxx}^{(2)} \;+ \underbrace{\left(w_{0,x} + z_2\,\varepsilon_{zz,x}^{(2)}\right)\left(w_{0,xx} + z_2\,\varepsilon_{zz,xx}^{(2)}\right)}\right]\int_{z_1}^{z_2} \overline{C}_{11}^{(1)}(z)\; dz +$$

$$+\left(w_{0,xxx} + z_2\,\varepsilon_{zz,xxx}^{(2)}\right)\int_{z_1}^{z_2} \overline{C}_{11}^{(1)}(z)\; z\; dz. \tag{5.11.23}$$

The substitution of equation (5.2.6) for $u^{(3)}$ and equation (5.11.8) for ${}^{H}\sigma_{xx}^{(3)}$ into equation (5.11.17) for $\sigma_{xz}^{(3)}$ yields:

$$\sigma_{xz}^{(3)} = \underbrace{I_2(x,t) + I_3(x,t)}_{\sigma_{xz}^{(2)}(z_3)} +$$

$$+\rho^{(3)} z_3 (z - z_3)\left[\left(2\ddot{\varepsilon}_{xz}^{(2)} - \ddot{w}_{0,x}\right) - \frac{1}{2}\ddot{\varepsilon}_{zz,x}^{(2)} z_3\right]$$

$$-\rho^{(3)}\frac{1}{2}(z - z_3)^2\left(\ddot{w}_{0,x} + \ddot{\varepsilon}_{zz,x}^{(2)} z_3\right)$$

$$-\left[\; 2z_3\,\varepsilon_{xz,xx}^{(2)} + \tfrac{1}{2}z_3^2\,\varepsilon_{zz,xxx}^{(2)} + \underbrace{\left(w_{0,x} + z_3\,\varepsilon_{zz,x}^{(2)}\right)\left(w_{0,xx} + z_3\,\varepsilon_{zz,xx}^{(2)}\right)}\;\right]\int_{z_3}^{z} \overline{C}_{11}^{(3)}(z)\; dz +$$

$$+\left(w_{0,xxx} + z_3\,\varepsilon_{zz,xxx}^{(2)}\right)\int_{z_3}^{z} \overline{C}_{11}^{(3)}(z)\; z\; dz, \tag{5.11.24}$$

where

$$I_3(x,t) = \rho^{(2)}\left[\left(2\ddot{\varepsilon}_{xz}^{(2)} - \ddot{w}_{0,x}\right)\frac{1}{2}\left(z_3^2 - z_2^2\right) - \frac{1}{6}\ddot{\varepsilon}_{zz,x}^{(2)}\left(z_3^3 - z_2^3\right)\right] - \underbrace{w_{0,x}w_{0,xx}}\int_{z_2}^{z_3} \overline{C}_{11}^{(2)}(z)\; dz$$

$$-\left(2\varepsilon_{xz,xx}^{(2)} - w_{0,xxx} + \underbrace{w_{0,xx}\varepsilon_{zz,x}^{(2)} + w_{0,x}\varepsilon_{zz,xx}^{(2)}}\right)\int_{z_2}^{z_3} \overline{C}_{11}^{(2)}(z)\; z\; dz$$

$$-\left(-\frac{1}{2}\varepsilon_{zz,xxx}^{(2)} + \underbrace{\varepsilon_{zz,x}^{(2)}\varepsilon_{zz,xx}^{(2)}}\right)\int_{z_2}^{z_3} \overline{C}_{11}^{(2)}(z)\; z^2\; dz - \overline{C}_{13}^{(2)}\varepsilon_{zz,x}^{(2)}(z_3 - z_2). \tag{5.11.25}$$

From equations

$$\sigma_{yz}^{(1)} = \int_{z_1}^{z} \left(\rho^{(1)}\ddot{v}^{(1)} - {}^{H}\sigma_{yx,x}^{(1)} - {}^{H}\sigma_{yy,y}^{(1)}\right) dz \quad \text{(eqn. 3.3.11)}$$

where we set $v = 0$ and ${}^{H}\sigma_{yy,y}^{(1)} = 0$ (because of cylindrical bending), and from equation (5.11.12) for $\sigma_{yx}^{(1)}$, we obtain:

$$\sigma_{yz}^{(1)}(x,z,t) = -\int_{z_1}^{z} {}^{H}\sigma_{yx,x}^{(1)} dz =$$

$$= -\left[2z_2\ \varepsilon_{xz,xx}^{(2)} + \tfrac{1}{2}z_2^2\ \varepsilon_{zz,xxx}^{(2)} + \underbrace{\left(w_{0,x} + z_2\ \varepsilon_{zz,x}^{(2)}\right)\left(w_{0,xx} + z_2\ \varepsilon_{zz,xx}^{(2)}\right)}\right] \int_{z_1}^{z} \overline{C}_{16}^{(1)}(z)\ dz +$$

$$+ \left(w_{0,xxx} + z_2\ \varepsilon_{zz,xxx}^{(2)}\right) \int_{z_1}^{z} \overline{C}_{16}^{(1)}(z)\ z\ dz. \tag{5.11.26}$$

From equation

$$\sigma_{yz}^{(2)} = \int_{z_1}^{z_2} \left(\rho^{(1)}\ddot{v}^{(2)} - {}^{H}\sigma_{yx,x}^{(1)} - {}^{H}\sigma_{yy,y}^{(1)}\right) dz +$$

$$+ \int_{z_2}^{z} \left(\rho^{(2)}\ddot{v}^{(3)} - {}^{H}\sigma_{yx,x}^{(2)} - {}^{H}\sigma_{yy,y}^{(2)}\right) dz \quad \text{(eqn 3.13.12)}$$

where we set $v = 0$ and derivatives with respect to y equal to zero (because of cylindrical bending), and from equation (5.11.13) for $\sigma_{yx}^{(2)}$, we obtain:

$$\sigma_{yz}^{(2)}(x,z,t) = \underbrace{\int_{z_1}^{z_2} \left(-\ {}^{H}\sigma_{yx,x}^{(1)}\right) dz}_{H^{(2)}(x,t) \equiv \sigma_{yz}^{(1)}(z_2)} - \int_{z_2}^{z} {}^{H}\sigma_{yx,x}^{(2)}\ dz =$$

$$= H^{(2)}(x,t)\ \ - \underbrace{w_{0,x}w_{0,xx}} \int_{z_2}^{z} \overline{C}_{16}^{(2)}(z)\ dz$$

$$-\left(2\varepsilon^{(2)}_{xz,xx} - w_{0,xxx} + \underbrace{w_{0,xx}\varepsilon^{(2)}_{zz,x} + w_{0,x}\varepsilon^{(2)}_{zz,xx}}\right)\int\limits_{z_2}^{z} \overline{C}^{(2)}_{16}(z)\ z\ dz +$$

$$+\left(\frac{1}{2}\varepsilon^{(2)}_{zz,xxx} - \underbrace{\varepsilon^{(2)}_{zz,x}\ \varepsilon^{(2)}_{zz,xx}}\right)\int\limits_{z_2}^{z} \overline{C}^{(2)}_{16}(z)\ z^2\ dz - \varepsilon^{(2)}_{zz,x}\int\limits_{z_2}^{z}\overline{C}^{(2)}_{36}(z)\ dz, \qquad (5.11.27)$$

where

$$H^{(2)}(x,t) \equiv \sigma^{(1)}_{yz}\bigg|_{z=z_2} \equiv \int\limits_{z_1}^{z_2}\left(-\ {}^{H}\sigma^{(1)}_{yx,x}\right)\ dz =$$

$$= -\left[2z_2\ \varepsilon^{(2)}_{xz,xx} + \tfrac{1}{2}z_2^2\ \varepsilon^{(2)}_{zz,xxx} + \underbrace{\left(w_{0,x} + z_2\ \varepsilon^{(2)}_{zz,x}\right)\left(w_{0,xx} + z_2\ \varepsilon^{(2)}_{zz,xx}\right)}\right]\int\limits_{z_1}^{z_2}\overline{C}^{(1)}_{16}(z)\ dz +$$

$$+\left(w_{0,xxx} + z_2\ \varepsilon^{(2)}_{zz,xxx}\right)\int\limits_{z_1}^{z_2}\overline{C}^{(1)}_{16}(z)\ z\ dz. \qquad (5.11.28)$$

Analogously, from equation (3.3.13), where we set $v = 0$ and the derivatives with respect to y equal to zero (because of cylindrical bending), and from equation (5.11.14) for $\sigma^{(3)}_{yx}$, we obtain:

$$\sigma^{(3)}_{yz}(x,z,t) = \overbrace{\underbrace{\int\limits_{z_1}^{z_2}\left(-\ {}^{H}\sigma^{(1)}_{yx,x}\right)\ dz}_{H^{(2)}(x,t)} + \underbrace{\int\limits_{z_2}^{z_3}\left(-\ {}^{H}\sigma^{(2)}_{yx,x}\right)\ dz}_{H^{(3)}(x,t)}}^{\sigma^{(2)}_{yz}(z_3)} + \int\limits_{z_3}^{z}\left(-\ {}^{H}\sigma^{(3)}_{yx,x}\right)\ dz =$$

$$= H^{(2)}(x,t) + H^{(3)}(x,t)$$

$$-\left[\ 2z_3\ \varepsilon^{(2)}_{xz,xx} + \tfrac{1}{2}z_3^2\ \varepsilon^{(2)}_{zz,xxx} + \underbrace{\left(w_{0,x} + z_3\ \varepsilon^{(2)}_{zz,x}\right)\left(w_{0,xx} + z_3\ \varepsilon^{(2)}_{zz,xx}\right)}\ \right]\int\limits_{z_3}^{z}\overline{C}^{(3)}_{16}(z)\,dz +$$

$$+\left(w_{0,xxx} + z_3\ \varepsilon^{(2)}_{zz,xxx}\right)\int\limits_{z_3}^{z}\overline{C}^{(3)}_{16}(z)\ z\ dz, \qquad (5.11.29)$$

where

$$H^{(3)}(x,t) = -\underbrace{w_{0,x}w_{0,xx}}\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{16}(z)\ dz$$

$$
-\left(2\varepsilon^{(2)}_{xz,xx}-w_{0,xxx}+\underbrace{w_{0,xx}\varepsilon^{(2)}_{zz,x}+w_{0,x}\varepsilon^{(2)}_{zz,xx}}\right)\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{16}(z)\ z\ dz+
$$

$$
+\left(\frac{1}{2}\varepsilon^{(2)}_{zz,xxx}-\underbrace{\varepsilon^{(2)}_{zz,x}\ \varepsilon^{(2)}_{zz,xx}}\right)\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{16}(z)\ z^2\ dz-\varepsilon^{(2)}_{zz,x}\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{36}(z)\ dz. \qquad (5.11.30)
$$

If one substitutes equation

$$
w^{(1)}(x,t)=w_0(x,t)+\varepsilon^{(2)}_{zz}(x,t)\ z_2 \qquad \text{(eqn 5.2.1)}
$$

into equation (5.11.6) for $\sigma^{(1)}_{xx}$, and equation (5.11.15) for $\sigma^{(1)}_{xz}$ into equation (5.11.18) for $\sigma^{(1)}_{zz}$, one can receive:

$$
\sigma^{(1)}_{zz}(x,z,t)=s\left[w_0+\varepsilon^{(2)}_{zz}\ z_2\ \right]+\rho^{(1)}\left[\ddot{w}_0+\ddot{\varepsilon}^{(2)}_{zz}z_2+g\right](z-z_1)
$$

$$
\underbrace{-\left[2z_2\ \varepsilon^{(2)}_{xz,xx}+\frac{1}{2}z_2^2\ \varepsilon^{(2)}_{zz,xxx}+\left(w_{0,x}+z_2\ \varepsilon^{(2)}_{zz,x}\right)\left(w_{0,xx}+z_2\ \varepsilon^{(2)}_{zz,xx}\right)\right]\left(w_{0,x}+\varepsilon^{(2)}_{zz,x}\ z_2\right)\int\limits_{z_1}^{z}\overline{C}^{(1)}_{11}(z)\ dz}
$$

$$
\underbrace{+\left(w_{0,xxx}+z_2\ \varepsilon^{(2)}_{zz,xxx}\right)\left(w_{0,x}+\varepsilon^{(2)}_{zz,x}\ z_2\right)\int\limits_{z_1}^{z}\overline{C}^{(1)}_{11}(z)\ z\ dz}
$$

$$
\underbrace{-\ \left[2z_2\ \varepsilon^{(2)}_{xz,x}+\frac{1}{2}z_2^2\ \varepsilon^{(2)}_{zz,xx}+\frac{1}{2}\left(w_{0,x}+z_2\ \varepsilon^{(2)}_{zz,x}\right)^2\right]\left(w_{0,xx}+\varepsilon^{(2)}_{zz,xx}\ z_2\right)\int\limits_{z_1}^{z}\overline{C}^{(1)}_{11}(z)\ dz}
$$

$$
\underbrace{+\left(w_{0,xx}+z_2\ \varepsilon^{(2)}_{zz,xx}\right)\left(w_{0,xx}+\varepsilon^{(2)}_{zz,xx}\ z_2\right)\int\limits_{z_1}^{z}\overline{C}^{(1)}_{11}(z)\ z\ dz}
$$

$$
-\rho^{(1)}\frac{1}{2}(z_1-z)^2\left[\left(2\ \ddot{\varepsilon}^{(2)}_{xz,x}-\ddot{w}_{0,xx}\right)z_2-\frac{1}{2}\ddot{\varepsilon}^{(2)}_{zz,xx}\ z_2^2-\frac{1}{3}(2z_1+z-3z_2)\left(\ddot{\varepsilon}^{(2)}_{zz,xx}\ z_2+\ddot{w}_{0,xx}\right)\right]
$$

$$
+\left[\ 2z_2\ \varepsilon^{(2)}_{xz,xxx}+\tfrac{1}{2}z_2^2\ \varepsilon^{(2)}_{zz,xxxx}\ \ +\underbrace{\left(w_{0,x}+z_2\ \varepsilon^{(2)}_{zz,xx}\right)\left(w_{0,xxx}+z_2\ \varepsilon^{(2)}_{zz,xxx}\right)}\right]\int\limits_{z_1}^{z}\int\limits_{z_1}^{z}\overline{C}^{(1)}_{11}(z)\ dz\ dz
$$

$$-\left(w_{0,xxxx}+z_2\,\varepsilon^{(2)}_{zz,xxxx}\right)\int\limits_{z_1}^{z}\int\limits_{z_1}^{z}\overline{C}^{(1)}_{11}(z)\;z\;dz\;dz. \tag{5.11.31}$$

From equation (5.11.19) we receive:

$$\sigma^{(2)}_{zz}(x,z,t)=J_2(x,t)+\rho^{(2)}(z-z_2)\left[\frac{1}{2}(z+z_2)\,\ddot{\varepsilon}^{(2)}_{zz}+\ddot{w}_0+g\right]$$

$$\underbrace{-\left(w_{0,x}\right)^2 w_{0,xx}\int\limits_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\;dz}$$

$$\underbrace{-\,w_{0,x}\left(2\varepsilon^{(2)}_{xz,xx}-w_{0,xxx}+w_{0,xx}\varepsilon^{(2)}_{zz,x}+w_{0,x}\varepsilon^{(2)}_{zz,xx}\right)\int\limits_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\;z\;dz}$$

$$\underbrace{-\,w_{0,x}\left(-\frac{1}{2}\varepsilon^{(2)}_{zz,xxx}+\varepsilon^{(2)}_{zz,x}\;\varepsilon^{(2)}_{zz,xx}\right)\int\limits_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\;z^2\;dz}$$

$$\underbrace{-\;\varepsilon^{(2)}_{zz,x}w_{0,x}w_{0,xx}\int\limits_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\;z\;dz}$$

$$\underbrace{-\,\varepsilon^{(2)}_{zz,x}\left(2\varepsilon^{(2)}_{xz,xx}-w_{0,xxx}+w_{0,xx}\varepsilon^{(2)}_{zz,x}+w_{0,x}\varepsilon^{(2)}_{zz,xx}\right)\int\limits_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\;z^2\;dz}$$

$$\underbrace{-\,\varepsilon^{(2)}_{zz,x}\left(-\frac{1}{2}\varepsilon^{(2)}_{zz,xxx}+\varepsilon^{(2)}_{zz,x}\;\varepsilon^{(2)}_{zz,xx}\right)\int\limits_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\;z^3\;dz}$$

$$\underbrace{-\,w_{0,x}\;\varepsilon^{(2)}_{zz,x}\int\limits_{z_2}^{z}\overline{C}^{(2)}_{13}(z)\;dz+\varepsilon^{(2)}_{zz,x}\;\varepsilon^{(2)}_{zz,x}\int\limits_{z_2}^{z}\overline{C}^{(2)}_{13}(z)\;z\;dz}$$

$$\underbrace{-\frac{1}{2}(w_{0,x})^2 w_{0,xx}\int\limits_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\;dz+\frac{1}{2}(w_{0,x})^2\varepsilon^{(2)}_{zz,xx}\int\limits_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\;z\;dz}$$

$$-\underbrace{\left(2\varepsilon^{(2)}_{xz,x} - w_{0,xx} + w_{0,x}\varepsilon^{(2)}_{zz,x}\right) w_{0,xx}\int_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\; z\;dz}_{}$$

$$-\underbrace{\left(2\varepsilon^{(2)}_{xz,x} - w_{0,xx} + w_{0,x}\varepsilon^{(2)}_{zz,x}\right) \varepsilon^{(2)}_{zz,xx}\int_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\; z^2\;dz}_{}$$

$$-\underbrace{\left[-\frac{1}{2}\varepsilon^{(2)}_{zz,xx} + \frac{1}{2}\left(\varepsilon^{(2)}_{zz,x}\right)^2\right] w_{0,xx}\int_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\; z^2\;dz}_{}$$

$$-\underbrace{\left[-\frac{1}{2}\varepsilon^{(2)}_{zz,xx} + \frac{1}{2}\left(\varepsilon^{(2)}_{zz,x}\right)^2\right] \varepsilon^{(2)}_{zz,xx}\int_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\; z^3\;dz}_{}$$

$$\underbrace{-\varepsilon^{(2)}_{zz} w_{0,xx}\int_{z_2}^{z}\overline{C}^{(2)}_{13}(z)\;dz + \varepsilon^{(2)}_{zz}\varepsilon^{(2)}_{zz,xx}\int_{z_2}^{z}\overline{C}^{(2)}_{13}(z)\; z\;dz}_{}$$

$$-\frac{\partial I_2(x,t)}{\partial x}(z - z_2)$$

$$-\rho^{(2)}\left[\left(2\ddot{\varepsilon}^{(2)}_{xz,x} - \ddot{w}_{0,xx}\right)(z + 2z_2) - \ddot{\varepsilon}^{(2)}_{zz,xx}\left(z^2 + 2zz_2 + 3z_2^2\right)\frac{1}{4}\right]\frac{1}{6}(z - z_2)^2$$

$$+\underbrace{\left(\left(w_{0,xx}\right)^2 + w_{0,x}w_{0,xxx}\right)}_{}\int_{z_2}^{z}\int_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\;dz\;dz$$

$$+\left[2\varepsilon^{(2)}_{xz,xxx} - w_{0,xxxx} + \underbrace{w_{0,xxx}\varepsilon^{(2)}_{zz,x} + 2w_{0,xx}\varepsilon^{(2)}_{zz,xx} + w_{0,x}\varepsilon^{(2)}_{zz,xxx}}_{}\right]\int_{z_2}^{z}\int_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\; z\;dz\;dz$$

$$+\left(-\frac{1}{2}\varepsilon^{(2)}_{zz,xxxx} + \underbrace{\varepsilon^{(2)}_{zz,xx}\varepsilon^{(2)}_{zz,xx} + \varepsilon^{(2)}_{zz,x}\varepsilon^{(2)}_{zz,xxx}}_{}\right)\int_{z_2}^{z}\int_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\; z^2\;dz\;dz$$

$$-\varepsilon^{(2)}_{zz,xx}\int\limits_{z_2}^{z}\int\limits_{z_2}^{z}\overline{C}^{(2)}_{13}\left(z\right)\ dz\ dz, \tag{5.11.32}$$

where

$$J_2\left(x,t\right)\equiv\sigma^{(1)}_{zz}(z_2)=$$

$$=s\left[w_0+\varepsilon^{(2)}_{zz}\ z_2\ \right]+\rho^{(1)}\left[\ddot{w}_0+\ddot{\varepsilon}^{(2)}_{zz}z_2+g\right]\left(z_2-z_1\right)$$

$$-\underbrace{\left[2z_2\ \varepsilon^{(2)}_{xz,xx}+\frac{1}{2}z_2^2\ \varepsilon^{(2)}_{zz,xxx}+\left(w_{0,x}+z_2\ \varepsilon^{(2)}_{zz,x}\right)\left(w_{0,xx}+z_2\ \varepsilon^{(2)}_{zz,xx}\right)\right]\left(w_{0,x}+\varepsilon^{(2)}_{zz,x}\ z_2\right)\int\limits_{z_1}^{z_2}\overline{C}^{(1)}_{11}\left(z\right)\ dz}$$

$$+\underbrace{\left(w_{0,xxx}+z_2\ \varepsilon^{(2)}_{zz,xxx}\right)\left(w_{0,x}+\varepsilon^{(2)}_{zz,x}\ z_2\right)\int\limits_{z_1}^{z_2}\overline{C}^{(1)}_{11}\left(z\right)\ z\ dz}$$

$$-\ \underbrace{\left[2z_2\ \varepsilon^{(2)}_{xz,x}+\frac{1}{2}z_2^2\ \varepsilon^{(2)}_{zz,xx}+\frac{1}{2}\left(w_{0,x}+z_2\ \varepsilon^{(2)}_{zz,x}\right)^2\right]\left(w_{0,xx}+\varepsilon^{(2)}_{zz,xx}\ z_2\right)\int\limits_{z_1}^{z_2}\overline{C}^{(1)}_{11}\left(z\right)\ dz}$$

$$+\underbrace{\left(w_{0,xx}+z_2\ \varepsilon^{(2)}_{zz,xx}\right)\left(w_{0,xx}+\varepsilon^{(2)}_{zz,xx}\ z_2\right)\int\limits_{z_1}^{z_2}\overline{C}^{(1)}_{11}\left(z\right)\ z\ dz}$$

$$-\rho^{(1)}\frac{1}{2}\left(z_1-z_2\right)^2\left[\left(2\ \ddot{\varepsilon}^{(2)}_{xz,x}-\ddot{w}_{0,xx}\right)z_2-\frac{1}{2}\ddot{\varepsilon}^{(2)}_{zz,xx}\ z_2^2-\frac{1}{3}\left(2z_1-2z_2\right)\left(\ddot{\varepsilon}^{(2)}_{zz,xx}\ z_2+\ddot{w}_{0,xx}\right)\right]$$

$$+\left[2z_2\ \varepsilon^{(2)}_{xz,xxx}+\frac{1}{2}z_2^2\ \varepsilon^{(2)}_{zz,xxxx}+\underbrace{\left(w_{0,x}+z_2\ \varepsilon^{(2)}_{zz,xx}\right)\left(w_{0,xxx}+z_2\ \varepsilon^{(2)}_{zz,xxx}\right)}\right]\int\limits_{z_1}^{z_2}\int\limits_{z_1}^{z}\overline{C}^{(1)}_{11}\left(z\right)\ dz\ dz$$

$$-\left(w_{0,xxxx}+z_2\ \varepsilon^{(2)}_{zz,xxxx}\right)\int\limits_{z_1}^{z_2}\int\limits_{z_1}^{z}\overline{C}^{(1)}_{11}\left(z\right)\ z\ dz\ dz \tag{5.11.33}$$

and

$$\frac{\partial I_2\left(x,t\right)}{\partial x}=$$

$$= \rho^{(1)}(z_2 - z_1)\left[\left(2\,\ddot{\varepsilon}^{(2)}_{xz,x} - \ddot{w}_{0,xx}\right) z_2 - \frac{1}{2}\ddot{\varepsilon}^{(2)}_{zz,xx}\; z_2^2 + \left(\ddot{\varepsilon}^{(2)}_{zz,xx}\; z_2 + \ddot{w}_{0,xx}\right)\frac{1}{2}(z_2 - z_1)\right]$$

$$-\ \left(2z_2\ \varepsilon^{(2)}_{xz,xxx} + \tfrac{1}{2}z_2^2\ \varepsilon^{(2)}_{zz,xxxx}\right)\int\limits_{z_1}^{z_2}\overline{C}^{(1)}_{11}(z)\ dz$$

$$-\underbrace{\left[\left(w_{0,xx} + z_2\ \varepsilon^{(2)}_{zz,xx}\right)\left(w_{0,xx} + z_2\ \varepsilon^{(2)}_{zz,xx}\right) + \left(w_{0,x} + z_2\ \varepsilon^{(2)}_{zz,x}\right)\left(w_{0,xxx} + z_2\ \varepsilon^{(2)}_{zz,xxx}\right)\right]}\int\limits_{z_1}^{z_2}\overline{C}^{(1)}_{11}(z)\ dz +$$

$$+\left(w_{0,xxxx} + z_2\ \varepsilon^{(2)}_{zz,xxxx}\right)\int\limits_{z_1}^{z_2}\overline{C}^{(1)}_{11}(z)\ z\ dz. \qquad (5.11.34)$$

From formula (5.11.20) we receive:

$$\sigma^{(3)}_{zz}(x, z, t) = J_2(x, t) + J_3(x, t) +$$

$$+\rho^{(3)}\left(\ddot{w}_0 + \ddot{\varepsilon}^{(2)}_{zz} z_3 + g\right)(z - z_3)$$

$$-\underbrace{\left(2z_3\ \varepsilon^{(2)}_{xz,xx} + \frac{1}{2}z_3^2\ \varepsilon^{(2)}_{zz,xxx}\right)\left(w_{0,x} + \varepsilon^{(2)}_{zz,x}\ z_3\right)\int\limits_{z_3}^{z}\overline{C}^{(3)}_{11}\ dz}$$

$$-\underbrace{\left(w_{0,x} + z_3\ \varepsilon^{(2)}_{zz,x}\right)\left(w_{0,xx} + z_3\ \varepsilon^{(2)}_{zz,xx}\right)\left(w_{0,x} + \varepsilon^{(2)}_{zz,x}\ z_3\right)\int\limits_{z_3}^{z}\overline{C}^{(3)}_{11}\ dz}$$

$$+\underbrace{\left(w_{0,xxx} + z_3\ \varepsilon^{(2)}_{zz,xxx}\right)\ \left(w_{0,x} + \varepsilon^{(2)}_{zz,x}\ z_3\right)\int\limits_{z_3}^{z}\overline{C}^{(3)}_{11} z\ dz}$$

$$-\underbrace{\left(2z_3\ \varepsilon^{(2)}_{xz,x} + \frac{1}{2}z_3^2\ \varepsilon^{(2)}_{zz,xx}\right)\left(w_{0,xx} + \varepsilon^{(2)}_{zz,xx}\ z_3\right)\int\limits_{z_3}^{z}\overline{C}^{(3)}_{11}\ dz}$$

$$-\underbrace{\frac{1}{2}\left(w_{0,x} + z_3\ \varepsilon^{(2)}_{zz,x}\right)^2\left(w_{0,xx} + \varepsilon^{(2)}_{zz,xx}\ z_3\right)\int\limits_{z_3}^{z}\overline{C}^{(3)}_{11}\ dz}$$

$$+\underbrace{\left(w_{0,xx}+z_3\ \varepsilon^{(2)}_{zz,xx}\right)^2\int\limits_{z_3}^{z}\overline{C}^{(3)}_{11}\,z\ dz}$$

$$-\left(\frac{\partial\ I_2\left(x,t\right)}{\partial x}+\frac{\partial\ I_3\left(x,t\right)}{\partial x}\right)\left(z-z_3\right)$$

$$-\rho^{(3)}\frac{1}{2}z_3\left(z-z_3\right)^2\left[\left(2\ddot{\varepsilon}^{(2)}_{xz,x}-\ddot{w}_{0,xx}\right)-\frac{1}{2}\ddot{\varepsilon}^{(2)}_{zz,xx}z_3\right]$$

$$+\rho^{(3)}\frac{1}{6}\left(z-z_3\right)^3\left(\ddot{w}_{0,xx}+\ddot{\varepsilon}^{(2)}_{zz,xx}z_3\right)$$

$$+\left(2z_3\ \varepsilon^{(2)}_{xz,xxx}+\frac{1}{2}z_3^2\ \varepsilon^{(2)}_{zz,xxxx}\right)\int\limits_{z_3}^{z}\int\limits_{z_3}^{z}\overline{C}^{(3)}_{11}\left(z\right)\ dz\ dz$$

$$+\underbrace{\left(w_{0,xx}+z_3\ \varepsilon^{(2)}_{zz,xx}\right)\left(w_{0,xx}+z_3\ \varepsilon^{(2)}_{zz,xx}\right)\int\limits_{z_3}^{z}\overline{C}^{(3)}_{11}\left(z\right)\ dz}$$

$$+\underbrace{\left(w_{0,x}+z_3\ \varepsilon^{(2)}_{zz,x}\right)\left(w_{0,xxx}+z_3\ \varepsilon^{(2)}_{zz,xxx}\right)\int\limits_{z_3}^{z}\int\limits_{z_3}^{z}\overline{C}^{(3)}_{11}\left(z\right)\ dz\ dz}$$

$$-\left(w_{0,xxxx}+z_3\ \varepsilon^{(2)}_{zz,xxxx}\right)\int\limits_{z_3}^{z}\int\limits_{z_3}^{z}\overline{C}^{(3)}_{11}\left(z\right)\ z\ dz\ dz, \tag{5.11.35}$$

where

$$J_3\left(x,t\right)=$$

$$=\rho^{(2)}\left(z_3-z_2\right)\left[\frac{1}{2}\left(z_3+z_2\right)\ddot{\varepsilon}^{(2)}_{zz}+\ddot{w}_0+g\right]$$

$$\underbrace{-\left(w_{0,x}\right)^2w_{0,xx}\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{11}\left(z\right)\ dz}$$

$$\underbrace{-\,w_{0,x}\left(2\varepsilon^{(2)}_{xz,xx}-w_{0,xxx}+w_{0,xx}\varepsilon^{(2)}_{zz,x}+w_{0,x}\varepsilon^{(2)}_{zz,xx}\right)\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{11}\left(z\right)\;z\;dz}$$

$$\underbrace{-\,w_{0,x}\left(-\frac{1}{2}\varepsilon^{(2)}_{zz,xxx}+\varepsilon^{(2)}_{zz,x}\;\varepsilon^{(2)}_{zz,xx}\right)\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{11}\left(z\right)\;z^2\;dz}$$

$$\underbrace{-\;\varepsilon^{(2)}_{zz,x}w_{0,x}w_{0,xx}\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{11}\left(z\right)\;z\;dz}$$

$$\underbrace{-\,\varepsilon^{(2)}_{zz,x}\left(2\varepsilon^{(2)}_{xz,xx}-w_{0,xxx}+w_{0,xx}\varepsilon^{(2)}_{zz,x}+w_{0,x}\varepsilon^{(2)}_{zz,xx}\right)\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{11}\left(z\right)\;z^2\;dz}$$

$$\underbrace{-\,\varepsilon^{(2)}_{zz,x}\left(-\frac{1}{2}\varepsilon^{(2)}_{zz,xxx}+\varepsilon^{(2)}_{zz,x}\;\varepsilon^{(2)}_{zz,xx}\right)\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{11}\left(z\right)\;z^3\;dz}$$

$$\underbrace{-\,w_{0,x}\;\varepsilon^{(2)}_{zz,x}\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{13}\left(z\right)\;dz+\varepsilon^{(2)}_{zz,x}\;\varepsilon^{(2)}_{zz,x}\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{13}\left(z\right)\;z\;dz}$$

$$\underbrace{-\frac{1}{2}(w_{0,x})^2w_{0,xx}\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{11}\left(z\right)\;dz+\frac{1}{2}(w_{0,x})^2\varepsilon^{(2)}_{zz,xx}\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{11}\left(z\right)\;z\;dz}$$

$$\underbrace{-\left(2\varepsilon^{(2)}_{xz,x}-w_{0,xx}+w_{0,x}\varepsilon^{(2)}_{zz,x}\right)w_{0,xx}\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{11}\left(z\right)\;z\;dz}$$

$$\underbrace{-\left(2\varepsilon^{(2)}_{xz,x}-w_{0,xx}+w_{0,x}\varepsilon^{(2)}_{zz,x}\right)\varepsilon^{(2)}_{zz,xx}\int\limits_{z_2}^{z_3}\overline{C}^{(2)}_{11}\left(z\right)\;z^2\;dz}$$

$$-\underbrace{\left[-\frac{1}{2}\varepsilon^{(2)}_{zz,xx}+\frac{1}{2}\left(\varepsilon^{(2)}_{zz,x}\right)^2\right]w_{0,xx}\int_{z_2}^{z_3}\overline{C}^{(2)}_{11}(z)\ z^2\ dz}$$

$$\underbrace{-\left[-\frac{1}{2}\varepsilon^{(2)}_{zz,xx}+\frac{1}{2}\left(\varepsilon^{(2)}_{zz,x}\right)^2\right]\varepsilon^{(2)}_{zz,xx}\int_{z_2}^{z_3}\overline{C}^{(2)}_{11}(z)\ z^3\ dz}$$

$$\underbrace{-\varepsilon^{(2)}_{zz}w_{0,xx}\int_{z_2}^{z_3}\overline{C}^{(2)}_{13}(z)\ dz+\varepsilon^{(2)}_{zz}\varepsilon^{(2)}_{zz,xx}\int_{z_2}^{z_3}\overline{C}^{(2)}_{13}(z)\ z\ dz}$$

$$-\frac{\partial I_2(x,t)}{\partial x}(z_3-z_2)$$

$$-\rho^{(2)}\left[\left(2\ddot{\varepsilon}^{(2)}_{xz,x}-\ddot{w}_{0,xx}\right)(z_3+2z_2)-\ddot{\varepsilon}^{(2)}_{zz,xx}\left(z^2+2zz_2+3z_2^2\right)\frac{1}{4}\right]\frac{1}{6}(z_3-z_2)^2$$

$$+\underbrace{\left(\left(w_{0,xx}\right)^2+w_{0,x}w_{0,xxx}\right)\int_{z_2}^{z_3}\int_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\ dz\ dz}$$

$$+\left[2\varepsilon^{(2)}_{xz,xxx}-w_{0,xxxx}+\underbrace{w_{0,xxx}\varepsilon^{(2)}_{zz,x}+2w_{0,xx}\varepsilon^{(2)}_{zz,xx}+w_{0,x}\varepsilon^{(2)}_{zz,xxx}}\right]\int_{z_2}^{z_3}\int_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\ z\ dz\ dz$$

$$+\left(-\frac{1}{2}\varepsilon^{(2)}_{zz,xxxx}+\underbrace{\varepsilon^{(2)}_{zz,xx}\varepsilon^{(2)}_{zz,xx}+\varepsilon^{(2)}_{zz,x}\varepsilon^{(2)}_{zz,xxx}}\right)\int_{z_2}^{z_3}\int_{z_2}^{z}\overline{C}^{(2)}_{11}(z)\ z^2\ dz\ dz$$

$$-\varepsilon^{(2)}_{zz,xx}\int_{z_2}^{z_3}\int_{z_2}^{z}\overline{C}^{(2)}_{13}(z)\ dz\ dz, \tag{5.11.36}$$

and

$$\frac{\partial\ I_3(x,t)}{\partial x}=$$

$$= \rho^{(2)} \left[\left(2\ddot{\varepsilon}^{(2)}_{xz,x} - \ddot{w}_{0,xx} \right) \frac{1}{2} \left(z_3^2 - z_2^2 \right) - \frac{1}{6} \ddot{\varepsilon}^{(2)}_{zz,xx} \left(z_3^3 - z_2^3 \right) \right]$$

$$- \underbrace{\left(w_{0,xx} w_{0,xx} + w_{0,x} w_{0,xxx} \right)}_{} \int_{z_2}^{z_3} \overline{C}^{(2)}_{11} (z) \; dz$$

$$- \left[2\varepsilon^{(2)}_{xz,xxx} - w_{0,xxxx} + \underbrace{w_{0,xxx}\varepsilon^{(2)}_{zz,x} + 2w_{0,xx}\varepsilon^{(2)}_{zz,xx} + w_{0,x}\varepsilon^{(2)}_{zz,xxx}}_{} \right] \int_{z_2}^{z_3} \overline{C}^{(2)}_{11} (z) \; z \; dz$$

$$- \left(-\frac{1}{2}\varepsilon^{(2)}_{zz,xxxx} + \underbrace{\varepsilon^{(2)}_{zz,xx}\varepsilon^{(2)}_{zz,xx} + \varepsilon^{(2)}_{zz,x}\varepsilon^{(2)}_{zz,xxx}}_{} \right) \int_{z_2}^{z_3} \overline{C}^{(2)}_{11} (z) \; z^2 \; dz$$

$$-\overline{C}^{(2)}_{13} \varepsilon^{(2)}_{zz,xx} \left(z_3 - z_2 \right). \tag{5.11.37}$$

5.11.2 Computation of spacial derivatives of the field variables

The formulas for the stresses contain derivatives of the field variables w_0, $\varepsilon^{(2)}_{xz}$, $\varepsilon^{(2)}_{zz}$. In the finite element formulation the functions w_0 and $\varepsilon^{(2)}_{zz}$ are approximated by the Hermit interpolation polynomials of the third degree, and the function $\varepsilon^{(2)}_{xz}$ is approximated by the Lagrange polynomial of the first degree. Therefore, the derivatives $\frac{\partial w_0}{\partial x}$, $\frac{\partial^2 w_0}{\partial x^2}$, $\frac{\partial^3 w_0}{\partial x^3}$, $\frac{\partial \varepsilon^{(2)}_{zz}}{\partial x}$, $\frac{\partial^2 \varepsilon^{(2)}_{zz}}{\partial x^2}$, $\frac{\partial^3 \varepsilon^{(2)}_{zz}}{\partial x^3}$, $\frac{\partial \varepsilon^{(2)}_{xz}}{\partial x}$ can be and will be computed as the derivatives of the interpolation polynomials that were used for the finite element formulation.

The values of w_0, $\frac{\partial w_0}{\partial x}$, $\varepsilon^{(2)}_{zz}$, $\frac{\partial \varepsilon^{(2)}_{zz}}{\partial x}$ and $\varepsilon^{(2)}_{xz}$ are most accurate at the nodes (because these variables are carried as nodal variables), and they can be taken directly from the finite element solution. Let A_i and A_{i+1} be the nodal points of the $i-$th element. Then the average (over the element) value of $\frac{\partial w_0}{\partial x}$, that is used to compute an average stress in the element, will be computed as

$$\overline{\frac{\partial w_0}{\partial x}} = \frac{1}{2} \left[\frac{\partial w_0}{\partial x} (A_i) + \frac{\partial w_0}{\partial x} (A_{i+1}) \right] = \frac{1}{2} \left[w_0' (0) + w_0' (l) \right] = \frac{1}{2} (d_2 + d_4). \tag{5.11.38}$$

According to the more convenient numbering scheme of the element degrees of freedom introduced in equations (5.10.84), $d_2 = \theta_2$, $d_4 = \theta_7$. Therefore,

$$\overline{\frac{\partial w_0}{\partial x}} = \frac{1}{2} (\theta_2 + \theta_7). \tag{5.11.39}$$

Similarly, the average (over the element) values of $\varepsilon_{zz}^{(2)}$, $\frac{\partial \varepsilon_{zz}^{(2)}}{\partial x}$ and $\varepsilon_{xz}^{(2)}$ are

$$\overline{\varepsilon_{zz}^{(2)}} = \frac{1}{2}\left[\varepsilon_{zz}^{(2)}(0) + \varepsilon_{zz}^{(2)}(l)\right] = \frac{1}{2}(\theta_4 + \theta_9), \tag{5.11.40}$$

$$\overline{\frac{\partial \varepsilon_{zz}^{(2)}}{\partial x}} = \frac{1}{2}\left[\frac{\partial \varepsilon_{zz}^{(2)}}{\partial x}(0) + \frac{\partial \varepsilon_{zz}^{(2)}}{\partial x}(l)\right] = \frac{1}{2}(\theta_5 + \theta_{10}), \tag{5.11.41}$$

$$\overline{\varepsilon_{xz}^{(2)}} = \frac{1}{2}\left[\varepsilon_{xz}^{(2)}(0) + \varepsilon_{xz}^{(2)}(l)\right] = \frac{1}{2}(\theta_3 + \theta_8), \tag{5.11.42}$$

The second derivatives $\frac{\partial^2 w_0}{\partial x^2}$ and $\frac{\partial^2 \varepsilon_{zz}^{(2)}}{\partial x^2}$ will be computed at the Gauss points , whose coordinates in the local (element) coordinate system are $\overline{x}_1 = \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right) l$ and $\overline{x}_2 = \left(\frac{1}{2} - \frac{1}{6}\sqrt{3}\right) l$, because at the Gauss points (and maybe at some other points too), the derivatives $\frac{\partial^2 w_0}{\partial x^2}$ and $\frac{\partial^2 \varepsilon_{zz}}{\partial x^2}$, computed from interpolation polynomials used in the FE formulation, are most accurate (explanation of that is in Appendix 5-D). Then, the average (over the element) value of $\frac{\partial^2 w_0}{\partial x^2}$, that is used to compute an average stress in the element, will be computed as

$$\overline{\frac{\partial^2 w_0}{\partial x^2}} = \frac{1}{2}\left[\frac{\partial^2 w_0}{\partial x^2}(\overline{x}_1) + \frac{\partial^2 w_0}{\partial x^2}(\overline{x}_2)\right]. \tag{5.11.43}$$

In the finite element formulation, the following polynomial approximation of the function w_0 was used:

$$w_0 = \left\{\begin{array}{c} 1 - \frac{3\overline{x}^2}{l^2} + \frac{2\overline{x}^3}{l^3} \\ x - \frac{2\overline{x}^2}{l} + \frac{\overline{x}^3}{l^2} \\ \frac{3\overline{x}^2}{l^2} - \frac{2\overline{x}^3}{l^3} \\ -\frac{\overline{x}^2}{l} + \frac{\overline{x}^3}{l^2} \end{array}\right\}^T \left\{\begin{array}{c} w_0(0) \\ \frac{dw_0}{d\overline{x}}(0) \\ w_0(l) \\ \frac{dw_0}{dx}(l) \end{array}\right\} \quad \text{(eqn 5.10.14).}$$

From the last equation we obtain

$$\frac{\partial^2 w}{\partial x^2} = \left\{\begin{array}{c} -\frac{6}{l^2} + 12\frac{\overline{x}}{l^3} \\ -\frac{4}{l} + 6\frac{\overline{x}}{l^2} \\ \frac{6}{l^2} - 12\frac{\overline{x}}{l^3} \\ -\frac{2}{l} + 6\frac{\overline{x}}{l^2} \end{array}\right\}^T \left\{\begin{array}{c} w_0(0) \\ \frac{dw_0}{d\overline{x}}(0) \\ w_0(l) \\ \frac{dw_0}{dx}(l) \end{array}\right\} = \left\{\begin{array}{c} -\frac{6}{l^2} + 12\frac{\overline{x}}{l^3} \\ -\frac{4}{l} + 6\frac{\overline{x}}{l^2} \\ \frac{6}{l^2} - 12\frac{\overline{x}}{l^3} \\ -\frac{2}{l} + 6\frac{\overline{x}}{l^2} \end{array}\right\}^T \left\{\begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_6 \\ \theta_7 \end{array}\right\}, \tag{5.11.44}$$

and, therefore, at the Gauss points $\overline{x}_1 = \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) l$ and $\overline{x}_2 = \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right) l$ we have

$$\frac{\partial^2 w}{\partial x^2}(\overline{x}_1) = \left\{ \begin{array}{c} -2\frac{\sqrt{3}}{l^2} \\ -\frac{1+\sqrt{3}}{l} \\ 2\frac{\sqrt{3}}{l^2} \\ -\frac{-1+\sqrt{3}}{l} \end{array} \right\}^T \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_6 \\ \theta_7 \end{array} \right\}, \quad \frac{\partial^2 w}{\partial x^2}(\overline{x}_2) = \left\{ \begin{array}{c} 2\frac{\sqrt{3}}{l^2} \\ \frac{-1+\sqrt{3}}{l} \\ -2\frac{\sqrt{3}}{l^2} \\ \frac{1+\sqrt{3}}{l} \end{array} \right\}^T \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_6 \\ \theta_7 \end{array} \right\}. \tag{5.11.45}$$

Substitution of equations (5.11.45) into equation (5.11.43) yields:

$$\overline{\frac{\partial^2 w}{\partial x^2}} = \frac{1}{2}\left(\left\{ \begin{array}{c} -2\frac{\sqrt{3}}{l^2} \\ -\frac{1+\sqrt{3}}{l} \\ 2\frac{\sqrt{3}}{l^2} \\ -\frac{-1+\sqrt{3}}{l} \end{array} \right\}^T \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_6 \\ \theta_7 \end{array} \right\} + \left\{ \begin{array}{c} 2\frac{\sqrt{3}}{l^2} \\ \frac{-1+\sqrt{3}}{l} \\ -2\frac{\sqrt{3}}{l^2} \\ \frac{1+\sqrt{3}}{l} \end{array} \right\}^T \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_6 \\ \theta_7 \end{array} \right\} \right) =$$

$$= \left\{ \begin{array}{c} 0 \\ -\frac{1}{l} \\ 0 \\ \frac{1}{l} \end{array} \right\}^T \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_6 \\ \theta_7 \end{array} \right\} = \frac{1}{l}(\theta_7 - \theta_2). \tag{5.11.46}$$

The same way, from the polynomial approximation

$$\varepsilon_{zz}^{(2)} = \left\{ \begin{array}{c} 1 - \frac{3\overline{x}^2}{l^2} + \frac{2\overline{x}^3}{l^3} \\ x - \frac{2\overline{x}^2}{l} + \frac{\overline{x}^3}{l^2} \\ \frac{3\overline{x}^2}{l^2} - \frac{2\overline{x}^3}{l^3} \\ -\frac{\overline{x}^2}{l} + \frac{\overline{x}^3}{l^2} \end{array} \right\}^T \left\{ \begin{array}{c} \varepsilon_{zz}^{(2)}(0) \\ \frac{d\varepsilon_{zz}^{(2)}}{d\overline{x}}(0) \\ \varepsilon_{zz}^{(2)}(l) \\ \frac{d\varepsilon_{zz}^{(2)}}{dx}(l) \end{array} \right\} \qquad \text{(eqn 5.10.18)}$$

used in the finite element formulation, one can obtain

$$\overline{\frac{\partial^2 \varepsilon_{zz}^{(2)}}{\partial x^2}} = \left\{ \begin{array}{c} 0 \\ -\frac{1}{l} \\ 0 \\ \frac{1}{l} \end{array} \right\}^T \left\{ \begin{array}{c} \varepsilon_{zz}^{(2)}(0) \\ \frac{d\varepsilon_{zz}^{(2)}}{dx}(0) \\ \varepsilon_{zz}^{(2)}(l) \\ \frac{d\varepsilon_{zz}^{(2)}}{dx}(l) \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ -\frac{1}{l} \\ 0 \\ \frac{1}{l} \end{array} \right\}^T \left\{ \begin{array}{c} \theta_4 \\ \theta_5 \\ \theta_9 \\ \theta_{10} \end{array} \right\} = \frac{1}{l}(\theta_{10} - \theta_5). \tag{5.11.47}$$

The third derivatives $\frac{\partial^3 w_0}{\partial x^3}$ and $\frac{\partial^3 \varepsilon_{zz}^{(2)}}{\partial x^3}$, computed from the interpolation polynomials, used in the FE formulation, are constant in the finte element, and they are most accurate in the middle of

the finite element, at the point $\overline{x} = \frac{l}{2}$ and maybe at some other points too (explanation of that is Appendix 5-D). From polynomial approximations (5.10.17) and (5.10.18) we obtain:

$$\overline{\frac{\partial^3 w}{\partial x^3}} = \frac{\partial^3 w}{\partial x^3} = \left\{ \begin{array}{c} \frac{12}{l^3} \\ \frac{6}{l^2} \\ -\frac{12}{l^3} \\ \frac{6}{l^2} \end{array} \right\}^T \left\{ \begin{array}{c} w_0(0) \\ w_0'(0) \\ w_0(l) \\ w_0'(l) \end{array} \right\} = \left\{ \begin{array}{c} \frac{12}{l^3} \\ \frac{6}{l^2} \\ -\frac{12}{l^3} \\ \frac{6}{l^2} \end{array} \right\}^T \left\{ \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_6 \\ \theta_7 \end{array} \right\} =$$

$$= \frac{12}{l^3}\left(\theta_1 - \theta_6\right) + \frac{6}{l^2}\left(\theta_2 + \theta_7\right) \tag{5.11.48}$$

and

$$\overline{\frac{\partial^3 \varepsilon_{zz}^{(2)}}{\partial x^3}} = \frac{\partial^3 \varepsilon_{zz}^{(2)}}{\partial x^3} = \left\{ \begin{array}{c} \frac{12}{l^3} \\ \frac{6}{l^2} \\ -\frac{12}{l^3} \\ \frac{6}{l^2} \end{array} \right\}^T \left\{ \begin{array}{c} \varepsilon_{zz}^{(2)}(0) \\ \frac{d\varepsilon_{zz}^{(2)}}{dx}(0) \\ \varepsilon_{zz}^{(2)}(l) \\ \frac{d\varepsilon_{zz}^{(2)}}{dx}(l) \end{array} \right\} = \left\{ \begin{array}{c} \frac{12}{l^3} \\ \frac{6}{l^2} \\ -\frac{12}{l^3} \\ \frac{6}{l^2} \end{array} \right\}^T \left\{ \begin{array}{c} \theta_4 \\ \theta_5 \\ \theta_9 \\ \theta_{10} \end{array} \right\} =$$

$$= \frac{12}{l^3}\left(\theta_4 - \theta_9\right) + \frac{6}{l^2}\left(\theta_5 + \theta_{10}\right). \tag{5.11.49}$$

The first derivative $\frac{\partial \varepsilon_{xz}^{(2)}}{\partial x}$, computed from the interpolation polynomial, used in the FE formulation, is constant in the finite element, and is most accurate in the middle of the finite element and maybe at some other points too (explanation of this is in Appendix 5-D). From the polynomial approximation (5.10.11) we receive:

$$\overline{\frac{\partial \varepsilon_{xz}^{(2)}}{\partial x}} = \frac{\partial \varepsilon_{xz}^{(2)}}{\partial x} = \left\{ \begin{array}{c} -\frac{1}{l} \\ \frac{1}{l} \end{array} \right\}^T \left\{ \begin{array}{c} \varepsilon_{xz}^{(2)}(0) \\ \varepsilon_{xz}^{(2)}(l) \end{array} \right\} = \left\{ \begin{array}{c} -\frac{1}{l} \\ \frac{1}{l} \end{array} \right\}^T \left\{ \begin{array}{c} \theta_3 \\ \theta_8 \end{array} \right\} = \frac{1}{l}\left(\theta_8 - \theta_3\right). \tag{5.11.50}$$

The derivatives $\frac{\partial^4 w_0}{\partial x^4}$, $\frac{\partial^4 \varepsilon_{zz}^{(2)}}{\partial x^4}$ and $\frac{\partial^2 \varepsilon_{xz}^{(2)}}{\partial x^2}$, taken as the fourth derivatives of the interpolation polynomials, that were used in the finite element formulation, are equal to zero, that can be wrong for a particular problem. Therefore, these derivatives are computed numerically at the nodal points by a finite difference scheme, using the nodal values of w_0, $\varepsilon_{zz}^{(2)}$ and $\varepsilon_{xz}^{(2)}$, obtained from the finite element solution. The average over the element values of these derivatives will be computed as

$$\overline{\frac{\partial^4 w_0}{\partial x^4}} = \frac{1}{2}\left[\frac{\partial^4 w_0}{\partial x^4}\left(A_i\right) + \frac{\partial^4 w_0}{\partial x^4}\left(A_{i+1}\right)\right], \tag{5.11.51}$$

$$\overline{\frac{\partial^4 \varepsilon_{zz}^{(2)}}{\partial x^4}} = \frac{1}{2}\left[\frac{\partial^4 \varepsilon_{zz}^{(2)}}{\partial x^4}(A_i) + \frac{\partial^4 \varepsilon_{zz}^{(2)}}{\partial x^4}(A_{i+1})\right], \tag{5.11.52}$$

$$\overline{\frac{\partial^2 \varepsilon_{xz}^{(2)}}{\partial x^2}} = \frac{1}{2}\left[\frac{\partial^2 \varepsilon_{xz}^{(2)}}{\partial x^2}(A_i) + \frac{\partial^2 \varepsilon_{xz}^{(2)}}{\partial x^2}(A_{i+1})\right]. \tag{5.11.53}$$

5.12 Damage Progression and Time Integration

When a failure occurs in a single layer of a composite laminate, a composite structure can still carry a load. Therefore, a subsequent failure prediction is required to determine a dynamic response of the structure in the presence of some damage. This problem is dealt with by assuming that within a finite element where the damage occurs the original material characteristics of the damaged ply can be replaced with degraded material characteristics. The degraded material properties are assumed to be small fractions of the properties of the undamaged material, but not equal to zero, in order to avoid ill-conditioning of the finite element equations. For example, a degraded value of the Young's modulus E_d of the damaged ply within a finite element is computed as

$$E_{1d} = (src)\, E_1, \tag{5.12.1}$$

where E_1 is an original value of the Young's modulus and (src) is a stiffness reduction coefficient. The stiffness reduction coefficient is set to be as small as possible, but its smallness is limited by the need to avoid numerical difficulties that can be caused by the large difference of material constants of adjacent finite elements. Such values of the stiffness reduction coefficients are found by numerical experimentation.

The face sheets of the sandwich plate are made of laminated composite plates, that can fail in different modes: due to matrix cracking, fiber fracture, fiber matrix debonds and delamination. Therefore, for accurate prediction of failure in the face sheets, one needs to use a failure criterion that takes account of the microstructure of the composite laminates and the variety of modes of failure that can occur due to this microstructure. A set of failure criteria, designed for this purpose, were suggested by Hashin (1980). Therefore, for the face sheets the Hashin's criteria will be used in this study.

The core of the sandwich plate, made of polymeric foam or a honeycomb structure, is modelled as a homogeneous isotropic or transversely isotropic medium. Such a medium has fewer modes of failure, namely crushing under compression and cracking under tension. Therefore, for the failure analysis of the core, it is more appropriate to use a failure criterion that does not take account of the microstructure of the material. One such criterion is the Tsai-Wu criterion, and it will be used for the core in our study. The core, that is uniform before the beginning of the damage, becomes nonuniform in the thickness direction (as well as in longitudinal direction) when the damage starts to progress in the thickness direction. For this reason, we will divide the core into the nominal

layers, and we will check the failure criterion in the middle of each such layer.

At each time step the average (over a finite element length) stresses in each element and in each layer are used in the failure criteria. The expressions for stresses in terms of the variables w_0, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{zz}^{(2)}$ and their derivatives were developed in subsection 5.11.1 of this chapter. In order to compute the average (over the finite element's length) stresses, the average values of the field variables and their derivatives must be used in these expressions. The computation of the average (over the element) values of the field variables and their derivatives is discussed in subsection 5.11.2 of this chapter.

5.12.1 The Tsai-Wu criterion

The Tsai-Wu failure criterion (Azzi and Tsai, 1965. Wu, 1974) is used for the core. Let X_T, Y_T, Z_T be the lamina normal strengths in tension along the (1, 2, 3) directions, X_C, Y_C, Z_C – lamina normal strengths in compression and S_{23}, S_{13}, S_{12} – shear strengths in the (23, 13, 12) planes respectively. In the Tsai-Wu criterion, failure is assumed to occur if the following condition is satisfied:

$$F \equiv \sum_{i=1}^{6} F_i\sigma_i + \sum_{i=1}^{6}\sum_{j=1}^{6} F_{ij}\sigma_i\sigma_j \geq 1, \tag{5.12.2}$$

where

$$\sigma_1 = \sigma_{11},\ \sigma_2 = \sigma_{22},\ \sigma_3 = \sigma_{33},\ \sigma_4 = \sigma_{23},\ \sigma_5 = \sigma_{13},\ \sigma_6 = \sigma_{12}, \tag{5.12.3}$$

$$F_1 = \frac{1}{X_T} - \frac{1}{X_C},\ F_2 = \frac{1}{Y_T} - \frac{1}{Y_C},\ F_3 = \frac{1}{Z_T} - \frac{1}{Z_C},$$

$$F_{11} = \frac{1}{X_T X_C},\ F_{22} = \frac{1}{Y_C Y_T},\ F_{33} = \frac{1}{Z_T Z_C},$$

$$F_{44} = \frac{1}{S_{23}},\ F_{55} = \frac{1}{S_{13}},\ F_{66} = \frac{1}{S_{12}},$$

$$F_{12} = F_{21} = -\frac{1}{2}\frac{1}{\sqrt{X_T X_C Y_T Y_C}},\ F_{13} = F_{31} = -\frac{1}{2}\frac{1}{\sqrt{X_T X_C Z_T Z_C}},$$

$$F_{23} = F_{32} = -\frac{1}{2}\frac{1}{\sqrt{Z_T Z_C Y_T Y_C}}. \tag{5.12.4}$$

If the failure occurs, then the following expressions are used to determine the failure mode:

$$H_1 = F_1\sigma_1 + F_{11}\sigma_1^2,\ H_2 = F_2\sigma_2 + F_{22}\sigma_2^2,\ H_3 = F_3\sigma_3 + F_{33}\sigma_3^2,$$

$$H_4 = F_{44}\sigma_4^2,\ H_5 = F_{55}\sigma_5^2,\ H_6 = F_{66}\sigma_6^2. \tag{5.12.5}$$

The largest H_i is selected as a quantity that determines the dominant failure mode, and the corresponding engineering elastic constant is reduced. The correspondence between H_i and engineering elastic constants is the following:

$$H_1 \rightarrow E_1,$$

$$H_2 \rightarrow E_2,$$

$$H_3 \rightarrow E_3,$$

$$H_4 \rightarrow G_{23},$$

$$H_5 \rightarrow G_{13},$$

$$H_6 \rightarrow G_{12},$$

The **method of reduction of values of engineering elastic constants of the core**, using the Tsai-Wu failure criteria is described below.

Compute the failure index $F \equiv \sum_{i=1}^{6} F_i\sigma_i + \sum_{i=1}^{6}\sum_{j=1}^{6} F_{ij}\sigma_i\sigma_j$. If failure occurs, i.e. if $F \geq 1$, then in each layer of the core of each finite element, at each time step find the maximum of H_1, H_2, H_3, H_4, H_5, H_6.

a) If H_1 is the maximum among H_i, then set

$$E_{1d} = (src)\, E_1, \tag{5.12.6}$$

where (src) is a stiffness reduction coefficient.

b) If H_2 is the maximum among H_i, then set

$$E_{2d} = (src)\, E_2. \tag{5.12.7}$$

c) If H_3 is the maximum among H_i, then set

$$E_{3d} = (src)\, E_3. \tag{5.12.8}$$

d) If H_4 is the maximum among H_i then set

$$G_{23d} = (src)\, G_{23}. \tag{5.12.9}$$

e) If H_5 is the maximum among H_i then set

$$G_{13d} = (src)\, G_{13}. \tag{5.12.10}$$

f) If H_6 is the maximum among H_i, then set

$$G_{12d} = (src)\, G_{12}. \tag{5.12.11}$$

A value of the stiffness reduction coefficient needs to be chosen very small, but not lower than a certain limiting value, below which the ill-conditioning of the finite element equations can occur. This limiting value of the stiffness reduction coefficient can be found by numerical experiments with a particular model. In the numerical example in the subsequent section 5.14, the stiffness reduction coefficient (src) was chosen to be 0.001.

5.12.2 The Hashin's criteria

The Hashin's criteria (Hashin, 1980) will be used for the face sheets. The Hashin's criteria and the method of reducing the values of engineering elastic constants of the face sheets are described below.

The **fiber failure in tension** (fiber breakage) in the face sheets in a layer of a face sheet of a finite element is predicted when

$$\sigma_{11} > 0 \text{ and } \frac{\sigma_{11}}{X_T} + \frac{\sigma_{12} + \sigma_{13}}{S_{12}} \geq 1. \tag{5.12.12}$$

When fiber failure in tension is predicted in a layer, the load carrying capacity of that layer is almost completely eliminated. Therefore, the values of all the elastic constants that characterize the in-plane deformation of the plate in cylindrical bending are reduced to some very low values, i.e. it is set

$$E_{1d} = (src)\,E_1,\ G_{13d} = (src)\,G_{13},\ \nu_{13d} = (src)\,\nu_{13},\ \nu_{12d} = (src)\,\nu_{12}, \tag{5.12.13}$$

where (src) is a stiffness reduction coefficient. As it was mentioned earlier, the value of the stiffness reduction coefficient is chosen to be as small as possible, but not lower than a certain limit value under which the ill-conditioning of the FE equations occurs.

The **fiber failure in compression** in a layer of the face sheets of a finite element is predicted when

$$\sigma_{11} < 0 \text{ and } \left(\frac{\sigma_{11}}{X_C}\right)^2 \geq 1. \tag{5.12.14}$$

In the works of Schuerch (1966), Hermann, Mason, Chan (1967), Sadovski, Pu, Hussain (1967), Karpenko, Terletzki, Liashchenko (1972), Greszczuk (1974) and other authors, the compressive fiber mode of failure is interpreted as a failure caused by instability (buckling) of fibers in the matrix. These and other works were included into the monographs of Broutman and Krock (1967), Rosen and Dow (1975). More recently, the failure of composite materials under compression due to instability of fibers was considered in the monograph of Guz (1989).

For compressive fiber failure, it is assumed that the material constants E_2, E_3, G_{12}, G_{13}, responsible for transverse load carrying capacity, are reduced to some very low values. Therefore, it is set:

$$E_{2d} = (src)\,E_2,\ E_{3d} = (src)\,E_3,\ G_{12d} = (src)\,G_{12},\ G_{13d} = (src)\,G_{13}, \tag{5.12.15}$$

where (src) is a stiffness reduction coefficient. Besides, it is assumed that if the buckling of the fibers occurs, the layer still has some residual strength in the direction of the fibers. Therefore, the original Young's modulus in the fiber direction E_1 is replaced with some reduced value E_{1d} by the formula

$$E_{1d} = (SRC)\,E_1, \tag{5.12.16}$$

where (SRC) is another stiffness reduction coefficient, whose value is larger than the value of the stiffness reduction coefficient (src):

$$(src) < (SRC)\,. \tag{5.12.17}$$

The **matrix failure in the face sheets** is predicted when

$$F_t^2 = \left(\frac{\sigma_{22}+\sigma_{33}}{Y_T}\right)^2 + \frac{(\sigma_{23})^2 - \sigma_{22}\sigma_{33}}{(S_{23})^2} + \frac{(\sigma_{12})^2 + (\sigma_{13})^2}{(S_{12})^2} \geq 1 \text{ and } \sigma_{22}+\sigma_{33} > 0, \tag{5.12.18}$$

or when

$$F_c^2 = \frac{1}{Y_C}\left[\left(\frac{Y_C}{2S_{23}}\right)^2 - 1\right] + \frac{(\sigma_{22}+\sigma_{33})^2}{4\,(S_{23})^2} +$$

$$+\frac{(\sigma_{23})^2 - \sigma_{22}\sigma_{33}}{(S_{23})^2} + \frac{(\sigma_{12})^2 + (\sigma_{13})^2}{(S_{12})^2} \geq 1 \text{ and } \sigma_{22}+\sigma_{33} < 0. \tag{5.12.19}$$

In this case, the degraded stiffness properties are:

$$E_{3d} = (src)\,E_3,\;\; G_{23d} = (src)\,G_{23},\;\; G_{13d} = (src)\,G_{13},$$

$$E_{2d} = (src)\,E_2, G_{12d} = (src)\,G_{12},$$

$$\nu_{12d} = (src)\,\nu_{12}, \nu_{23d} = (src)\,\nu_{23}.$$

The **delamination** (separation of the plies) occurs when

$$\left(\frac{\sigma_{33}}{Z_t}\right)^2 \geq 1 \quad \text{and} \quad \sigma_{33} > 0. \tag{5.12.20}$$

In this case, the degraded material properties are:

$$E_{3d} = (src)\,E_3,\;\; G_{23d} = (src)\,G_{23},\;\; G_{13d} = (src)\,G_{13}, \nu_{23d} = (src)\,\nu_{23}.$$

5.12.3 The algorithm of modeling the damage progression

Now, the algorithm of damage progression will be presented without the details of how it is imbedded into the time integration scheme. These details will be discussed in the subsequent subsection.

1) At each time step compute average (over an element length) stresses σ_{xx}, σ_{xy}, σ_{yy}, σ_{xz}, σ_{yz}, σ_{zz} in the problem coordinate system in all finite elements, in the middle of each ply of the face sheets (at $z = \frac{\xi_k+\xi_{k+1}}{2}$) and in the middle of each nominal layer of the core. The method of computing the average (over an element length) values of derivatives of the field variables, that enter into the formulas for the average stresses, was presented in section 5.11 of chapter 5.

2) Transform the stresses to the principle material coordinates, i.e. compute σ_{11}, σ_{22}, σ_{33}, σ_{12}, σ_{13}, σ_{23} by formulas (Reddy, 1996):

$$\sigma_{11} = (\sigma_{xx}\cos\theta + \sigma_{xy}\sin\theta)\cos\theta + (\sigma_{xy}\cos\theta + s\sigma_{yy})\sin\theta, \tag{5.12.21}$$

$$\sigma_{12} = -(\sigma_{xx}\cos\theta + \sigma_{xy}\sin\theta)\sin\theta + (\sigma_{xy}\cos\theta + \sigma_{yy}\sin\theta)\cos\theta, \tag{5.12.22}$$

$$\sigma_{13} = \sigma_{xz}\cos\theta + \sigma_{yz}\sin\theta, \tag{5.12.23}$$

$$\sigma_{22} = (\sigma_{xx}\sin\theta - \sigma_{xy}\cos\theta)\sin\theta + (-s\sigma_{xy} + \sigma_{yy}\cos\theta)\cos\theta, \tag{5.12.24}$$

$$\sigma_{23} = -\sigma_{xz}\sin\theta + \sigma_{yz}\cos\theta, \tag{5.12.25}$$

$$\sigma_{33} = \sigma_{zz}, \tag{5.12.26}$$

where θ is angle of fiber orientation with respect to the x-axis of the problem coordinate system.

3) Substitute the stresses in the material coordinate system into the failure criteria. The Hashin criteria will be used for the face sheets and the Tsai-Wu criterion will be used for the core. If the failure occurs, reduce the appropriate engineering constants of the face sheets and the core, using the methods, described above.

4) Using the modified values of engineering elastic constants, for each layer of each finite element that fails recompute elastic constants ${}^{\alpha}\overline{C}_{ij}^{(k)}$, element stiffness matrices, global stiffness matrix and

restart the analysis at the same time step, i.e. return to the 1-st step.[6]

5) If failure does not occur, proceed to the next time step.

The analysis will continue for a time duration, specified by a user, or until all finite elements fail.

The flow-chart of this algorithm is presented in Figure 5.1

5.12.4 Time-history analysis by Newmark method with account of damage progression

Let us introduce the following notations:

$\{\Theta\}\Big|_{t=t_n} \equiv \{\Theta\}_n$ – vector of nodal variables, evaluated at moment of time t_n, $\{\Theta\}\Big|_{t=t_{n+1}} \equiv \{\Theta\}_{n+1}$ – vector of nodal variables, evaluated at moment of time t_{n+1}, $\tau \equiv t_{n+1} - t_n$.

Then, the Taylor expansion of $\{\Theta\}$ about a point t_n, with four terms, evaluated at point t_{n+1}, has the form:

$$\{\Theta\}_{n+1} \approx \{\Theta\}_n + \left\{\dot{\Theta}\right\}_n \tau + \frac{1}{2}\left\{\ddot{\Theta}\right\}_n \tau^2 + \frac{1}{6}\{\dddot{\Theta}\}_n \tau^3 \ . \tag{5.12.27}$$

The quantity $\frac{1}{6}\{\dddot{\Theta}\}_n$ in the last term can be written approximately as follows (Englemann, 1988):

$$\frac{1}{6}\{\dddot{\Theta}\}_n \approx \beta \frac{\left\{\ddot{\Theta}\right\}_{n+1} - \left\{\ddot{\Theta}\right\}_n}{\tau}, \tag{5.12.28}$$

where β is a free parameter that controls the accuracy and stability of the method. Therefore, equation (5.12.27) takes the form

$$\{\Theta\}_{n+1} \approx \{\Theta\}_n + \left\{\dot{\Theta}\right\}_n \tau + \frac{1}{2}\left\{\ddot{\Theta}\right\}_n \tau^2 + \beta\tau^2 \left(\left\{\ddot{\Theta}\right\}_{n+1} - \left\{\ddot{\Theta}\right\}_n\right) \approx$$

$$\approx \{\Theta\}_n + \tau\left\{\dot{\Theta}\right\}_n + \tau^2\left(\frac{1}{2} - \beta\right)\left\{\ddot{\Theta}\right\}_n + \tau^2\beta\left\{\ddot{\Theta}\right\}_{n+1}, \tag{5.12.29}$$

Analogously, expanding the vector $\left\{\dot{\Theta}\right\}$ in Taylor series, keeping three terms in the expansion and evaluating $\left\{\dot{\Theta}\right\}$ at moment of time t_{n+1}, one can obtain (Englemann, 1988):

$$\left\{\dot{\Theta}\right\}_{n+1} \approx \left\{\dot{\Theta}\right\}_n + \left\{\ddot{\Theta}\right\}_n \tau + \gamma\left(\left\{\ddot{\Theta}\right\}_{n+1} - \left\{\ddot{\Theta}\right\}_n\right)\tau^2 \approx$$

[6] When failure occurs, the stress field changes instantly due to the change of material properties. This redistribution of the stresses may cause additional failure to occur. Therefore, in case of failure, the time incrementation must be stopped, and analysis must be run again for the same time interval to determine the new failure. If the new failure does not occur, the analysis can go on to the next time step.

$$\approx \left\{\dot{\Theta}\right\}_n + \tau(1-\gamma\tau)\left\{\ddot{\Theta}\right\}_n + \tau^2\gamma\left\{\ddot{\Theta}\right\}_{n+1}, \qquad (5.12.30)$$

where γ is another free parameter that controls the accuracy and stability of the method.

Equations of motion of the system in terms of the global nodal variables $\{\Theta\}$, in which vectors $\{\Theta\}$, $\left\{\dot{\Theta}\right\}$, $\left\{\ddot{\Theta}\right\}$ are evaluated at a moment of time t_{n+1}, are

$$[M]\left\{\ddot{\Theta}\right\}_{n+1} + [C]\left(\left\{\dot{\Theta}\right\}_{n+1} - \left\{\dot{\Theta}\right\}_1\right) + [K]\{\Theta\}_{n+1} + \{Q\}_{n+1} = \{P\}, \qquad (5.12.31)$$

where $\{Q\}$ is a nonlinear part of the internal force vector, whose components are defined as $\frac{\partial (U_{nl})_{system}}{\partial \Theta_i}$, where $(U_{nl})_{system}$ is the whole system's part of the strain energy, that is not quadratic with respect to the nodal parameters Θ_i. This part of the strain energy appears due to the nonlinear terms in the von-Karman strain-displacement relations. In equation (5.12.31) the vector $[K]\{\Theta\}$ is a linear part of the internal force vector, and, therefore, the matrix $[K]$ is a stiffness matrix of a geometrically linearly formulated problem. The stiffness matrix $[K]$ does not depend on the nodal unknowns. In equation (5.12.31) the load vector $\{P\}$ is due to the gravity force, therefore it does not depend on time. At the initial moment of time t_1, when the platform touches the elastic foundation, but the foundation is not compressed yet, the damping in the platform is absent. This is taken into account by writing in equation (5.12.31) the term, responsible for damping, in the form $[C]\left(\left\{\dot{\Theta}\right\}_{n+1} - \left\{\dot{\Theta}\right\}_1\right)$, with initial velocity $\left\{\dot{\Theta}\right\}_1$ subtracted from the velocity $\left\{\dot{\Theta}\right\}_{n+1}$. As a result of this, the equation of motion (5.12.31), written for the initial moment of time t_1, takes the form:

$$[M]\left\{\ddot{\Theta}\right\}_1 + \underbrace{[K]\{\Theta\}_1 + \{Q\}_1}_{0} = \{P\}. \qquad (5.12.32)$$

The internal force vector at the initial moment of time is equal to a zero-vector, because at the initial moment of time all components of the generalized displacement vector $\{\Theta\}$ are equal to zero:

$$\{\Theta\}_1 = \{0\}. \qquad (5.12.33)$$

Therefore, equation (5.12.32) takes the form:

$$[M]\left\{\ddot{\Theta}\right\}_1 = \{P\}. \qquad (5.12.34)$$

The global vector of nodal parameters at the initial moment of time, $\left\{\ddot{\Theta}\right\}_1$, computed from equation (5.12.34), is such that those components of this vector, that are the second time derivatives of the

nodal transverse displacements at $t = t_1$, $\ddot{w}_0(t_1)$, are equal to $9.8\frac{m}{s^2}$, as it is expected to be. If in equation (5.12.31) the term responsible for damping was written as $[C]\left\{\dot{\Theta}\right\}_{n+1}$ instead of $[C]\left(\left\{\dot{\Theta}\right\}_{n+1} - \left\{\dot{\Theta}\right\}_1\right)$, then the initial acceleration $\ddot{w}_0(t_1)$ would be computed from equation $[M]\left\{\ddot{\Theta}\right\}_1 = \{P\} - [C]\left\{\dot{\Theta}\right\}_1$ and, therefore, would take on very high values, different from the acceleration of free fall $9.8\frac{m}{s^2}$.

Substitution of equations (5.12.29) and (5.12.30) into equation (5.12.31) yields:

$$\left([M] + [C]\tau^2\gamma + [K]\tau^2\beta\right)\left\{\ddot{\Theta}\right\}_{n+1} + \{Q\}_{n+1} +$$

$$+[C]\left(\left\{\dot{\Theta}\right\}_n + \tau(1-\gamma\tau)\left\{\ddot{\Theta}\right\}_n - \left\{\dot{\Theta}\right\}_1\right) +$$

$$+[K]\left(\{\Theta\}_n + \tau\left\{\dot{\Theta}\right\}_n + \tau^2\left(\frac{1}{2} - \beta\right)\left\{\ddot{\Theta}\right\}_n\right) = \{P\} \tag{5.12.35}$$

From equation (5.12.29) we find

$$\left\{\ddot{\Theta}\right\}_{n+1} = \frac{1}{\tau^2\beta}\{\Theta\}_{n+1} - \frac{1}{\tau^2\beta}\left(\{\Theta\}_n + \tau\left\{\dot{\Theta}\right\}_n + \tau^2\left(\frac{1}{2} - \beta\right)\left\{\ddot{\Theta}\right\}_n\right). \tag{5.12.36}$$

Substitution of equation (5.12.36) into equation (5.12.35) yields

$$\left([M]\frac{1}{\tau^2\beta} + [C]\frac{\gamma}{\beta} + [K]\right)\{\Theta\}_{n+1} + \{Q\}_{n+1}$$

$$-\frac{1}{\tau^2\beta}\left(\{\Theta\}_n + \tau\left\{\dot{\Theta}\right\}_n + \tau^2\left(\frac{1}{2} - \beta\right)\left\{\ddot{\Theta}\right\}_n\right)\left([M] + [C]\tau^2\gamma + [K]\tau^2\beta\right) +$$

$$+[C]\left(\left\{\dot{\Theta}\right\}_n + \tau(1-\gamma\tau)\left\{\ddot{\Theta}\right\}_n - \left\{\dot{\Theta}\right\}_1\right) +$$

$$+[K]\left(\{\Theta\}_n + \tau\left\{\dot{\Theta}\right\}_n + \tau^2\left(\frac{1}{2} - \beta\right)\left\{\ddot{\Theta}\right\}_n\right) - \{P\} = \{0\}. \tag{5.12.37}$$

Now, assuming that we know the values of $\{\Theta\}_n$, $\left\{\dot{\Theta}\right\}_n$,and$\left\{\ddot{\Theta}\right\}_n$, we need to find the values of $\{\Theta\}_{n+1}$, $\left\{\dot{\Theta}\right\}_{n+1}$,and$\left\{\ddot{\Theta}\right\}_{n+1}$. Components of vector $\{Q\}_{n+1}$, that enters into equation (5.12.37),

depend nonlinearly on components of the vector of nodal parameters $\{\Theta\}_{n+1}$. Therefore, equation (5.12.37) is a nonlinear system of equations with respect to components of the vector $\{\Theta\}_{n+1}$. These nonlinear equations will be solved by a direct iteration (Picard) method (Reddy, 1996). Let us introduce the following notations:

$$\left[\hat{K}\right] \equiv [M]\,\frac{1}{\tau^2\beta} + [C]\,\frac{\gamma}{\beta} + [K]\ , \tag{5.12.38}$$

$$\begin{aligned}\left\{\hat{F}\right\}_n = &-\frac{1}{\tau^2\beta}\left(\{\Theta\}_n + \tau\left\{\dot{\Theta}\right\}_n + \tau^2\left(\frac{1}{2}-\beta\right)\left\{\ddot{\Theta}\right\}_n\right)\left([M] + [C]\,\tau^2\gamma + [K]\,\tau^2\beta\right) + \\ &+ [C]\left(\left\{\dot{\Theta}\right\}_n + \tau\,(1-\gamma\tau)\left\{\ddot{\Theta}\right\}_n - \left\{\dot{\Theta}\right\}_1\right) + \\ &+ \left[K^{(l)}\right]\left(\{\Theta\}_n + \tau\left\{\dot{\Theta}\right\}_n + \tau^2\left(\frac{1}{2}-\beta\right)\left\{\ddot{\Theta}\right\}_n\right) - \{P\}\,,\end{aligned} \tag{5.12.37}$$

Then, equation (5.12.37) takes the form

$$\left[\hat{K}\right]\{\Theta\}_{n+1} = -\left\{\hat{F}\right\}_n - \{Q\}_{n+1}\ , \tag{5.12.38}$$

or

$$\{\Theta\}_{n+1} = -\left[\hat{K}\right]^{-1}\left(\left\{\hat{F}\right\}_n + \{Q\}_{n+1}\right) \tag{5.12.39}$$

The direct iteration method is based on computing a sequence of vectors

$$\{\Theta\}^{(1)}_{n+1}\,,\ \{\Theta\}^{(2)}_{n+1}\,,\ \{\Theta\}^{(3)}_{n+1}\,, \ldots \tag{5.12.40}$$

by the recurrence formula

$$\{\Theta\}^{(r+1)}_{n+1} = -\left[\hat{K}\right]^{-1}\left(\left\{\hat{F}\right\}_n + \{Q\}^{(r)}_{n+1}\right)\ , \tag{5.12.41}$$

where the vector $\{Q\}^{(r)}_{n+1}$ is the vector $\{Q\}_{n+1}$ evaluated at $\{\Theta\}_{n+1} = \{\Theta\}^{(r)}_{n+1}$. The components of the matrix $\left[\hat{K}\right]$ and the vector $\left\{\hat{F}\right\}_n$ do not depend on the unknowns, i.e. on the components of the vector $\{\Theta\}_{n+1}$. If the sequence of vectors $\{\Theta\}^{(1)}_{n+1}$, $\{\Theta\}^{(2)}_{n+1}$, $\{\Theta\}^{(3)}_{n+1}$,... converges to some vector $\left\{\tilde{\Theta}\right\}_{n+1}$, then this vector $\left\{\tilde{\Theta}\right\}_{n+1}$ is a solution of the system of equations (5.12.38). Since the inversion of matrix $\left[\hat{K}\right]$ is not an effective computational procedure, it is more convenient to find each next term of the recurrence sequence (5.12.40) by solving a system of linear algebraic equations

$$\left[\hat{K}\right]\{\Theta\}^{(r+1)}_{n+1} = -\left\{\hat{F}\right\}_n - \{Q\}^{(r)}_{n+1} \tag{5.12.42}$$

for the components of the vector $\{\Theta\}_{n+1}^{(r+1)}$. The components of the vector $\{\Theta\}_{n+1}^{(r+1)}$ found in each iteration, are used to evaluate the nonlinear part of the internal force vector $\{Q\}_{n+1}$, which is then used in the next iteration to obtain the next improved approximation of the vector $\{\Theta\}_{n+1}$. In other words, in the next iteration the system of linear algebraic equations

$$\left[\hat{K}\right]\{\Theta\}_{n+1}^{(r+2)} = -\left\{\hat{F}\right\}_n - \{Q\}_{n+1}^{(r+1)} \ . \tag{5.12.43}$$

is solved for the vector $\{\Theta\}_{n+1}^{(r+2)}$. In the FE program, that is developed for the analysis of the problem, the first term of the iteration sequence $\{\Theta\}_{n+1}^{(1)}$, $\{\Theta\}_{n+1}^{(2)}$, $\{\Theta\}_{n+1}^{(3)}$, ... is set equal to a zero-vector at all time intervals:

$$\{\Theta\}_{n+1}^{(1)} = \{0\} \tag{5.12.44}$$

for n=1,2,3,... . With such a choice of initial guess of the solution vector, the convergence of the iteration sequence (5.12.40) is achieved successfully unless the number of the damaged plies is high. But if the number of the damaged plies is large, the program needs to be stopped anyway. Iteration is stopped if a norm of vector $\{\Theta\}_{n+1}^{(r+1)} - \{\Theta\}_{n+1}^{(r)}$ (a difference of solution vectors in two successive approximations), divided by the norm of vector $\{\Theta\}_{n+1}^{(r+1)}$ is less than some number (tolerance):

$$\frac{\left\|\{\Theta\}_{n+1}^{(r+1)} - \{\Theta\}_{n+1}^{(r)}\right\|}{\left\|\{\Theta\}_{n+1}^{(r+1)}\right\|} < tolerance \tag{5.12.44}$$

As a norm of a vector, we used a square root of sum of squares of its components. Let $(\Theta_i)_{n+1}^{(r)}$ be an i-th component of the approximate solution vector obtained in an iteration with a number r at a moment of time with a number $n+1$.Then the criterion (5.12.44) for stopping the iterations will be written as follows:

$$\frac{\sqrt{\left[(\Theta_i)_{n+1}^{(r+1)} - (\Theta_i)_{n+1}^{(r)}\right]^2}}{\sqrt{\left[(\Theta_i)_{n+1}^{(r+1)}\right]^2}} < tolerance \tag{5.12.45}$$

In the example problem in the subsequent section 5.14, the value of tolerance is chosen to be 0.001. So, in the problems **without** damage progression taken into account, the algorithm of the Newmark time integration scheme, combined with the direct iteration method of solving the nonlinear algebraic equations, can be summarized as follows:

I) **At the first time interval** $[t_1,\ t_2]$ **:**

Set the vectors of initial generalized displacements $\{\Theta_1\}$ and velocities $\{\Theta\}_1$ equal to the initial conditions. In our finite element formulation we have the following nodal variables at each node: w_0, $\frac{\partial w_0}{\partial x}$, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{zz}^{(2)}$, $\frac{\partial \varepsilon_{zz}^{(2)}}{\partial x}$. Therefore, for a platform, dropped on elastic foundation, at the initial moment of time $t = t_1$ we set at each node

$$w_0 = 0, \frac{\partial w_0}{\partial x} = 0, \varepsilon_{xz}^{(2)} = 0, \varepsilon_{zz}^{(2)} = 0, \frac{\partial \varepsilon_{zz}^{(2)}}{\partial x} = 0 \ ; \tag{5.12.46}$$

$$\frac{\partial w_0}{\partial t} = initial\ veocity,\ \frac{\partial}{\partial t}\left(\frac{\partial w_0}{\partial x}\right) = 0,\ \frac{\partial \varepsilon_{xz}^{(2)}}{\partial t} = 0,\ \frac{\partial \varepsilon_{zz}^{(2)}}{\partial t} = 0,\ \frac{\partial}{\partial t}\left(\frac{\partial \varepsilon_{zz}^{(2)}}{\partial x}\right) = 0\ . \tag{5.12.47}$$

The vector $\left\{\ddot{\Theta}\right\}_1$ of initial generalized accelerations is found from the equation (5.12.34), repeated here as equation (5.12.48):

$$[M]\left\{\ddot{\Theta}\right\}_1 = \{P\}. \tag{5.12.48}$$

II) **At the n-th time interval** $[t_n,\ \ t_{n+1}]$ the vectors $\{\Theta\}_n$, $\left\{\dot{\Theta}\right\}_n$, $\left\{\ddot{\Theta}\right\}_n$ are known, and it is necessary to find the vectors $\{\Theta\}_{n+1}$, $\left\{\dot{\Theta}\right\}_{n+1}$, $\left\{\ddot{\Theta}\right\}_{n+1}$. For this purpose the following algorithm is used.

1) Set iteration counter $r = 1$, and set the initial approximation for the vector of nodal parameters at $t = t_{n+1}$ as

$$\{\Theta\}_{n+1}^{(1)} = \{0\},$$

2) Evaluate $\{Q\}_{n+1}^{(r)}$, i.e. evaluate $\{Q\}_{n+1}$ at $\{\Theta\}_{n+1} = \{\Theta\}_{n+1}^{(r)}$ and solve a linear system of algebraic equations for the components of the vector $\{\Theta\}_{n+1}^{(r+1)}$

$$\left[\hat{K}\right]\{\Theta\}_{n+1}^{(r+1)} = -\left\{\hat{F}\right\}_n - \{Q\}_{n+1}^{(r)}$$

Evaluate the acceleration vector of the current iteration by the formula

$$\left\{\ddot{\Theta}\right\}_{n+1}^{(r+1)} = \frac{1}{\tau^2\beta}\left(\{\Theta\}_{n+1}^{(r+1)} - \{\Theta\}_n - \tau\left\{\dot{\Theta}\right\}_n - \tau^2\left(\frac{1}{2} - \beta\right)\left\{\ddot{\Theta}\right\}_n\right) \tag{5.12.49}$$

(equation (5.12.49) is obtained by expressing $\left\{\ddot{\Theta}\right\}_{n+1}$ from equation (5.12.29)). Evaluate the velocity vector of the current iteration by the formula

$$\left\{\dot{\Theta}\right\}_{n+1}^{(i+1)} = \left\{\dot{\Theta}\right\}_n + \tau(1 - \gamma\tau)\left\{\ddot{\Theta}\right\}_n + \tau^2\gamma\left\{\ddot{\Theta}\right\}_{n+1}^{(i+1)}. \tag{5.12.50}$$

(equation (5.12.50) is obtained from equation (5.12.30).

3) Check if the vectors $\{\Theta\}_{n+1}^{(r+1)}$ and $\{\Theta\}_{n+1}^{(r)}$ satisfy the convergence criterion of equation (5.12.45)

$$\frac{\sqrt{\left[(\Theta_i)_{n+1}^{(r+1)} - (\Theta_i)_{n+1}^{(r)}\right]^2}}{\sqrt{\left[(\Theta_i)_{n+1}^{(r+1)}\right]^2}} < tolerance \qquad \text{(eqn 5.12.45)}$$

If the convergence criterion is not satisfied, then begin a new iteration within this time interval, i.e. set $r = r + 1$ and go to the step 2. If the convergence criterion is satisfied, go to the next step.

4) Set the vector of nodal parameters and the vectors of the first and second time derivatives of the nodal parameters equal to the corresponding vectors obtained in the iteration at which the convergence criterion of the step 3 was satisfied, i.e. set

$$\{\Theta\}_{n+1} = \{\Theta\}_{n+1}^{(r+1)} \ , \qquad (5.12.51)$$

$$\left\{\dot{\Theta}\right\}_{n+1} = \left\{\dot{\Theta}\right\}_{n+1}^{(r+1)} \ , \qquad (5.12.52)$$

$$\left\{\ddot{\Theta}\right\}_{n+1} = \left\{\ddot{\Theta}\right\}_{n+1}^{(r+1)} \ . \qquad (5.12.53)$$

for use in the next time step and for computation of stresses at $t = t_{n+1}$.

5) Compute average stresses in all plies of each finite element at $t = t_{n+1}$, using the vectors $\{\Theta\}_{n+1}$, $\left\{\dot{\Theta}\right\}_{n+1}$ and $\left\{\ddot{\Theta}\right\}_{n+1}$,obtained in the 4-th step. Then set $n = n + 1$, i.e. go to the next time interval.

Analysis goes on for all time steps, the number of which is specified by a user, or until all plies in all finite elements fail. If a number of the damaged plies is large, the iterative procedure of solving the nonlinear algebraic equations (5.12.43) can fail to lead to convergence of the sequence of approximate solutions, i.e. the termination criterion (5.12.45) of the iteration process will not be satisfied. This serves as an indicator that the number of the damaged plies is high and also leads to stopping the finite element program.

If the **damage is taken into account**, then the 5-th step of the above algorithm will be modified as follows:

5′) Compute average stresses in all plies of each finite element at $t = t_{n+1}$, using the vectors $\{\Theta\}_{n+1}$, $\left\{\dot{\Theta}\right\}_{n+1}$ and $\left\{\ddot{\Theta}\right\}_{n+1}$,obtained in the 4-th step. Substitute these stresses into the failure

criteria. If failure occurs in a ply of a finite element, modify material elastic constants of this ply, modify the element stiffness matrix $[k]$ and the nonlinear internal force vector $\{q\}_{n+1} = \left(\frac{\partial U_{nl}}{\partial\{\theta\}}\right)_{n+1}$ of the finite element to which the damaged ply belongs and assemble the global stiffness matrix $[K]$ and global nonlinear internal force vector $\{Q\}_{n+1}$ with account of modifications to the element stiffness matrices and element nonlinear internal force vectors due to the damage. Then go to the step 2. If failure does not occur in any ply of any finite element, then set $n = n + 1$, i.e. go to the next time interval.

5.13 Verification of results of the finite element program

In this section we will consider some static and dynamic problems, for which exact elasticity solutions exist, and compare results of these exact solutions with the results produced by the finite element program, based on the layerwise theory of sandwich plates developed in this chapter.

5.13.1 Comparison of exact solution for a homogeneous isotropic simply supported plate and the FE solution of the same problem, based on the layerwise plate theory.

Let us consider a **static** problem of cylindrical bending of a simply supported homogeneous isotropic plate of length L, height h and width b (Figure 2.2). The plate is under a uniform load, acting on the upper surface with intensity (force per unit length) q_u. By q_u we denote not an absolute value of the load intensity, but a projection of the load intensity on the z-axis, i.e. q_u can be positive or negative, depending on the direction of the load. Let $\frac{q_u}{b} = Q = -1 \times 10^5 \frac{N}{m^2}$, $h = 0.022m$, $L = 1m$, $x = 0.5m$, where q_u is force per unit length on the upper surface, b is width of the plate. In this problem, the exact solution for stresses is (Appendix 2-A):

$$\sigma_{xz} = \frac{6}{h^3} Q \left(x - \frac{L}{2} \right) \left(z^2 - \frac{h^2}{4} \right), \tag{5.13.1}$$

$$\sigma_{zz} = -\frac{1}{2h^3} Q \left(2z + h\right)^2 \left(z - h\right), \tag{5.13.2}$$

$$\sigma_{xx} = -\frac{6}{h^3} Q x \left(x - L\right) z + \frac{4}{h^3} Q z^3 - \frac{3}{5h} Q z. \tag{5.13.3}$$

In the finite element model, 50 elements of equal length were used. The stresses were computed as the average stresses over the length of the elements. The tables of comparison of stresses, obtained from the exact and the finite element solutions, are shown below.

Table 5.1: Comparison of exact and FE solutions for stress σ_{xx} in a homogeneous isotropic simply supported plate

x (m)	z (m)	σ_{xx} $(\times 10^6 \frac{N}{m^2})$	σ_{xx} $(\times 10^6 \frac{N}{m^2})$
		exact	plate theory
0.5	−0.011	154.98	154.87 error 0.07%
0.5	−0.0105	147.93	147.84 error 0.06%
0.5	−0.010	140.88	140.80 error 0.06%
0.5	−0.008	112.69	112.64 error 0.04%
0.5	−0.005	70.427	70.402 error 0.03%
0.5	−0.002	28.169	28.164 error 0.02%
0.5	0.0	0	−0.026
0.5	0.002	−28.169	−28.164 error 0.02%
0.5	0.005	−70.427	−70.402 error 0.04%
0.5	0.008	−112.69	−112.64 error 0.04%
0.5	0.010	−140.88	−140.80 error 0.06%
0.5	0.0105	−147.93	−147.84 error 0.06%
0.5	0.011	−154.98	−154.87 error 0.07%

Table 5.2: Comparison of exact and FE solutions for stress σ_{xz} in a homogeneous isotropic simply supported plate

x (m)	z (m)	σ_{xz} $(\times 10^6 \frac{N}{m^2})$ exact	σ_{xz} $(\times 10^6 \frac{N}{m^2})$ plate theory
0.8	−0.011	0	0.0
0.8	−0.0105	0.1817	0.1882 error 3.6%
0.8	−0.010	0.355	0.3677 error 3.6%
0.8	−0.008	0.9636	0.9981 error %
0.8	−0.005	1.6228	1.6810 error 3.6%
0.8	−0.002	1.9778	2.048 error 3.5%
0.8	0.0	2.0455	2.1188 error 3.6%
0.8	0.002	1.9778	2.048 error 3.5%
0.8	0.005	1.6228	1.6810 error 3.6%
0.8	0.008	0.9636	0.9981 error 3.6%
0.8	0.010	0.355	0.3677 error 3.6%
0.8	0.0105	0.1817	0.1883 error 3.6%
0.8	0.011	0	0.00002

Table 5.3: Comparison of exact and FE solutions for stress σ_{zz} in a homogeneous isotropic simply supported plate

x (m)	z (m)	σ_{zz} $(\times 10^4 \frac{N}{m^2})$ exact	σ_{zz} $(\times 10^4 \frac{N}{m^2})$ plate theory
0.8	−0.011	0	0
0.8	−0.0105	−0.015261	−0.0145 error 4.9%
0.8	−0.010	−0.060105	−0.0571 error 4.9%
0.8	−0.008	−0.50714	−0.4814 error 5%
0.8	−0.005	−1.8257	−1.7327 error 5%
0.8	−0.002	−3.6514	−3.4646 error 5.1%
0.8	0	−5.0	−4.7434 error 5.1%
0.8	0.002	−6.3486	−6.0260 error 5.1%
0.8	0.005	−8.1743	−7.777 error 4.9%
0.8	0.008	−9.4929	−9.0206 error 5%
0.8	0.010	−9.9399	−9.4489 error 4.9%
0.8	0.0105	−9.9847	−9.4924 error 4.9%
0.8	0.011	−10.0	−9.5079 error 4.9%

So, the FE program allows one to achieve high accuracy of computation of the in-plane stress σ_{xx} and satisfactory computational accuracy of the transverse stresses σ_{xz} and σ_{zz}. The lower accuracy of the transverse stresses is explained by the fact that these stresses are computed by integration of the pointwise equilibrium equations, and this procedure requires computation of the higher-order derivatives[7] by a finite difference scheme. The results of stress computation presented in the tables above, confirm an idea discussed in chapter 2, that the transverse stresses obtained by integration of the equilibrium equations, satisfy the boundary conditions on both the upper surface and lower surface of the plate[8] despite the fact that the number of constants of integration is fewer

[7] of the order higher than the degree of the interpolation polynomials used in the finite element formulation

[8] i.e. the transverse stresses at the external surfaces are equal to the loads applied at these surfaces

than the number of the boundary conditions. The satisfaction of the stress boundary conditions on the lower surface is exact, because these boundary conditions were used in integration of the equilibrium equations, and the satisfaction of the stress boundary conditions on the upper surface is approximate, because the field variables w_0, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{zz}^{(2)}$ that enter into the formulas for the transverse stresses (section 5.11 of chapter 5)[9] are computed approximately by the FE method.

Now, let us consider a **dynamic** problem of a plate falling on simple supports and compare the values of the transverse displacement at the middle surface ($z = 0$) and at $x = \frac{L}{2}$, as a function of time, obtained from the exact and finite element solutions. In this example problem, the material properties and geometric dimensions are

$$E = 114.8 \times 10^9 \frac{N}{m^2},\ \nu = 0.3,\ \rho = 1614 \frac{kg}{m^3},\ L = 1m,\ h = 0.06m.$$

The plate falls on simple supports with velocity $-10\frac{m}{s}$. In this example problem, the exact elasticity solution for w_0, with 25 terms in the series expansion, is (Appendix 5-E):

w_0 =- 0.009128805307 sin(1395.05t)+ 0.0003289319625 sin(12902.7t)-
0.00007686926503 sin(33127.8t)+ 0.00003079797019 sin(59061.4t)-
0.00001601025860 sin(88362.8t)+ 0.000009680766718 sin(119567.0t)-
0.000006451456069 sin(151811.0t)+ 0.000004598394395 sin(184592.0t)-
0.000003441628335 sin(217620.0t)+ 0.000002672743865 sin(250724.0t)-
0.000002136375939 sin(283800.0t)+ 0.000001747401522 sin(316802.0t)-
0.000001456401848 sin(349694.0t)+ 0.000001232994575 sin(382460.0t)-
0.000001057703130 sin(415098.0t)+ 0.0000009175877618 sin(447610.0t)
- 0.0000008038299104 sin(479992.0t)+ 0.0000007101573229 sin(512252.0t)
- 0.0000006321067301 sin(544398.0t)+ 0.0000005663630004 sin(576434.0t)-
0.0000005104607016 sin(608364.0t)- 0.0000004625190558 sin(640194.0t)
- 0.0000004210857347 sin(671938.0t)+0.0000003850322280 sin(703586.0t)
- 0.0000003534568308 sin(735156.0t)

The displacement w_0 as a function of time, obtained from the finite element analysis, is

Time	w (z= 0, x=L/2)
.0000	.0000E+00
.0001	-.1024E-02

[9]these formulas for the transverse stresses are obtained by integration of the pointwise equilibrium equations

.0002	-.2303E-02
.0003	-.3777E-02
.0004	-.5104E-02
.0005	-.6143E-02
.0006	-.6904E-02
.0007	-.7398E-02
.0008	-.8021E-02
.0009	-.8500E-02
.0010	-.8823E-02
.0011	-.9036E-02
.0012	-.8918E-02
.0013	-.8442E-02
.0014	-.7760E-02
.0015	-.6954E-02
.0016	-.6246E-02
.0017	-.5560E-02
.0018	-.4632E-02
.0019	-.3318E-02
.0020	-.3200E-02
.0021	-.3632E-03
.0022	.8289E-03
.0023	.1855E-02
.0024	.2918E-02
.0025	.4228E-02
.0026	.5522E-02
.0027	.6531E-02
.0028	.7404E-02
.0029	.7953E-02
.0030	.8301E-02
.0031	.8549E-02
.0032	.8706E-02
.0033	.8852E-02

.0034	.8771E-02
.0035	.8389E-02
.0036	.7532E-02
.0037	.6491E-02
.0038	.5625E-02
.0039	.4854E-02
.0040	.3921E-02
.0041	.2786E-02
.0042	.1223E-02
.0043	-.2953E-03
.0044	-.1544E-02
.0045	-.2687E-02
.0046	-.3719E-02
.0047	-.4786E-02
.0048	-.5858E-02
.0049	-.6891E-02
.0050	-.7768E-02
.0051	-.8385E-02

The graphs of the exact and the finite element solutions for w_0 as a function of time are shown in figure 5.4. These two graphs are close to each other.

Now, let us consider a **dynamic** problem of a plate falling on simple supports and compare the values of the transverse displacement at the middle surface ($z = 0$) and at $t = 0.0004s$, as a function of x-coordinate, obtained from the exact and finite element solutions. In this example problem, the material properties and geometric dimensions are

$$E = 114.8 \times 10^9 \frac{N}{m^2},\ \rho = 1614 \frac{kg}{m^3},\ \nu = 0.3,\ L = 1m,\ h = 0.06m.$$

The plate falls on simple supports with velocity $-10\frac{m}{s^2}$. In this example problem, the exact elasticity solution for w_0, with 25 terms in the series expansion, is (Appendix 5-E):

w_0 = 0.0000005282171331 sin(116.2389282x)+ 0.0000006388611505 sin(59.69026043x)
- 0.000001005826267 sin(84.82300166x)- 0.004833771658 sin(3.141592654x)
+ 0.0000005351651669 sin(122.5221135x)-0.0000004744973290 sin(91.10618697x)
+ 0.000005544528261 sin(40.84070450x)+ 0.00003073761580 sin(21.99114858x)

- 0.000001521866275 sin(72.25663104x)+ 0.000002731048304 sin(53.40707512x)+
0.000006258042552 sin(34.55751919x)- 0.00000002457121402 sin(97.38937227x)+
0.000004598192420 sin(47.12388981x)- 0.00004861818949 sin(15.70796327x)
- 0.0000008763241083 sin(65.97344573x)+ 0.0000003719067811 sin(147.6548547x)
+ 0.0000004560154750 sin(109.9557429x)- 0.000001452117018 sin(78.53981635x)
+0.0000002829038361 sin(103.6725576x)+0.0000003351240114 sin(153.9380400x)-
0.0000004621886845 sin(135.0884841x)+0.00001134587104 sin(28.27433389x)
+ 0.0002963728581 sin(9.424777962x)+ 0.0000004150847268 sin(141.3716694x)
+ 0.0000005062957414 sin(128.8052988x).

In this expression, the terms are written not in ascending order of coefficients of x under the "sin" sign, i.e. not in ascending order of summation index k in the formula (5-E.74). The finite element solution for the same problem is presented in the table below:

x-coordinate	w (t=0.0004s, z=0)
0.0	0.00000E+00;
0.05	-7.24664E-04;
0.10	-1.40675E-03;
0.15	-1.99268E-03;
0.20	-2.60446E-03;
0.25	-3.29689E-03;
0.30	-3.92329E-03;
0.35	-4.38506E-03;
0.40	-4.66854E-03;
0.45	-4.81039E-03;
0.50	-4.85239E-03;
0.55	-4.81161E-03;
0.60	-4.67059E-03;
0.65	-4.38700E-03;
0.70	-3.92346E-03;
0.75	-3.29359E-03;
0.80	-2.59886E-03;
0.85	-1.99009E-03;

0.90 -1.43711E-03;

0.95 -6.87228E-04;

1.0 0.00000E+00;

The graphs of displacement $w_0(x)$ as a function of x-coordinate, obtained from the exact and the finite element solutions, are shown in Figure 5.5. These two graphs are close to each other.

Now, let us consider again the problem of a plate falling on simple supports and compare the values of the stress σ_{xx} at the upper surface $\left(z = \frac{h}{2}\right)$ and at $x = \frac{L}{2}$, as a function of time, obtained from the exact and finite element solutions. In this example problem, the material properties and geometric dimensions are

$$E = 114.8 \times 10^9 \frac{N}{m^2},\ \rho = 1614 \frac{kg}{m^3},\ \nu = 0.3,\ L = 1m,\ h = 0.06m.$$

The plate falls on simple supports with velocity $-10\frac{m}{s^2}$. In this example problem, the analytical solution for σ_{xx}, with 25 terms in the series expansion, is (Appendix 5-E):

$\sigma_{xx} =$

-307276473.2 sin(1395.05t)+ 108412712.3 sin(12902.7t)

-69200847.15 sin(33127.8t)+ 53045550.32 sin(59061.4t)

-44665057.84 sin(88362.8t)+ 39735103.54 sin(119567.0t)

-36639293.88 sin(151811.0t)+34641592.27 sin(184592.0t)

-33362189.14 sin(217620.0t)+ 32597764.0 sin(250724.0t)

-32217053.76 sin(283800.0t)+ 32130246.4 sin(316802.0t)

-32286840.13 sin(349694.0t)+ 32659958.90 sin(382460.0t)

-33216080.31 sin(415098.0t)+ 33947871.87 sin(447610.0t)

-34841555.15 sin(479992.0t)+ 35895713.33 sin(512252.0t)

-37089957.48 sin(544398.0t)+ 38455409.80 sin(576434.0t)

-39978697.92 sin(608364.0t)-41658403.41 sin(640194.0t)

-43517015.88 sin(671938.0t)+ 45541253.86 sin(703586.0t)

-47771451.41 sin(735156.0t)

On the time interval $0 \le t \le 0.005\ s$, the above formula for the stress σ_{xx} can be represented by the following least-square polynomial approximation (in order to smooth out the small oscillations due to truncation of the Fourier series):

$$\sigma_{xx} = -5.16214 \times 10^{11} t - 9.09721 \times 10^{13} t^2 + 3.51795 \times 10^{17} t^3$$

$$-1.26565 \times 10^{20}t^4 + 1.5273 \times 10^{22}t^5 - 5.05268 \times 10^{23}t^6. \tag{5.13.4}$$

The stress σ_{xx} as a function of time, obtained from the finite element analysis, is:

time	sigma_xx (x=0.5, z=0.03)
0.0	0.0
1.0E-4	300590795.9
2.0E-4	-136863991.1
3.0E-4	-285641192.0
4.0E-4	-231020454.7
5.0E-4	-368096028.6
6.0E-4	-310867463.8
7.0E-4	9814421.003
8.0E-4	-438507427.3
9.0E-4	-291994594.9
10.0E-4	-270109686.1
11.0E-4	-44108278.28
12.0E-4	-371968043.5
13.0E-4	-372064327.8
14.0E-4	-252926687.6
15.0E-4	-226476659.7
16.0E-4	-223540794.1
17.0E-4	-114323702.8
18.0E-4	-499221197.3
19.0E-4	-239680400.7
20.0E-4	-253103767.2
21.0E-4	44260144.22
22.0E-4	-367739967.3
23.0E-4	-291313468.1
24.0E-4	34464105.86
25.0E-4	69105029.93
26.0E-4	504973155.0

27.0E-4	181440238.1
28.0E-4	327516616.8
29.0E-4	171564055.1
30.0E-4	465921279.0
31.0E-4	483077118.1
32.0E-4	133516492.0
33.0E-4	9867897.009
34.0E-4	350106295.7
35.0E-4	218206438.9
36.0E-4	478119085.7
37.0E-4	150719904.3
38.0E-4	-53134854.76
39.0E-4	31817117.24
40.0E-4	255088912.6
41.0E-4	313136614.6
42.0E-4	122606780.4
43.0E-4	-87272429.67
44.0E-4	-124044299.8
45.0E-4	86437846.18
46.0E-4	-11355005.43
47.0E-4	83989098.10
48.0E-4	- 97318192.67
49.0E-4	-60958656.28
50.0E-4	-252631951.8

The least-square polynomial approximation of this data, produced by the FE program, is

$$\sigma_{xx} = -4.39327 \times 10^{11} t - 1.27106 \times 10^{14} t^2 +$$

$$+3.42043 \times 10^{17} t^3 - 1.17946 \times 10^{20} t^4 + 1.33736 \times 10^{22} t^5 - 3.57798 \times 10^{23} t^6. \qquad (5.13.5)$$

The graphs of polynomials (5.13.4) and (5.13.5), representing the analytical and FE solutions for stress σ_{xx} as functions of time, are shown in Figure 5.6. These two graphs are sufficiently close to each other.

5.13.2 Comparison of exact and FE solutions for a simply supported sandwich plate with isotropic face sheets and the core

Let us consider a sandwich plate with steel face sheets and an isotropic core made of foam. We assume the following properties of the face sheets and the core:

core: Young's modulus $E_2 = 1.0192 \times 10^8 \frac{N}{m^2}$, Poisson's ratio $\nu = 0.3$, thickness $t = 2 \times 10^{-2} m$, mass density $\rho_c = 2 \times 10^2 \frac{kg}{m^3}$;

face sheets: Young's modulus $E_1 = 1.9796 \times 10^{11} \frac{N}{m^2}$, Poisson's ratio $\nu = 0.3$, thickness of each face sheet $\frac{h}{2} - \frac{t}{2} = 1 \times 10^{-3} m$, mass density $\rho_1 = 7.8 \times 10^3 \frac{kg}{m^3}$.

The total thickness of the plate is $h = 2.2 \times 10^{-2} m$. The plate is under the load $\frac{q_u}{b} = -1 \times 10^5 \frac{N}{m^2}$.

The exact analytical solution for stresses in a static simply supported isotropic sandwich plate, loaded uniformly on the upper surface, has the form of equations (2-E.43) – (2-E.51) of Appendix 2-E. The tables below show the results of comparison of the stresses, obtained for this problem by exact analytical method and by the FE method.

Table 5.4: Comparison of exact and FE solutions for a simply supported sandwich plate with isotropic face sheets and the core for stress σ_{xx} at $x = \frac{L}{2}$. Thickness of the plate is $h = 0.022m$, thickness of each face sheet is $0.001m$, thickness of the core is $t = 0.02m$, length L varies.

L (m)	$\frac{h}{L}$	σ_{xx} at $z = -\frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{z_3+z_4}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$	
		exact	plate theory	exact	plate theory	exact	plate theory
0.05	0.44	1.556	1.555 error0.06 %	−1.484	−1.481 error 0.2 %	−1.556	−1.555 error0.06 %
0.1	0.22	6.222	6.221 error 0.02%	−5.938	−5.922 error 0.3 %	−6.222	6.221 error 0.02%
0.2	0.11	24.887	24.875 error 0.05%	−23.75	−23.69 error 0.25 %	−24.887	−24.875 error 0.05%
0.3	0.07	55.99	55.97 error0.04 %	−53.45	−53.23 error 0.4 %	−55.99	−55.97 error 0.04 %
0.4	0.055	99.54	99.49 error 0.05 %	−95.02	−94.64 error 0.4 %	−99.54	−99.49 error 0.05 %
0.5	0.044	155.5	155.4 error 0.06%	−148.5	−147.91 error 0.4 %	−155.5	−155.4 error 0.06 %
0.6	0.037	223.97	223.75 error 0.1 %	−213.8	−212.74 error 0.5 %	−223.97	−223.75 error 0.1 %
0.7	0.031	304.85	304.69 error 0.05 %	−291.0	−289.3 error 0.6 %	−304.85	−304.69 error 0.05 %
0.8	0.0275	398.2	399.18 error 0.2 %	−380.1	−378.3 error 0.5 %	−398.2	399.18 error 0.2 %
0.9	0.024	503.9	504.5 error 0.1 %	−481.0	−477.5 error 0.7 %	−503.9	504.5 error 0.1 %
1	0.022	622.1	624.4 error 0.4 %	−593.9	−587.55 error 1.1 %	−622.1	−624.4 error 0.4 %
1.1	0.02	752.8	756.6 error 0.5 %	−718.58	−698.7 error 2.8 %	−752.8	−756.6 error 0.5 %
1.2	0.018	895.9	873.2 error 2.5 %	−855.2	−790.85 error 7.5 %	−895.9	−873.2 error 2.5 %

This table shows that the finite element program allows one to achieve a high accuracy of computation of the in-plane stress σ_{xx}. As the thickness-to-length ratio decreases, the accuracy of σ_{xx} computed by the FE program decreases slightly, but remains acceptable for a very wide range of the thickness-to-length ratios. The upper faces are under compression (stress σ_{xx} is negative), and the lower faces are in tension (stress σ_{xx} is positive) as expected.

Table 5.5: Comparison of exact and FE solutions for a simply supported sandwich plate with isotropic face sheets and the core for stress σ_{xx} at $x = \frac{L}{2}$ $(L = 0.5m)$. Thickness of the plate is h=0.022m, thickness of the face sheet τ varies

τ (m)	$\frac{\tau}{h}$	σ_{xx} at $z = -\frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{z_3+z_4}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xx} at $z = \frac{h}{2}$ $(\times 10^6 \frac{N}{m^2})$	
		exact	plate theory	exact	plate theory	exact	plate theory
0.001	0.045	155.5	155.3 error 0.13%	−148.5	−147.68 error 0.55%	−155.5	−155.3 error 0.13%
0.002	0.09	85.60	85.39 error 0.2%	−77.82	−77.48 error 0.4%	−85.60	−85.39 error 0.2%
0.003	0.14	62.94	62.78 error 0.25%	−54.35	−54.14 error 0.4%	−62.94	−62.78 error 0.25%
0.004	0.18	52.18	52.04 error 0.27%	−42.69	−42.52 error 0.4%	−52.18	−52.04 error 0.27%
0.005	0.18	46.245	46.12 error 0.27%	−35.728	−35.64 error 0.25%	−46.245	−46.12 error 0.27%
0.006	0.27	42.76	42.63 error 0.3%	−31.09	−30.97 error 0.4%	−42.76	−42.63 error 0.3%
0.010	0.45	38.78	38.64 error 0.4%	−21.14	−21.07 error 0.3%	−38.78	−38.64 error 0.4%

From the last table we see that as the relative thickness of the face sheets increases, the accuracy of the in-plane stress σ_{xx} decreases slightly, but remains sufficiently high in a wide range of the ratios of the face sheet's thickness to the total thickness of the plate.

Table 5.6: Comparison of exact and FE solutions for a simply supported sandwich plate with isotropic face sheets and the core. Variation of stress σ_{xx} in the thickness direction at $x = L/2$ of a plate with length $L = 1m$, thickness of each face sheet $\tau = 0.001m$, thickness of the core $t = 0.02m$

x (m)	z (m)	σ_{xx} $(\times 10^6 \frac{N}{m^2})$	σ_{xx} $(\times 10^6 \frac{N}{m^2})$
		exact	plate theory
0.5	−0.0110	622.14	619.27 error 0.46%
0.5	−0.0108	610.83	609.51 error 0.22%
0.5	−0.0106	599.52	598.21 error .022%
0.5	−0.0104	588.21	586.91 error 0.22%
0.5	−0.0102	576.9	574.17 error 0.47%
0.5	−0.0100	565.59	564.30 error 0.23%
0.5	−0.009999	0.29119	0.2825 error 3%
0.5	−0.0060	0.1747	0.1662 error 4.9 %
0.5	−0.0020	0.05823	0.0556 error 4.5%
0.5	0.0	0	−0.006
0.5	0.0020	-0.05823	0.0556 error 4.5%
0.5	0.0060	-0.1747	−0.1664 error 4.7%
0.5	0.009999	-0.29119	−0.2836 error 2.6%
0.5	0.0100	-565.59	−565.30 error 0.05%
0.5	0.0102	-576.9	−575.6 error 0.22%
0.5	0.0104	-588.21	−586.91 error 0.22%
0.5	0.0106	-599.52	−598.21 error 0.22%
0.5	0.0108	-610.83	−609.513 error 0.22%
0.5	0.0110	-622.14	−622.44 error 0.05%

This data is shown graphically in Figure 5.7. This comparison shows that the in-plane stress σ_{xx} in the face sheets is computed by the finite element with high accuracy. In the core, the relative error in computation of the stress σ_{xx}, is higher, but is acceptable. The values of the stress σ_{xx} in the core are very low, and this is the reason why the relative error is higher in the core than in the face sheets, despite the fact that the absolute error in the core is small. At the middle surface of the plate ($z = 0$), the exact value of σ_{xx} is equal to zero, and this leads to the infinite relative error at this location regardless of the smallness of the approximate solution. This suggests that if the exact values of stresses are very small, the relative error can be not a good measure of accuracy.

Table 5.7: Comparison of exact and FE solutions for a simply supported sandwich plate with isotropic face sheets and the core for stress σ_{xz} at $x = 0.8L$. Thickness of the plate is $h = 0.022m$, thickness of each face sheet is $0.001m$, length L varies

L (m)	$\frac{h}{L}$	σ_{xz} at $z = \frac{z_1+z_2}{2}$ $(10^6 \frac{N}{m^2})$		σ_{xz} at $z = \frac{z_2+z_3}{2}$ $(10^6 \frac{N}{m^2})$		σ_{xz} at $z = \frac{z_3+z_4}{2}$ $(10^6 \frac{N}{m^2})$	
		exact	plate theory	exact	plate theory	exact	plate theory
0.05	0.44	0.0365	0.0338 error 7.4%	0.0714	0.0660 error 7.5%	0.0365	0.03376 error 7.5%
0.1	0.22	0.0730	0.0675 error 7.5%	0.1429	0.1321 error 7.5%	0.0730	0.0678 error 7.4%
0.2	0.11	0.1459	0.1357 error 7.0%	0.2857	0.2654 error 7.1%	0.1459	0.1382 error 5.3%
0.3	0.07	0.2189	0.2191 error 0.09%	0.4286	0.4289 error 0.07%	0.2189	0.2123 error 3.0%
0.4	0.055	0.2918	0.2725 error 6.6%	0.5715	0.5331 error 6.7%	0.2918	0.2892 error 0.9%
0.5	0.044	0.3648	0.3649 error 0.03%	0.7144	0.7143 error 0.01%	0.3648	0.3625 error 0.6%
0.6	0.037	0.4378	0.4409 error 0.7%	0.8573	0.8630 error 0.7%	0.4378	0.4502 error 2.8%
0.7	0.031	0.5107	0.5172 error 1.3%	1.0001	1.0127 error 1.26%	0.5107	0.5320 error 4.2%
0.8	0.0275	0.5837	0.5681 error 2.7%	1.1430	1.1126 error 2.6%	0.5837	0.5715 error 2.1%
0.9	0.024	0.6566	0.6756 error 2.9%	1.2859	1.3229 error 2.8 %	0.6566	0.6800 error 3.6%
1	0.022	0.7296	0.7367 error 0.9%	1.4288	1.4426 error 1.0%	0.7296	0.7391 error 1.3%
1.1	0.02	0.8026	0.7879 error 1.8%	1.5716	1.5429 error 1.8%	0.8026	0.7902 error 1.5%

The accuracy of computation of the transverse shear stress σ_{xz} is good in the wide range of the thickness-to-length ratios. For very short plates (high thickness-to-length ratios), the relative errors are larger than 7%, despite the fact that the absolute errors are small. This is due to the fact that in short plates the exact values of the stress σ_{xz} are very small, and, as it was mentioned earlier, the relative error in computation of small values can be not a good criterion of accuracy.

Table 5.8: Comparison of exact and FE solutions for a simply supported sandwich plate with isotropic face sheets and the core for stress σ_{xz} at $x = 0.8L$ ($L = 1m$). Thickness of the plate is $h = 0.022m$, thickness of the face sheets τ varies.

τ (m)	$\frac{\tau}{h}$	σ_{xz} at $z = \frac{z_1+z_2}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xz} at $z = \frac{z_2+z_3}{2}$ $(\times 10^6 \frac{N}{m^2})$		σ_{xz} at $z = \frac{z_3+z_4}{2}$ $(\times 10^6 \frac{N}{m^2})$	
		exact	plate theory	exact	plate theory	exact	plate theory
0.001	0.045	0.72961	0.7367 error 1.0%	1.4288	1.4426 error 1.0%	0.72961	0.7391 error 1.3%
0.002	0.09	0.78439	0.7986 error 1.8%	1.4956	1.5227 error 1.8%	0.78439	0.8051 error 2.6%
0.003	0.14	0.84451	0.8662 error 2.6%	1.5663	1.6068 error 2.3%	0.84451	0.8148 error 3.5%
0.004	0.18	0.91077	0.9401 error 3.2%	1.64	1.6949 error 3.3%	0.91077	0.9512 error 4.4 %
0.005	0.18	0.98365	0.9985 error 1.5%	1.7154	1.7405 error 1.5%	0.98365	1.0299 error 4.7%
0.006	0.27	1.0634	1.0723 error 0.84%	1.7912	1.8010 error 0.54%	1.0634	1.1123 error 4.6%
0.010	0.45	1.438	1.4767 error 2.7%	2.0301	2.0843 error 2.7%	1.438	1.4895 error 3.4%

The accuracy of computation of stress σ_{xz} is good for a wide range of ratios of the face sheet's thickness to the total thickness of the plate. The closer to the upper surface of the plate, the lower the accuracy. This is due to the fact that expressions for the stress σ_{xz} in the face sheets and the core are found by integration of equilibrium equations:

$$\sigma_{xz}^{(1)} = \underbrace{\sigma_{xz}^{(1)}(z_1)}_{0} + \int\limits_{z_1}^{z} \left(\rho^{(1)}\ddot{u}^{(1)} - {}^{H}\sigma_{xx,x}^{(1)} - {}^{H}\sigma_{xy,y}^{(1)}\right) dz \quad (z_1 \le z \le z_2), \tag{5.13.6}$$

$$\sigma_{xz}^{(2)} = \sigma_{xz}^{(2)}(z_2) + \int\limits_{z_2}^{z} \left(\rho^{(1)}\ddot{u}^{(2)} - {}^{H}\sigma_{xx,x}^{(2)} - {}^{H}\sigma_{xy,y}^{(2)}\right) dz \quad (z_2 \le z \le z_3), \tag{5.13.7}$$

$$\sigma_{xz}^{(3)} = \sigma_{xz}^{(3)}(z_3) + \int\limits_{z_3}^{z} \left(\rho^{(1)}\ddot{u}^{(3)} - {}^{H}\sigma_{xx,x}^{(3)} - {}^{H}\sigma_{xy,y}^{(3)}\right) dz \quad (z_3 \le z \le z_4), \tag{5.13.8}$$

The integration is performed in the direction from the lower surface to the upper surface. This leads to exact satisfaction of the boundary condition at the lower surface ($\sigma_{xz}^{(1)}(z_1) = 0$) regardless of

accuracy of computation of the in-plane stresses, and to approximate satisfaction of the boundary conditions at the upper surface $\left(\sigma_{xz}^{(2)}(z_4) = 0\right)$, if the in-plane stresses are computed approximately. Therefore, the accuracy of computation of the transverse stresses σ_{xz} deteriorates slightly as the observation point moves from the lower surface to the upper surface of the plate. Besides, the accuracy of the transverse stress σ_{xz} computation is lower than the accuracy of the in-plane stress computation. This is due to the fact that the computation of the transverse stresses by the integration of equilibrium equations requires computation of the derivatives of the field variables of the order higher than the degree of the interpolation polynomials. This is done by a finite difference scheme applied to the nodal values of the field variables. But with the increase in the order of a derivative, the accuracy of numerical differentiation is reduced. To overcome this deterioration of accuracy of computation of the higher order derivatives, a large number of elements must be used. The same is true for the transverse stress σ_{zz}, as will be seen in the subsequent text.

Table 5.9: Comparison of exact and FE solutions for a simply supported sandwich plate with isotropic face sheets and the core. Variation of stress σ_{xz} in the thickness direction for a plate with length $L = 1m$, thickness of each face sheet $\tau = 0.001m$, thickness of the core $t = 0.02m$, at $x = 0.8L$.

x (m)	z (m)	σ_{xz} $(\times 10^6 \frac{N}{m^2})$ exact	σ_{xz} $(\times 10^6 \frac{N}{m^2})$ plate theory
0.8	−0.0110	0	0
0.8	−0.0108	0.29591	0.2988 error 1%
0.8	−0.0106	0.58640	0.5921 error 1%
0.8	−0.0104	0.87145	0.8799 error 1%
0.8	−0.0102	1.1511	1.1622 error 1%
0.8	−0.0100	1.4253	1.4390 error 1%
0.8	−0.009999	1.4253	1.4390 error 1%
0.8	−0.0060	1.4275	1.4413 error 1%
0.8	−0.0020	1.4286	1.4424 error 1%
0.8	0.0	1.4288	1.4426 error 1%
0.8	0.0020	1.4286	1.4424 error 1%
0.8	0.0060	1.4275	1.4413 error 1%
0.8	0.009999	1.4253	1.4390 error 1%
0.8	0.0100	1.4253	1.4426 error 1%
0.8	0.0102	1.1511	1.1654 error 1.2%
0.8	0.0104	0.87145	0.8826 error 1.3%
0.8	0.0106	0.5864	0.5943 error 1.3%
0.8	0.0108	0.29591	0.3005 error 1.5%
0.8	0.011	0.0	0.0012

This data is shown graphically in Figure 5.8. This comparison shows that the through-the-

thickness variation of the stress σ_{xz} is accurately computed by the FE program. The accuracy of the stress σ_{xz} computation deteriorates slightly as the observation point moves from the lower surface to the upper surface for the reason mentioned above. Besides, the accuracy of the stress σ_{xz} is somewhat lower than the accuracy of the in-plane stress σ_{xx}. The reason of this (as it was mentioned above) is the need to evaluate the higher order derivatives of the field variables by a finite difference scheme in order to compute the stress σ_{xz}.

Table 5.10: Comparison of exact and FE solutions for a simply supported sandwich plate with isotropic face sheets and the core for stress σ_{zz} at $x = L/2$. Thickness of the plate is $h = 0.022m$, thickness of each face sheet is $0.001m$, length L varies

L (m)	$\frac{h}{L}$	σ_{zz} at $z = -\frac{h}{2}$ $(\times 10^5 \frac{N}{m^2})$		σ_{zz} at $z = \frac{z_2+z_3}{2}$ $(\times 10^5 \frac{N}{m^2})$		σ_{zz} at $z = \frac{h}{2}$ $(\times 10^5 \frac{N}{m^2})$	
		exact	plate theory	exact	plate theory	exact	plate theory
0.05	0.44	0	0	-0.5	-0.5004 error 0.08%	-1	-1.0466 error 4.7%
0.1	0.22	0	0	-0.5	-0.4997 error 0.06%	-1	-1.0451 error 4.5%
0.2	0.11	0	0	-0.5	-0.5077 error 1.54%	-1	-1.0413 error 4.1%
0.3	0.07	0	0	-0.5	-0.5043 error 0.9%	-1	-1.0438 error 4.4%
0.4	0.055	0	0	-0.5	-0.5162 error 3.2%	-1	-1.0490 error 4.9%
0.5	0.044	0	0	-0.5	-0.4862 error 2.8%	-1	-1.0162 error 1.6%
0.6	0.037	0	0	-0.5	-0.4993 error 0.1%	-1	-1.0435 error 4.35%
0.7	0.031	0	0	-0.5	-0.4942 error 1.2%	-1	-1.0329 error 3.29%
0.8	0.0275	0	0	-0.5	-0.5147 error 2.9%	-1	-1.0457 error 4.6%
0.9	0.024	0	0	-0.5	-0.5026 error 0.5%	-1	-1.0406 error 4.1%
1	0.022	0	0	-0.5	-0.4859 error 2.8%	-1	-1.0156 error 1.6%
1.1	0.02	0	0	-0.5	-0.4953 error 0.9%	-1	-1.0352 error 3.5%
1.2	0.018	0	0	-0.5	-0.5001 error 0.02%	-1	-1.0453 error 4.5%

This comparison shows the following tendencies: the thickness-to-length ratio has little influence on the accuracy of the stress σ_{zz} computation; the accuracy decreases as the observation point moves from the lower surface to the upper surface (the reason of this was discussed above);

Table 5.11: Comparison of exact and FE solutions for a simply supported sandwich plate with isotropic face sheets and the core for stress σ_{zz} at $x = L/2$ (L=1m). Thickness of the plate is $h = 0.022m$, thickness τ of each face sheet varies

τ (m)	$\frac{\tau}{h}$	σ_{zz} at $z = -\frac{h}{2}$ $(\times 10^5 \frac{N}{m^2})$		σ_{zz} at $z = \frac{z_2+z_3}{2}$ $(\times 10^5 \frac{N}{m^2})$		σ_{zz} at $z = \frac{h}{2}$ $(\times 10^5 \frac{N}{m^2})$	
		exact	plate theory	exact	plate theory	exact	plate theory
0.001	0.045	0	0	-0.5	-0.4859 error 2.8%	-1	-1.0156 error 1.56%
0.002	0.09	0	0	-0.5	-0.4932 error 1.4%	-1	-0.9855 error 1.45%
0.003	0.14	0	0	-0.5	-0.5031 error 0.6%	-1	-1.024 error 2.4%
0.004	0.18	0	0	-0.5	-0.4879 error 2.4%	-1	-1.031 error 3.1%
0.005	0.18	0	0	-0.5	-0.5003 error 0.06%	-1	-1.039 error 3.9%
0.006	0.27	0	0	-0.5	-0.4845 error 3.1%	-1	-1.046 error 4.6%
0.010	0.45	0	0	-0.5	-0.4849 error 3.0%	-1	-1.071 error 7.1%

So, the accuracy is higher for the plates with thinner face sheets.

Table 5.12: Comparison of exact and FE solutions for a simply supported sandwich plate with isotropic face sheets and the core. Variation of stress σ_{zz} in the thickness direction of a plate with length $L = 1m$, thickness $h = 0.022m$, thickness of each face sheet $\tau = 0.001m$, thickness of the core $t = 0.02m$.

x (m)	z (m)	σ_{zz} $\left(\frac{N}{m^2}\right)$	
		exact	plate theory
0.5	−0.0110	0	0
0.5	−0.0108	−98.94	−96.13 error 2.8%
0.5	−0.0106	−393.35	−382.18 error 2.8%
0.5	−0.0104	−879.6	−854.65 error 2.8%
0.5	−0.0102	−1554.1	1510.02 error 2.8%
0.5	−0.0100	−2413.2	−2344.78 error 2.8%
0.5	−0.009999	−2417.9	−2354.90 error 2.5 %
0.5	−0.0060	-2.1433×10^4	-2.0832×10^4 error 2.8%
0.5	−0.0020	-4.0475×10^4	-3.9335×10^4 error 2.8 %
0.5	0.0	-5.0×10^4	-4.8590×10^4 error 2.8%
0.5	0.0020	-5.9525×10^4	-5.7846×10^4 error 2.8%
0.5	0.0060	-7.8567×10^4	-7.6349×10^4 error 2.8%
0.5	0.009999	-9.7582×10^4	-9.4826×10^4 error 2.8%
0.5	0.0100	-9.7587×10^4	-9.4837×10^4 error 2.8%
0.5	0.0102	-9.8446×10^4	-9.5641×10^4 error 2.8 %
0.5	0.0104	-9.9120×10^4	-9.6266×10^4 error 2.9%
0.5	0.0106	-9.9607×10^4	-9.6708×10^4 error 2.9%
0.5	0.0108	-9.9901×10^4	-9.6963×10^4 error 2.9%
0.5	0.011	-1.0×10^5	-1.0156×10^5 error 1.6 %

This data is shown graphically in Figure 5.9. The accuracy of computation is sufficiently high.

The comparison of the exact and finite element solutions made in this section, shows that 1) the simplified layerwise theory of the sandwich plate, developed in this chapter, leads to sufficiently high accuracy of stress computation for a wide range of geometric dimensions; 2) the finite element program developed on the basis of the simplified layerwise theory of the sandwich plates in cylindrical bending is a reliable tool for analysis of the sandwich plates if the conditions of cylindrical bending are met.

In the next section, this finite element program will be applied to stress and failure analysis of a composite cargo platform dropped on elastic foundation. It will be assumed that the conditions that allow the platform to be in cylindrical bending, are met.

5.14 An Example Problem: Finite Element Analysis, with Account of Damage Progression, of a Composite Sandwich Cargo Platform Dropped on Elastic Foundation

Let us consider a sandwich platform with laminated composite face sheets, made of AS4/3501-6 material, and a honeycomb core, made of Nomex HRH10-1/8-4.0. Both face sheets have the same thickness 0.0025m, and each of them consists of 25 plies with $0^0/90^0$ layup. The thickness of the core is 0.04m. The cargo of mass 500 kg on the upper surface is located symmetrically with respect to the middle of the plate's span, and has the length 0.2m. The moduli of the elastic Winkler foundations, considered in the example problems, are $6.7864 \times 10^7 \frac{N}{m^3}$ and $6.7864 \times 10^8 \frac{N}{m^3}$. We will consider a plate falling on the elastic foundation with the initial velocities $-1\frac{m}{s}$ and $-30\frac{m}{s}$. The values of coefficients α_1 and α_2 in the proportional damping matrix $[C] = \alpha_1\,[\mathrm{K}] + \alpha_2\,[\mathrm{M}]$ were chosen to be $\alpha_1 = 0.002$, $\alpha_2 = 0.2$. In this example problem we will compute all stresses as functions of time at the middle of the plate's span (i.e. at $x = \frac{L}{2} = 0.5$m) and at the plate's lower surface (i.e. at $x = -\frac{h}{2} = -0.0225$).

First, a nonlinear dynamic finite element analysis will be performed and a comparison will be made of stresses and the transverse displacement, obtained from the finite element program with damage analysis capability activated and deactivated, with different initial velocities. The input data (in SI units) can be summarized as follows:

Number of elements in FE mesh........= 40

Panel length ....= 1.00

Panel width ....= 5.00

Total number of nodes in FE mesh.....= 41

Number of DOF per node ....= 5

Number of plies in each face ..= 25

Number of core plies ....= 10

Face ply material properties	Core ply material properties
E1 = .145E+12	E1 = 0.804E+08
E2 = .970E+10	E2 = 0.804E+08
E3 = .970E+10	E3 = 0.101E+10

G12 = .600E+10	G12 = 0.322E+08
G13 = .600E+10	G13 = 0.120E+09
G23 = .360E+10	G23 = 0.758E+11
Nu12 = .300E+00	Nu12 = 0.250E+00
Nu13 = .300E+00	Nu13 = 0.200E-01
Nu23 = .300E+00	Nu23 = 0.200E-01
XT = .217E+10	XT = 0.100E+07
XC = .172E+10	XC = 0.100E+07
YT = .538E+08	YT = 0.100E+07
YC = .206E+09	YC = 0.100E+07
ZT = .538E+08	ZT = 0.383E+07
ZC = .206E+09	ZC = 0.383E+07
S12 = .121E+09	S12 = 0.178E+09
S13 = .121E+09	S13 = 0.178E+09
S23 = .893E+08	S23 = 0.142E+09

Face mass density = 0.161E+04

Face thickness = 0.100E-03

Core mass density = 0.139E+03

Core thickness = 0.400E-01

Rigid body mass = 0.500E+03

Coordinates of the beginning and the end of the cargo:

X1 = 0.4

X2 = 0.6

Foundation modulus = 0.679E+08

Time increment ....= 0.10000E-03

Total time ....= 0.40000E-01

Initial displacement ..= 0.00000E+00

Initial velocity...........= -1.0

Initial acceleration ...= 0.98100E+01

Parameters of proportional damping matrix:

Alpha1 ....= .20000E-02

Alpha2 ....= .20000E+00

Parameters of the Newmark method:

Gamma ....= 0.5

Beta ....= 0.25

Figures 5.10-5.14 show results of analysis with initial velocity $-1\frac{m}{s}$ and a foundation modulus $6.7864 \times 10^7 \frac{N}{m^3}$ (sand). In this case no damage occurs, therefore, the graphs of the stresses and the transverse displacement, computed with and without account of damage, coincide. Figure 5.13 shows the transverse displacements of the upper and lower surfaces as a function of time. In the first half-period, the absolute value of the transverse displacement of the upper surface is larger than the absolute value of the transverse displacement of the lower surface, that means that in the first half-period the thickness of the plate is smaller than its thickness in the undeformed state. In the second half-period the thickness of the plate is larger than its thickness in the undeformed state. This change of the plate's thickness was captured due to the fact that the direct transverse strain ε_{zz} was not assumed to be equal to zero.

Figures 5.15–5.18 show stresses and the transverse displacement in the platform that has initial velocity $-30\frac{m}{s}$ and falls on the same elastic foundation (with modulus $6.7864 \times 10^7 \frac{N}{m^2}$). Under this initial velocity the damage in the plate occurs at the moment of time $t = 0.14 \times 10^{-2} s$ (Figure 5.20). In the finite elements, that are located directly under the mass on the upper surface (for example the element #11, Figure 5.19) the damage occurs in both the core and the face sheets. The picture of damage progression in the thickness direction of the eleventh element is shown in Figure 5.20. We see that the failure of the core occurs first,and this failure is due to the vertical compression (crushing) of the core. This is due to the fact that the compression strength of the Nomex core in the thickness direction is the lowest as compared to all other strength characteristics of the faces and the core. At the moment of time $t = 0.14 \times 10^{-2} s$, when the damage starts to progress, there also occurs the tensile matrix failure in the ply of the lower face that is adjacent to the core.

As the failure in the 11-th element progresses with time in the thickness direction, the plies in the lower face sheets with 90-degree orientation experience the tensile matrix failure. (Figure 5.20). This occurs mainly due to the tensile (positive) stress σ_{xx} (σ_{22} for the plies with 90-degree fiber orientation) in the plies that are closer to the lower surface.

As the failure in the 11-th element progresses further, the fiber failure in compression occurs in the plies of the upper face with the 0-degree fiber orientation. This mode of failure starts closer to the upper surface and progresses downward as the compressive stress σ_{xx} (responsible for this mode of failure according to the criterion (5.2.14)) increases with time

The next mode of failure is the fiber failure in tension that occurs (according to the criterion (5.12.12)) in the lower face sheet in the plies with the 0-degree fiber orientation that are closer to the lower surface. This mode of failure occurs closer to the lower surface, that suggests that it is mainly due to the tensile stress σ_{xx}, but the stress σ_{xz} also contributes to the breakage of the fibers.

The last mode of failure in the 11-th element is matrix failure in compression (matrix crushing) in the 90-degree plies of the upper face sheet, that is predicted by the criterion (5.2.19)

The graphs in Figures 5.15 –5.18 show stresses and the transverse displacement in the 11-th element of the plate dropped on the sand foundation, computed with and without account of damage, in order to study the changes in the structural response due to the damage progression. When the failure of the face sheets and the core occurs and begins to progress, the stress σ_{xx} in the lower face sheet, in a finite element that contains the damaged face sheet (at a point $x = L/2$, $z = -h/2$, Figure 5.15), begins reducing rapidly with time until it reaches the zero value. This result is expected, since in this problem there are no external forces in the x-direction, acting on the plate. The stress σ_{xx} in the face sheets is due to the strains that appear in the face sheets because of bending, and is computed from the constitutive equations. Therefore, if the values of the stiffness coefficients in the constitutive equations reduce because of fiber failure in tension, the stress σ_{xx} also reduces.

The amplitude of stress σ_{zz} does not change significantly when the failure occurs, because it depends mainly on the external forces in z-direction, that do not change abruptly when the failure occurs. But the amplitudes of the stress σ_{zz} in the presence of damage (Figures 5.16) shift in the graphs to right, because the frequencies of vibration decrease, due to the decrease of the plate's stiffness.

The graphs of stress σ_{yy} as a function of time have the same shape as the corresponding graphs of the stress σ_{xx}, but the values of σ_{yy} are much lower than the values of σ_{xx} at the same moments of time. This can be explained as follows. From the constitutive equations (3.6.13), we receive the following stress-strain relations in case of plane strain (cylindrical bending) :

$$\sigma_{xx} = \overline{C}_{11}\varepsilon_{xx} + \overline{C}_{13}\varepsilon_{zz} \ , \tag{5.14.1}$$

$$\sigma_{zz} = \overline{C}_2\varepsilon_{xx} + \overline{C}_{33}\varepsilon_{zz} \ , \tag{5.14.2}$$

$$\sigma_{yy} = \overline{C}_{12}\varepsilon_{xx} + \overline{C}_{23}\varepsilon_{zz} \ . \tag{5.14.3}$$

If we express strains ε_{xx} and ε_{zz} in terms of stresses from equations (5.14.1) and (5.14.2), and substitute the resulting equations into the equation (5.14.3), we receive

$$\sigma_{yy} = \frac{1}{\overline{C}_{11}\overline{C}_{33} - \overline{C}_{13}^2}\left[\left(\overline{C}_{12}\overline{C}_{33} - \overline{C}_{23}\overline{C}_{13}\right)\sigma_{xx} + \left(\overline{C}_{23}\overline{C}_{11} - \overline{C}_{12}\overline{C}_{13}\right)\sigma_{zz}\right]. \tag{5.14.4}$$

In a ply with zero-degree fiber orientation, according to equations (3.6.4) – (3.6.9),

$$\left(\overline{C}_{12}\overline{C}_{33} - \overline{C}_{23}\overline{C}_{13}\right) =$$

$$= \frac{\left(\nu_{12}E_2 + \nu_{23}\nu_{13}E_3\right)\left(E_1 - \nu_{12}^2E_2\right)E_1E_2^2E_3 - \left(\nu_{23}E_1 + \nu_{13}\nu_{12}E_2\right)\left(\nu_{12}\nu_{23} + \nu_{13}\right)E_1E_2^2E_3^2}{\left(E_2E_1 - E_1\nu_{23}^2E_3 - \nu_{12}^2E_2^2 - 2\nu_{12}E_2\nu_{23}\nu_{13}E_3 - \nu_{13}^2E_2E_3\right)^2} \tag{5.14.5}$$

and

$$\left(\overline{C}_{23}\overline{C}_{11} - \overline{C}_{12}\overline{C}_{13}\right) =$$

$$= \frac{\left(\nu_{23}E_1 + \nu_{13}\nu_{12}E_2\right)\left(E_2 - \nu_{23}^2E_3\right)E_1^2E_2E_3 - \left(\nu_{12}E_2 + \nu_{23}\nu_{13}E_3\right)\left(\nu_{12}\nu_{23} + \nu_{13}\right)E_1^2E_2^2E_3}{\left(E_2E_1 - E_1\nu_{23}^2E_3 - \nu_{12}^2E_2^2 - 2\nu_{12}E_2\nu_{23}\nu_{13}E_3 - \nu_{13}^2E_2E_3\right)^2} \tag{5.14.6}$$

Therefore, in equation (5.14.6), coefficients of σ_{xx} and σ_{zz} are of the same order of magnitude, but the stress σ_{zz} on the lower surface of the plate is much lower than the stress σ_{xx} , according to Figures 5.15 and 5.16. Therefore, according to equation (5.14.4), the stress σ_{yy} is proportional to the stress σ_{xx}.

The transverse displacement w (Figure 5.18), computed with account of damage, has larger amplitudes than w, computed without account of damage, that is expected, because the damage leads to reduction of the plate's stiffness.

Figures 5.22–5.25 show the stresses and the transverse displacement of the plate that falls with the same initial velocity on the elastic foundation with a higher modulus. Comparing the graphs of Figures 5.22 and 5.15, we see that with the increase of the modulus of elastic foundation, the stress

σ_{xx} decreases, that is expected because the plate on the elastic foundation with higher modulus has smaller curvature. The stress σ_{zz} in the plate, falling on a stiffer foundation, is higher (Figures 5.23 and 5.16), as expected, because deceleration of the plate interacting with the stiffer foundation occurs at a higher rate, that leads to the larger forces of interaction of the plate with the cargo and with the foundation. The transverse displacement of the plate on the stiffer foundation is lower (compare Figures 5.25 and 5.18). When the plate falls on the stiffer foundation, the modes of failure and the sequence of occurrence of failure in time are approximately the same, but the failure begins earlier in time, as can be seen by comparing the Figure 5.26 to 5.20, Figure 5.27 to 5.21.

The developed finite element program allows to perform both linear and nonlinear analyses, based on linear strain-displacement relations and the von-Karman strain-displacement relations. Therefore, it is interesting to compare the stresses and displacements obtained from these two kinds of analysis. The question of appropriateness of such a comparison is discussed in Appendix 5-F. The results based on linear and nonlinear analyses (Figures 5.28 – 5.31) are somewhat different: the nonlinear analysis predicts a higher rate of decrease of the stresses σ_{xx} and σ_{yy} due to the failure and slightly higher amplitudes of the stress σ_{zz} and the transverse displacement w.

5.15 Appendix 5-A. Components of the stiffness matrix of the linearly formulated problem

Matrix $\left[k^{(1)}\right] = \left[k^{(1)}\right]^T$ in the first term of expression (5.10.35) has the following components:

$$k_{11}^{(1)} = \tfrac{12}{l^3} D_{22}^{(1)}, \quad k_{12}^{(1)} = \tfrac{6}{l^2} D_{22}^{(1)}, \quad k_{13}^{(1)} = -\tfrac{12}{l^3} D_{22}^{(1)}, \quad k_{14}^{(1)} = \tfrac{6}{l^2} D_{22}^{(1)},$$
$$k_{15}^{(1)} = 0, \; k_{16}^{(1)} = 0, \quad k_{17}^{(1)} = -6 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^3},$$
$$k_{18}^{(1)} = -3 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^2}, \quad k_{19}^{(1)} = 6 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^3}, \quad k_{1,10}^{(1)} = -3 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^2},$$

$$k_{22}^{(1)} = \tfrac{4}{l} D_{22}^{(1)}, \quad k_{23}^{(1)} = -\tfrac{6}{l^2} D_{22}^{(1)}, \quad k_{24}^{(1)} = \tfrac{2}{l} D_{22}^{(1)}, \quad k_{25}^{(1)} = -2 \tfrac{z_2}{l} D_{12}^{(1)},$$
$$k_{26}^{(1)} = 2 \tfrac{z_2}{l} D_{12}^{(1)}, \quad k_{27}^{(1)} = -3 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^2}, \quad k_{28}^{(1)} = -2 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l},$$
$$k_{29}^{(1)} = 3 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^2}, \quad k_{2,10}^{(1)} = -z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l},$$

$$k_{33}^{(1)} = \tfrac{12}{l^3} D_{22}^{(1)}, \quad k_{34}^{(1)} = -\tfrac{6}{l^2} D_{22}^{(1)}, \quad k_{35}^{(1)} = 0, \quad k_{36}^{(1)} = 0,$$
$$k_{37}^{(1)} = 6 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^3}, \quad k_{38}^{(1)} = 3 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^2},$$
$$k_{39}^{(1)} = -6 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^3}, \quad k_{3,10}^{(1)} = 3 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^2},$$

$$k_{44}^{(1)} = \tfrac{4}{l} D_{22}^{(1)}, \quad k_{45}^{(1)} = 2 \tfrac{z_2}{l} D_{12}^{(1)}, \quad k_{46}^{(1)} = -2 \tfrac{z_2}{l} D_{12}^{(1)}, \quad k_{47}^{(1)} = -3 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^2},$$
$$k_{48}^{(1)} = -z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l}, \quad k_{49}^{(1)} = 3 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l^2}, \quad k_{4,10}^{(1)} = -2 z_2 \tfrac{z_2 D_{12}^{(1)} - 2 D_{22}^{(1)}}{l},$$

$$k_{55}^{(1)} = \tfrac{4}{l} z_2^2 D_{11}^{(1)}, \quad k_{56}^{(1)} = -\tfrac{4}{l} z_2^2 D_{11}^{(1)}, \quad k_{57}^{(1)} = 0, \quad k_{58}^{(1)} = -z_2^2 \tfrac{-z_2 D_{11}^{(1)} + 2 D_{12}^{(1)}}{l},$$
$$k_{59}^{(1)} = 0, \quad k_{5,10}^{(1)} = z_2^2 \tfrac{-z_2 D_{11}^{(1)} + 2 D_{12}^{(1)}}{l},$$

$$k_{66}^{(1)} = \tfrac{4}{l} z_2^2 D_{11}^{(1)}, \quad k_{67}^{(1)} = 0, \quad k_{68}^{(1)} = z_2^2 \tfrac{-z_2 D_{11}^{(1)} + 2 D_{12}^{(1)}}{l}, \quad k_{69}^{(1)} = 0,$$
$$k_{6,10}^{(1)} = -z_2^2 \tfrac{-z_2 D_{11}^{(1)} + 2 D_{12}^{(1)}}{l},$$

$$k_{77}^{(1)} = -3 z_2^2 \tfrac{-z_2^2 D_{11}^{(1)} + 4 z_2 D_{12}^{(1)} - 4 D_{22}^{(1)}}{l^3}, \quad k_{78}^{(1)} = -\tfrac{3}{2} z_2^2 \tfrac{-z_2^2 D_{11}^{(1)} + 4 z_2 D_{12}^{(1)} - 4 D_{22}^{(1)}}{l^2},$$
$$k_{79}^{(1)} = 3 z_2^2 \tfrac{-z_2^2 D_{11}^{(1)} + 4 z_2 D_{12}^{(1)} - 4 D_{22}^{(1)}}{l^3}, \quad k_{7,10}^{(1)} = -\tfrac{3}{2} z_2^2 \tfrac{-z_2^2 D_{11}^{(1)} + 4 z_2 D_{12}^{(1)} - 4 D_{22}^{(1)}}{l^2},$$

$$k_{88}^{(1)} = -\tfrac{z_2^2}{l} \left(-z_2^2 D_{11}^{(1)} + 4 z_2 D_{12}^{(1)} - 4 D_{22}^{(1)} \right), \quad k_{89}^{(1)} = \tfrac{3}{2} z_2^2 \tfrac{-z_2^2 D_{11}^{(1)} + 4 z_2 D_{12}^{(1)} - 4 D_{22}^{(1)}}{l^2},$$
$$k_{8,10}^{(1)} = -\tfrac{1}{2} \tfrac{z_2^2}{l} \left(-z_2^2 D_{11}^{(1)} + 4 z_2 D_{12}^{(1)} - 4 D_{22}^{(1)} \right),$$

$k_{99}^{(1)} = -3z_2^2 \frac{-z_2^2 D_{11}^{(1)} + 4z_2 D_{12}^{(1)} - 4D_{22}^{(1)}}{l^3}, \quad k_{9,10}^{(1)} = \frac{3}{2} z_2^2 \frac{-z_2^2 D_{11}^{(1)} + 4z_2 D_{12}^{(1)} - 4D_{22}^{(1)}}{l^2},$

$k_{10,10}^{(1)} = -\frac{z_2^2}{l} \left(-z_2^2 D_{11}^{(1)} + 4z_2 D_{12}^{(1)} - 4D_{22}^{(1)} \right).$

Matrix $\left[k^{(3)}\right] = \left[k^{(3)}\right]^T$ in the first term of expression (5.10.37) has the following components:

$k_{11}^{(3)} = \frac{12}{l^3} D_{22}^{(3)}, \quad k_{12}^{(3)} = \frac{6}{l^2} D_{22}^{(3)}, \quad k_{13}^{(3)} = -\frac{12}{l^3} D_{22}^{(3)}, \quad k_{14}^{(3)} = \frac{6}{l^2} D_{22}^{(3)},$
$k_{15}^{(3)} = 0, \; k_{16}^{(3)} = 0, \quad k_{17}^{(3)} = -6z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^3},$
$k_{18}^{(3)} = -3z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^2}, \; k_{19}^{(3)} = 6z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^3}, \quad k_{1,10}^{(3)} = -3z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^2},$

$k_{22}^{(3)} = \frac{4}{l} D_{22}^{(3)}, \quad k_{23}^{(3)} = -\frac{6}{l^2} D_{22}^{(3)}, \quad k_{24}^{(3)} = \frac{2}{l} D_{22}^{(3)}, \quad k_{25}^{(3)} = -2\frac{z_3}{l} D_{12}^{(3)},$
$k_{26}^{(3)} = 2\frac{z_3}{l} D_{12}^{(3)}, \quad k_{27}^{(3)} = -3z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^2}, \quad k_{28}^{(3)} = -2z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l},$
$k_{29}^{(3)} = 3z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^2}, \quad k_{2,10}^{(3)} = -z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l},$

$k_{33}^{(3)} = \frac{12}{l^3} D_{22}^{(3)}, \quad k_{34}^{(3)} = -\frac{6}{l^2} D_{22}^{(3)}, \quad k_{35}^{(3)} = 0, \quad k_{36}^{(3)} = 0,$
$k_{37}^{(3)} = 6z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^3}, \quad k_{38}^{(3)} = 3z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^2},$
$k_{39}^{(3)} = -6z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^3}, \quad k_{3,10}^{(3)} = 3z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^2} \;,$

$k_{44}^{(3)} = \frac{4}{l} D_{22}^{(3)}, \quad k_{45}^{(3)} = 2\frac{z_3}{l} D_{12}^{(3)}, \quad k_{46}^{(3)} = -2\frac{z_3}{l} D_{12}^{(3)}, \quad k_{47}^{(3)} = -3z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^2},$
$k_{48}^{(3)} = -z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l}, \quad k_{49}^{(3)} = 3z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l^2}, \quad k_{4,10}^{(3)} = -2z_3 \frac{z_3 D_{12}^{(3)} - 2D_{22}^{(3)}}{l},$

$k_{55}^{(3)} = \frac{4}{l} z_3^2 D_{11}^{(3)}, \quad k_{56}^{(3)} = -\frac{4}{l} z_3^2 D_{11}^{(3)}, \quad k_{57}^{(3)} = 0, \quad k_{58}^{(3)} = -z_3^2 \frac{-z_3 D_{11}^{(3)} + 2D_{12}^{(3)}}{l},$
$k_{59}^{(3)} = 0, \quad k_{5,10}^{(3)} = z_3^2 \frac{-z_3 D_{11}^{(3)} + 2D_{12}^{(3)}}{l},$

$k_{66}^{(3)} = \frac{4}{l} z_3^2 D_{11}^{(3)}, \quad k_{67}^{(3)} = 0, \quad k_{68}^{(3)} = z_3^2 \frac{-z_3 D_{11}^{(3)} + 2D_{12}^{(3)}}{l}, \quad k_{69}^{(3)} = 0,$
$k_{6,10}^{(3)} = -z_3^2 \frac{-z_3 D_{11}^{(3)} + 2D_{12}^{(3)}}{l},$

$k_{77}^{(3)} = -3z_3^2 \frac{-z_3^2 D_{11}^{(3)} + 4z_3 D_{12}^{(3)} - 4D_{22}^{(3)}}{l^3}, \quad k_{78}^{(3)} = -\frac{3}{2} z_3^2 \frac{-z_3^2 D_{11}^{(3)} + 4z_3 D_{12}^{(3)} - 4D_{22}^{(3)}}{l^2},$
$k_{79}^{(3)} = 3z_3^2 \frac{-z_3^2 D_{11}^{(3)} + 4z_3 D_{12}^{(3)} - 4D_{22}^{(3)}}{l^3}, \quad k_{7,10}^{(3)} = -\frac{3}{2} z_3^2 \frac{-z_3^2 D_{11}^{(3)} + 4z_3 D_{12}^{(3)} - 4D_{22}^{(3)}}{l^2},$

$k_{88}^{(3)} = -\frac{z_3^2}{l} \left(-z_3^2 D_{11}^{(3)} + 4z_3 D_{12}^{(3)} - 4D_{22}^{(3)} \right), \quad k_{89}^{(3)} = \frac{3}{2} z_3^2 \frac{-z_3^2 D_{11}^{(3)} + 4z_3 D_{12}^{(3)} - 4D_{22}^{(3)}}{l^2},$

$$k_{8,10}^{(3)} = -\frac{1}{2}\frac{z_3^2}{l}\left(-z_3^2 D_{11}^{(3)} + 4z_3 D_{12}^{(3)} - 4D_{22}^{(3)}\right),$$

$$k_{99}^{(3)} = -3z_3^2\frac{-z_3^2 D_{11}^{(3)} + 4z_3 D_{12}^{(3)} - 4D_{22}^{(3)}}{l^3}, \quad k_{9,10}^{(3)} = \frac{3}{2}z_3^2\frac{-z_3^2 D_{11}^{(3)} + 4z_3 D_{12}^{(3)} - 4D_{22}^{(3)}}{l^2},$$

$$k_{10,10}^{(3)} = -\frac{z_3^2}{l}\left(-z_3^2 D_{11}^{(3)} + 4z_3 D_{12}^{(3)} - 4D_{22}^{(3)}\right).$$

Matrix $\left[\hat{k}^{(2)}\right] = \left[\hat{k}^{(2)}\right]^T$ in the first term of expression (5.10.42) has the following components:

$$\hat{k}^{(2)}_{11} = \tfrac{12}{l^3}\hat{D}^{(2)}_{22},\ \hat{k}^{(2)}_{12} = \tfrac{6}{l^2}\hat{D}^{(2)}_{22},\ \hat{k}^{(2)}_{13} = -\tfrac{12}{l^3}\hat{D}^{(2)}_{22},\ \hat{k}^{(2)}_{14} = \tfrac{6}{l^2}\hat{D}^{(2)}_{22},\ \hat{k}^{(2)}_{15} = 0,\ \hat{k}^{(2)}_{16} = 0\ ,$$

$$\hat{k}^{(2)}_{17} = \tfrac{6}{5}\tfrac{5\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l^3},\ \hat{k}^{(2)}_{18} = \tfrac{1}{10}\tfrac{30\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l^2},\quad \hat{k}^{(2)}_{19} = -\tfrac{6}{5}\tfrac{5\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l^3},\quad \hat{k}^{(2)}_{1,10} = \tfrac{1}{10}\tfrac{30\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l^2},$$

$$\hat{k}^{(2)}_{22} = \tfrac{4}{l}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{23} = -\tfrac{6}{l^2}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{24} = \tfrac{2}{l}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{25} = -\tfrac{2}{l}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{26} = \tfrac{2}{l}\hat{D}^{(2)}_{22},$$

$$\hat{k}^{(2)}_{27} = \tfrac{1}{10}\tfrac{30\hat{D}^{(2)}_{23}+11\hat{D}^{(2)}_{24}l^2}{l^2},\ \hat{k}^{(2)}_{28} = \tfrac{2}{15l}\left(\hat{D}^{(2)}_{24}l^2 + 15\hat{D}^{(2)}_{23}\right),\quad \hat{k}^{(2)}_{29} = -\tfrac{1}{10}\tfrac{30\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l^2},$$

$$\hat{k}^{(2)}_{2,10} = -\tfrac{1}{30}\tfrac{-30\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l},$$

$$\hat{k}^{(2)}_{33} = \tfrac{12}{l^3}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{34} = -\tfrac{6}{l^2}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{35} = 0,\quad \hat{k}^{(2)}_{36} = 0,\quad \hat{k}^{(2)}_{37} = -\tfrac{6}{5}\tfrac{5\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l^3},$$

$$\hat{k}^{(2)}_{38} = -\tfrac{1}{10}\tfrac{30\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l^2},\quad \hat{k}^{(2)}_{39} = \tfrac{6}{5}\tfrac{5\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l^3},\quad \hat{k}^{(2)}_{3,10} = -\tfrac{1}{10}\tfrac{30\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l^2},$$

$$\hat{k}^{(2)}_{44} = \tfrac{4}{l}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{45} = \tfrac{2}{l}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{46} = -\tfrac{2}{l}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{47} = \tfrac{1}{10}\tfrac{30\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l^2},$$

$$\hat{k}^{(2)}_{48} = -\tfrac{1}{30}\tfrac{-30\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l},\quad \hat{k}^{(2)}_{49} = -\tfrac{1}{10}\tfrac{30\hat{D}^{(2)}_{23}+11\hat{D}^{(2)}_{24}l^2}{l^2},\quad \hat{k}^{(2)}_{4,10} = \tfrac{2}{15l}\left(\hat{D}^{(2)}_{24}l^2 + 15\hat{D}^{(2)}_{23}\right),$$

$$\hat{k}^{(2)}_{55} = \tfrac{4}{l}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{56} = -\tfrac{4}{l}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{57} = -\hat{D}^{(2)}_{24},\quad \hat{k}^{(2)}_{58} = -\tfrac{1}{6}\tfrac{6\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l},\quad \hat{k}^{(2)}_{59} = -\hat{D}^{(2)}_{24},$$

$$\hat{k}^{(2)}_{5,10} = \tfrac{1}{6}\tfrac{6\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l},$$

$$\hat{k}^{(2)}_{66} = \tfrac{4}{l}\hat{D}^{(2)}_{22},\quad \hat{k}^{(2)}_{67} = \hat{D}^{(2)}_{24},\quad \hat{k}^{(2)}_{68} = \tfrac{1}{6}\tfrac{6\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l},\quad \hat{k}^{(2)}_{69} = \hat{D}^{(2)}_{24},\quad \hat{k}^{(2)}_{6,10} = -\tfrac{1}{6}\tfrac{6\hat{D}^{(2)}_{23}+\hat{D}^{(2)}_{24}l^2}{l},$$

$$\hat{k}^{(2)}_{77} = \tfrac{1}{35l^3}\left(42\hat{D}^{(2)}_{34}l^2 + 105\hat{D}^{(2)}_{33} + 13l^4\hat{D}^{(2)}_{44}\right),\quad \hat{k}^{(2)}_{78} = \tfrac{1}{210l^2}\left(126\hat{D}^{(2)}_{34}l^2 + 315\hat{D}^{(2)}_{33} + 11l^4\hat{D}^{(2)}_{44}\right),$$

$$\hat{k}^{(2)}_{79} = \tfrac{3}{70l^3}\left(-28\hat{D}^{(2)}_{34}l^2 - 70\hat{D}^{(2)}_{33} + 3l^4\hat{D}^{(2)}_{44}\right),\quad \hat{k}^{(2)}_{7,10} = -\tfrac{1}{420l^2}\left(-42\hat{D}^{(2)}_{34}l^2 - 630\hat{D}^{(2)}_{33} + 13l^4\hat{D}^{(2)}_{44}\right),$$

$$\hat{k}^{(2)}_{88} = \tfrac{1}{105l}\left(14\hat{D}^{(2)}_{34}l^2 + 105\hat{D}^{(2)}_{33} + l^4\hat{D}^{(2)}_{44}\right),\quad \hat{k}^{(2)}_{89} = \tfrac{1}{420l^2}\left(-42\hat{D}^{(2)}_{34}l^2 - 630\hat{D}^{(2)}_{33} + 13l^4\hat{D}^{(2)}_{44}\right),$$

$$\hat{k}^{(2)}_{8,10} = -\tfrac{1}{420l}\left(14\hat{D}^{(2)}_{34}l^2 - 210\hat{D}^{(2)}_{33} + 3l^4\hat{D}^{(2)}_{44}\right),$$

$$\hat{k}^{(2)}_{99} = \tfrac{1}{35l^3}\left(42\hat{D}^{(2)}_{34}l^2 + 105\hat{D}^{(2)}_{33} + 13l^4\hat{D}^{(2)}_{44}\right),\quad \hat{k}^{(2)}_{9,10} = -\tfrac{1}{210l^2}\left(126\hat{D}^{(2)}_{34}l^2 + 315\hat{D}^{(2)}_{33} + 11l^4\hat{D}^{(2)}_{44}\right),$$

$$\hat{k}^{(2)}_{10,10} = \tfrac{1}{105l}\left(14\hat{D}^{(2)}_{34}l^2 + 105\hat{D}^{(2)}_{33} + l^4\hat{D}^{(2)}_{44}\right).$$

Matrix $[\check{k}^{(2)}]$, that enters into the expression (5.10.45) for the strain energy of the core, has the form

$$
\underset{(10\times10)}{[\check{k}^{(2)}]} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{3}l\check{D}_{22}^{(2)} & \frac{1}{6}l\check{D}_{22}^{(2)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{6}l\check{D}_{22}^{(2)} & \frac{1}{3}l\check{D}_{22}^{(2)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

If the modulus s of the elastic foundation is constant, matrix $\left[k^{(f)}\right]$ in the expression (5.10.47) for the strain energy of the elastic foundation has the form :

$$
\left[k^{(f)}\right] =
$$

$$
bs\begin{bmatrix}
\frac{13}{35}l & \frac{11}{210}l^2 & \frac{9}{70}l & -\frac{13}{420}l^2 & 0 & 0 & \frac{13}{35}lz_2 & \frac{11}{210}l^2z_2 & \frac{9}{70}lz_2 & -\frac{13}{420}l^2z_2 \\
\frac{11}{210}l^2 & \frac{1}{105}l^3 & \frac{13}{420}l^2 & -\frac{1}{140}l^3 & 0 & 0 & \frac{11}{210}l^2z_2 & \frac{1}{105}l^3z_2 & \frac{13}{420}l^2z_2 & -\frac{1}{140}l^3z_2 \\
\frac{9}{70}l & \frac{13}{420}l^2 & \frac{13}{35}l & -\frac{11}{210}l^2 & 0 & 0 & \frac{9}{70}lz_2 & \frac{13}{420}l^2z_2 & \frac{13}{35}lz_2 & -\frac{11}{210}l^2z_2 \\
-\frac{13}{420}l^2 & -\frac{1}{140}l^3 & -\frac{11}{210}l^2 & \frac{1}{105}l^3 & 0 & 0 & -\frac{13}{420}l^2z_2 & -\frac{1}{140}l^3z_2 & -\frac{11}{210}l^2z_2 & \frac{1}{105}l^3z_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{13}{35}lz_2 & \frac{11}{210}l^2z_2 & \frac{9}{70}lz_2 & -\frac{13}{420}l^2z_2 & 0 & 0 & \frac{13}{35}lz_2^2 & \frac{11}{210}l^2z_2^2 & \frac{9}{70}lz_2^2 & -\frac{13}{420}l^2z_2^2 \\
\frac{11}{210}l^2z_2 & \frac{1}{105}l^3z_2 & \frac{13}{420}l^2z_2 & -\frac{1}{140}l^3z_2 & 0 & 0 & \frac{11}{210}l^2z_2^2 & \frac{1}{105}l^3z_2^2 & \frac{13}{420}l^2z_2^2 & -\frac{1}{140}l^3z_2^2 \\
\frac{9}{70}lz_2 & \frac{13}{420}l^2z_2 & \frac{13}{35}lz_2 & -\frac{11}{210}l^2z_2 & 0 & 0 & \frac{9}{70}lz_2^2 & \frac{13}{420}l^2z_2^2 & \frac{13}{35}lz_2^2 & -\frac{11}{210}l^2z_2^2 \\
-\frac{13}{420}l^2z_2 & -\frac{1}{140}l^3z_2 & -\frac{11}{210}l^2z_2 & \frac{1}{105}l^3z_2 & 0 & 0 & -\frac{13}{420}l^2z_2^2 & -\frac{1}{140}l^3z_2^2 & -\frac{11}{210}l^2z_2^2 & \frac{1}{105}l^3z_2^2
\end{bmatrix}.
$$

The the stiffness matrix of the finite element is

$$
\left[k^{(l)}\right] = \left[k^{(1)}\right] + \left[k^{(3)}\right] + \left[\widehat{k}^{(2)}\right] + \left[\check{k}^{(2)}\right] + \left[k^{(f)}\right].
$$

5.16 Appendix 5-B. Mass matrix.

The first term of expression (5.10.54) for the kinetic energy is (expression (5.10.57))

$$\frac{1}{2}\rho^{(1)}b\int_0^l \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right)^T \underset{(3\times3)}{\left[\widetilde{D}^{(1)}\right]}\left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(1)}\right]}\frac{\partial}{\partial t}\underset{(3\times1)}{\{f\}}\right)\,dx =$$

$$=\frac{1}{2}\underset{(1\times10)}{\left\{\dot{d}\right\}}^T\underset{(10\times10)}{\left[m^{(1)}\right]}\underset{(10\times1)}{\left\{\dot{d}\right\}},$$

where components of the matrix $\left[m^{(1)}\right]$ are

$$m_{11}^{(1)}=\tfrac{1}{35}\rho^{(1)}b\left(z_2-z_1\right)\tfrac{14z_2^2+14z_1z_2+13l^2+14z_1^2}{l},$$
$$m_{12}^{(1)}=\tfrac{1}{210}\rho^{(1)}b\left(z_2-z_1\right)\left(7z_2^2+7z_1z_2+11l^2+7z_1^2\right),$$
$$m_{13}^{(1)}=-\tfrac{1}{70}\rho^{(1)}b\left(z_2-z_1\right)\tfrac{28z_2^2+28z_1z_2-9l^2+28z_1^2}{l},$$
$$m_{14}^{(1)}=\tfrac{1}{420}\rho^{(1)}b\left(z_2-z_1\right)\left(14z_2^2+14z_1z_2-13l^2+14z_1^2\right),$$
$$m_{15}^{(1)}=\tfrac{1}{2}\rho^{(1)}bz_2\left(z_2^2-z_1^2\right),\ \ m_{16}^{(1)}=\tfrac{1}{2}\rho^{(1)}bz_2\left(z_2^2-z_1^2\right),$$
$$m_{17}^{(1)}=\tfrac{1}{70}\rho^{(1)}bz_2\left(z_2-z_1\right)\tfrac{7z_2^2+7z_1z_2+28z_1^2+26l^2}{l},$$
$$m_{18}^{(1)}=\tfrac{1}{840}\rho^{(1)}bz_2\left(z_2-z_1\right)\left(7z_2^2+7z_1z_2+28z_1^2+44l^2\right),$$
$$m_{19}^{(1)}=-\tfrac{1}{70}\rho^{(1)}bz_2\left(z_2-z_1\right)\tfrac{7z_2^2+7z_1z_2-9l^2+28z_1^2}{l},$$
$$m_{1,10}^{(1)}=\tfrac{1}{840}\rho^{(1)}bz_2\left(z_2-z_1\right)\left(7z_2^2+7z_1z_2+28z_1^2-26l^2\right),$$

$$m_{22}^{(1)}=\tfrac{1}{315}\rho^{(1)}bl\left(z_2-z_1\right)\left(14z_2^2+14z_1z_2+3l^2+14z_1^2\right),$$
$$m_{23}^{(1)}=\tfrac{1}{420}\rho^{(1)}b\left(z_2-z_1\right)\left(-14z_2^2-14z_1z_2+13l^2-14z_1^2\right)$$
$$m_{24}^{(1)}=-\tfrac{1}{1260}\rho^{(1)}bl\left(z_2-z_1\right)\left(14z_2^2+14z_1z_2+9l^2+14z_1^2\right),$$
$$m_{25}^{(1)}=-\tfrac{1}{12}\rho^{(1)}blz_2\left(z_2^2-z_1^2\right),$$
$$m_{26}^{(1)}=\tfrac{1}{12}\rho^{(1)}blz_2\left(z_2^2-z_1^2\right),$$
$$m_{27}^{(1)}=\tfrac{1}{840}\rho^{(1)}bz_2\left(z_2-z_1\right)\left(7z_2^2+7z_1z_2+28z_1^2+44l^2\right),$$
$$m_{28}^{(1)}=\tfrac{1}{630}\rho^{(1)}bz_2l\left(z_2-z_1\right)\left(7z_2^2+7z_1z_2+28z_1^2+6l^2\right),$$
$$m_{29}^{(1)}=-\tfrac{1}{840}\rho^{(1)}bz_2\left(z_2-z_1\right)\left(7z_2^2+7z_1z_2+28z_1^2-26l^2\right),$$
$$m_{2,10}^{(1)}=-\tfrac{1}{2520}\rho^{(1)}bz_2l\left(z_2-z_1\right)\left(7z_2^2+7z_1z_2+28z_1^2+18l^2\right),$$

$$m_{33}^{(1)}=\tfrac{1}{35}\rho^{(1)}b\tfrac{14z_2^3-14z_1^3+13z_2l^2-13l^2z_1}{l},$$
$$m_{34}^{(1)}=\rho^{(1)}b\left[\tfrac{1}{30}\left(z_1^3-z_2^3\right)+\tfrac{11}{210}l^2\left(z_1-z_2\right)\ \right],$$

$$m_{35}^{(1)} = -\tfrac{1}{2}\rho^{(1)}bz_2\left(z_2 - z_1\right)\left(z_2 + z_1\right),$$
$$m_{36}^{(1)} = -\tfrac{1}{2}\rho^{(1)}bz_2\left(z_2 - z_1\right)\left(z_2 + z_1\right),$$
$$m_{37}^{(1)} = -\tfrac{1}{70}\rho^{(1)}bz_2\left(z_2 - z_1\right)\tfrac{7z_2^2+7z_1z_2+28z_1^2-9l^2}{l},$$
$$m_{38}^{(1)} = -\tfrac{1}{840}\rho^{(1)}bz_2\left(z_2 - z_1\right)\left(7z_2^2 + 7z_1z_2 + 28z_1^2 - 26l^2\right),$$
$$m_{39}^{(1)} = \tfrac{1}{70}\rho^{(1)}bz_2\left(z_2 - z_1\right)\tfrac{7z_2^2+7z_1z_2+28z_1^2+26l^2}{l},$$
$$m_{3,10}^{(1)} = -\tfrac{1}{840}\rho^{(1)}bz_2\left(z_2 - z_1\right)\left(7z_2^2 + 7z_1z_2 + 28z_1^2 + 44l^2\right),$$

$$m_{44}^{(1)} = \tfrac{1}{315}\rho^{(1)}bl\left(z_2 - z_1\right)\left(14z_2^2 + 14z_1z_2 + 3l^2 + 14z_1^2\right),$$
$$m_{45}^{(1)} = \tfrac{1}{12}\rho^{(1)}blz_2\left(z_2 - z_1\right)\left(z_2 + z_1\right),$$
$$m_{46}^{(1)} = -\tfrac{1}{12}\rho^{(1)}blz_2\left(z_2 - z_1\right)\left(z_2 + z_1\right),$$
$$m_{47}^{(1)} = -\tfrac{1}{840}\rho^{(1)}bz_2\left(z_2 - z_1\right)\left(-7z_2^2 - 7z_1z_2 - 28z_1^2 + 26l^2\right),$$
$$m_{48}^{(1)} = -\tfrac{1}{2520}\rho^{(1)}blz_2\left(z_2 - z_1\right)\left(7z_2^2 + 7z_1z_2 + 28z_1^2 + 18l^2\right),$$
$$m_{49}^{(1)} = -\tfrac{1}{840}\rho^{(1)}bz_2\left(z_2 - z_1\right)\left(7z_2^2 + 7z_1z_2 + 28z_1^2 + 44l^2\right),$$
$$m_{4,10}^{(1)} = \tfrac{1}{630}\rho^{(1)}blz_2\left(z_2 - z_1\right)\left(7z_2^2 + 7z_1z_2 + 28z_1^2 + 6l^2\right),$$

$$m_{55}^{(1)} = \tfrac{4}{3}\rho^{(1)}blz_2^2\left(z_2 - z_1\right),$$
$$m_{56}^{(1)} = -\tfrac{2}{3}\rho^{(1)}blz_2^2\left(-z_2 + z_1\right),$$
$$m_{57}^{(1)} = \tfrac{1}{2}\rho^{(1)}bz_1z_2^2\left(z_2 - z_1\right),$$
$$m_{58}^{(1)} = \tfrac{1}{12}\rho^{(1)}blz_1z_2^2\left(-z_2 + z_1\right),$$
$$m_{59}^{(1)} = \tfrac{1}{2}\rho^{(1)}bz_1z_2^2\left(-z_2 + z_1\right),$$
$$m_{5,10}^{(1)} = -\tfrac{1}{12}\rho^{(1)}blz_1z_2^2\left(-z_2 + z_1\right),$$

$$m_{66}^{(1)} = -\tfrac{4}{3}\rho^{(1)}blz_2^2\left(-z_2 + z_1\right),$$
$$m_{67}^{(1)} = \tfrac{1}{2}\rho^{(1)}bz_1z_2^2\left(z_2 - z_1\right),$$
$$m_{68}^{(1)} = \tfrac{1}{12}\rho^{(1)}blz_1z_2^2\left(z_2 - z_1\right),$$
$$m_{69}^{(1)} = \tfrac{1}{2}\rho^{(1)}bz_1z_2^2\left(z_1 - z_2\right),$$
$$m_{6,10}^{(1)} = \tfrac{1}{12}\rho^{(1)}blz_1z_2^2\left(z_1 - z_2\right),$$

$$m_{77}^{(1)} = \tfrac{1}{840}\rho^{(1)}bz_2^2\left(z_2 - z_1\right)\left(28z_1^2 - 14z_2z_1 + 44l^2 + 7z_2^2\right),$$
$$m_{78}^{(1)} = \tfrac{1}{840}\rho^{(1)}bz_2^2\left(z_2 - z_1\right)\left(7z_2^2 - 14z_1z_2 + 44l^2 + 28z_1^2\right),$$
$$m_{79}^{(1)} = \tfrac{1}{70}\rho^{(1)}bz_2^2\left(z_2 - z_1\right)\tfrac{-7z_2^2+14z_1z_2+9l^2-28z_1^2}{l},$$
$$m_{7,10}^{(1)} = -\tfrac{1}{840}\rho^{(1)}bz_2^2\left(z_2 - z_1\right)\left(-7z_2^2 + 14z_1z_2 + 26l^2 - 28z_1^2\right),$$

$$m_{88}^{(1)} = \tfrac{1}{630}\rho^{(1)}blz_2^2\left(z_2 - z_1\right)\left(7z_2^2 - 14z_1z_2 + 6l^2 + 28z_1^2\right),$$

$m_{89}^{(1)} = \frac{1}{840}\rho^{(1)}bz_2^2\left(z_2 - z_1\right)\left(-7z_2^2 + 14z_1z_2 + 26l^2 - 28z_1^2\right),$
$m_{8,10}^{(1)} = -\frac{1}{2520}\rho^{(1)}blz_2^2\left(z_2 - z_1\right)\left(7z_2^2 - 14z_1z_2 + 18l^2 + 28z_1^2\right),$

$m_{99}^{(1)} = \frac{1}{70}\rho^{(1)}bz_2^2\left(z_2 - z_1\right)\frac{7z_2^2 - 14z_1z_2 + 26l^2 + 28z_1^2}{l},$
$m_{9,10}^{(1)} = -\frac{1}{840}\rho^{(1)}bz_2^2\left(z_2 - z_1\right)\left(7z_2^2 - 14z_1z_2 + 44l^2 + 28z_1^2\right),$

$m_{10,10}^{(1)} = \frac{1}{630}\rho^{(1)}blz_2^2\left(z_2 - z_1\right)\left(7z_2^2 - 14z_1z_2 + 6l^2 + 28z_1^2\right).$

The second term of expression (5.10.54) for the kinetic energy is (expression (5.10.61))

$$\frac{1}{2}\rho^{(2)}b\int_0^l\left(\underset{(4\times 3)}{\left[\widetilde{\partial}^{(2)}\right]}\frac{\partial}{\partial t}\underset{(3\times 1)}{\{f\}}\right)^T\underset{(4\times 4)}{\left[\widetilde{D}^{(2)}\right]}\left(\underset{(4\times 3)}{\left[\widetilde{\partial}^{(2)}\right]}\frac{\partial}{\partial t}\underset{(3\times 1)}{\{f\}}\right)dx =$$

$$= \frac{1}{2}\underset{(1\times 10)}{\left\{\dot{d}\right\}}^T\underset{(10\times 10)}{\left[m^{(2)}\right]}\underset{(10\times 1)}{\left\{\dot{d}\right\}},$$

where components of the matrix $\left[m^{(2)}\right]$ are

$$m_{11}^{(2)} = \frac{1}{35}\rho^{(2)}b\frac{14z_3^3 - 14z_2^3 + 13l^2z_3 - 13l^2z_2}{l},$$

$$m_{12}^{(2)} = \rho^{(2)}b\left(\frac{1}{30}z_3^3 - \frac{1}{30}z_2^3 + \frac{11}{210}l^2z_3 - \frac{11}{210}l^2z_2\right),$$

$$m_{13}^{(2)} = \frac{1}{70}\rho^{(2)}b\frac{-28z_3^3 + 28z_2^3 + 9l^2z_3 - 9l^2z_2}{l},$$

$$m_{14}^{(2)} = \rho^{(2)}b\left(\frac{1}{30}z_3^3 - \frac{1}{30}z_2^3 - \frac{13}{420}l^2z_3 + \frac{13}{420}l^2z_2\right),$$

$$m_{15}^{(2)} = \rho^{(2)}b\left(\frac{1}{3}z_3^3 - \frac{1}{3}z_2^3\right),$$

$$m_{16}^{(2)} = \rho^{(2)}b\left(\frac{1}{3}z_3^3 - \frac{1}{3}z_2^3\right),$$

$$m_{17}^{(2)} = \frac{1}{140}\rho^{(2)}b\frac{21z_3^4 - 21z_2^4 + 26l^2z_3^2 - 26l^2z_2^2}{l},$$

$$m_{18}^{(2)} = \rho^{(2)} b \left(\frac{1}{80} z_3^4 - \frac{1}{80} z_2^4 + \frac{11}{420} l^2 z_3^2 - \frac{11}{420} l^2 z_2^2 \right),$$

$$m_{19}^{(2)} = \frac{3}{140} \rho^{(2)} b \frac{-7z_3^4 + 7z_2^4 + 3l^2 z_3^2 - 3l^2 z_2^2}{l},$$

$$m_{1,10}^{(2)} = \rho^{(2)} b \left(\frac{1}{80} z_3^4 - \frac{1}{80} z_2^4 - \frac{13}{840} l^2 z_3^2 + \frac{13}{840} l^2 z_2^2 \right),$$

$$m_{22}^{(2)} = \rho^{(2)} b \left(\frac{2}{45} l z_3^3 - \frac{2}{45} l z_2^3 + \frac{1}{105} l^3 z_3 - \frac{1}{105} l^3 z_2 \right),$$

$$m_{23}^{(2)} = \rho^{(2)} b \left(-\frac{1}{30} z_3^3 + \frac{1}{30} z_2^3 + \frac{13}{420} l^2 z_3 - \frac{13}{420} l^2 z_2 \right),$$

$$m_{24}^{(2)} = \rho^{(2)} b \left(-\frac{1}{90} l z_3^3 + \frac{1}{90} l z_2^3 - \frac{1}{140} l^3 z_3 + \frac{1}{140} l^3 z_2 \right),$$

$$m_{25}^{(2)} = \rho^{(2)} b \left(-\frac{1}{18} l z_3^3 + \frac{1}{18} l z_2^3 \right),$$

$$m_{26}^{(2)} = \rho^{(2)} b \left(\frac{1}{18} l z_3^3 - \frac{1}{18} l z_2^3 \right),$$

$$m_{27}^{(2)} = \rho^{(2)} b \left(\frac{1}{80} z_3^4 - \frac{1}{80} z_2^4 + \frac{11}{420} l^2 z_3^2 - \frac{11}{420} l^2 z_2^2 \right),$$

$$m_{28}^{(2)} = \rho^{(2)} b \left(\frac{1}{60} l z_3^4 - \frac{1}{60} l z_2^4 + \frac{1}{210} l^3 z_3^2 - \frac{1}{210} l^3 z_2^2 \right),$$

$$m_{29}^{(2)} = \rho^{(2)} b \left(-\frac{1}{80} z_3^4 + \frac{1}{80} z_2^4 + \frac{13}{840} l^2 z_3^2 - \frac{13}{840} l^2 z_2^2 \right),$$

$$m_{2,10}^{(2)} = \rho^{(2)} b \left(-\frac{1}{240} l z_3^4 + \frac{1}{240} l z_2^4 - \frac{1}{280} l^3 z_3^2 + \frac{1}{280} l^3 z_2^2 \right),$$

$$m_{33}^{(2)} = \frac{1}{35} \rho^{(2)} b \frac{14z_3^3 - 14z_2^3 + 13l^2 z_3 - 13l^2 z_2}{l},$$

$$m_{34}^{(2)} = \rho^{(2)} b \left(-\frac{1}{30} z_3^3 + \frac{1}{30} z_2^3 - \frac{11}{210} l^2 z_3 + \frac{11}{210} l^2 z_2 \right),$$

$$m_{35}^{(2)} = \rho^{(2)} b \left(-\frac{1}{3} z_3^3 + \frac{1}{3} z_2^3 \right),$$

$$m_{36}^{(2)} = \rho^{(2)} b \left(-\frac{1}{3} z_3^3 + \frac{1}{3} z_2^3 \right),$$

$$m_{37}^{(2)} = \frac{3}{140} \rho^{(2)} b \frac{-7z_3^4 + 7z_2^4 + 3l^2 z_3^2 - 3l^2 z_2^2}{l},$$

$$m_{38}^{(2)} = \rho^{(2)} b \left(-\frac{1}{80} z_3^4 + \frac{1}{80} z_2^4 + \frac{13}{840} l^2 z_3^2 - \frac{13}{840} l^2 z_2^2 \right),$$

$$m_{39}^{(2)} = \frac{1}{140} \rho^{(2)} b \frac{21z_3^4 - 21z_2^4 + 26l^2 z_3^2 - 26l^2 z_2^2}{l},$$

$$m_{3,10}^{(2)} = \rho^{(2)} b \left(-\frac{1}{80} z_3^4 + \frac{1}{80} z_2^4 - \frac{11}{420} l^2 z_3^2 + \frac{11}{420} l^2 z_2^2 \right),$$

$$m_{44}^{(2)} = \rho^{(2)} b \left(\frac{2}{45} l z_3^3 - \frac{2}{45} l z_2^3 + \frac{1}{105} l^3 z_3 - \frac{1}{105} l^3 z_2 \right),$$

$$m_{45}^{(2)} = \rho^{(2)} b \left(\frac{1}{18} l z_3^3 - \frac{1}{18} l z_2^3 \right),$$

$$m_{46}^{(2)} = \rho^{(2)} b \left(-\frac{1}{18} l z_3^3 + \frac{1}{18} l z_2^3 \right),$$

$$m_{47}^{(2)} = \rho^{(2)} b \left(\frac{1}{80} z_3^4 - \frac{1}{80} z_2^4 - \frac{13}{840} l^2 z_3^2 + \frac{13}{840} l^2 z_2^2 \right),$$

$$m_{48}^{(2)} = \rho^{(2)} b \left(-\frac{1}{240} l z_3^4 + \frac{1}{240} l z_2^4 - \frac{1}{280} l^3 z_3^2 + \frac{1}{280} l^3 z_2^2 \right),$$

$$m_{49}^{(2)} = \rho^{(2)} b \left(-\frac{1}{80} z_3^4 + \frac{1}{80} z_2^4 - \frac{11}{420} l^2 z_3^2 + \frac{11}{420} l^2 z_2^2 \right),$$

$$m_{4,10}^{(2)} = \rho^{(2)} b \left(\frac{1}{60} l z_3^4 - \frac{1}{60} l z_2^4 + \frac{1}{210} l^3 z_3^2 - \frac{1}{210} l^3 z_2^2 \right),$$

$$m_{55}^{(2)} = \rho^{(2)} b \left(\frac{4}{9} l z_3^3 - \frac{4}{9} l z_2^3 \right),$$

$$m_{56}^{(2)} = \rho^{(2)} b \left(\frac{2}{9} l z_3^3 - \frac{2}{9} l z_2^3 \right),$$

$$m_{57}^{(2)} = \frac{1}{8} \rho^{(2)} b \left(z_3^4 - z_2^4 \right),$$

$$m_{58}^{(2)} = \frac{1}{48} l \rho^{(2)} b \left(z_2^4 - z_3^4 \right),$$

$$m_{59}^{(2)} = \frac{1}{8} \rho^{(2)} b \left(z_2^4 - z_3^4 \right),$$

$$m_{5,10}^{(2)} = \frac{1}{48} l \rho^{(2)} b \left(z_3^4 - z_2^4 \right),$$

$$m_{66}^{(2)} = \frac{4}{9} \rho^{(2)} b l \left(z_3^3 - z_2^3 \right),$$

$$m_{67}^{(2)} = \frac{1}{8} \rho^{(2)} b \left(z_3^4 - z_2^4 \right),$$

$$m_{68}^{(2)} = \frac{1}{48} \rho^{(2)} b l \left(z_3^4 - z_2^4 \right),$$

$$m_{69}^{(2)} = \frac{1}{8} \rho^{(2)} b \left(z_2^4 - z_3^4 \right),$$

$$m_{6,10}^{(2)} = \frac{1}{48} \rho^{(2)} b l \left(z_2^4 - z_3^4 \right),$$

$$m_{77}^{(2)} = \frac{1}{1050} \rho^{(2)} b \frac{63 z_3^5 - 63 z_2^5 + 130 l^2 z_3^3 - 130 l^2 z_2^3}{l},$$

$$m_{78}^{(2)} = \rho^{(2)} b \left(\frac{1}{200} z_3^5 - \frac{1}{200} z_2^5 + \frac{11}{630} l^2 z_3^3 - \frac{11}{630} l^2 z_2^3 \right),$$

$$m_{79}^{(2)} = \frac{3}{350} \rho^{(2)} b \frac{-7z_3^5 + 7z_2^5 + 5l^2 z_3^3 - 5l^2 z_2^3}{l},$$

$$m_{7,10}^{(2)} = \rho^{(2)} b \left(\frac{1}{200} z_3^5 - \frac{1}{200} z_2^5 - \frac{13}{1260} l^2 z_3^3 + \frac{13}{1260} l^2 z_2^3 \right),$$

$$m_{88}^{(2)} = \rho^{(2)} b \left(\frac{1}{150} l z_3^5 - \frac{1}{150} l z_2^5 + \frac{1}{315} l^3 z_3^3 - \frac{1}{315} l^3 z_2^3 \right),$$

$$m_{89}^{(2)} = \rho^{(2)} b \left(-\frac{1}{200} z_3^5 + \frac{1}{200} z_2^5 + \frac{13}{1260} l^2 z_3^3 - \frac{13}{1260} l^2 z_2^3 \right),$$

$$m_{8,10}^{(2)} = \rho^{(2)} b \left(-\frac{1}{600} l z_3^5 + \frac{1}{600} l z_2^5 - \frac{1}{420} l^3 z_3^3 + \frac{1}{420} l^3 z_2^3 \right),$$

$$m_{99}^{(2)} = \frac{1}{1050} \rho^{(2)} b \frac{63z_3^5 - 63z_2^5 + 130l^2 z_3^3 - 130l^2 z_2^3}{l},$$

$$m_{9,10}^{(2)} = \rho^{(2)} b \left(-\frac{1}{200} z_3^5 + \frac{1}{200} z_2^5 - \frac{11}{630} l^2 z_3^3 + \frac{11}{630} l^2 z_2^3 \right),$$

$$m_{10,10}^{(2)} = \rho^{(2)} b \left(\frac{1}{150} l z_3^5 - \frac{1}{150} l z_2^5 + \frac{1}{315} l^3 z_3^3 - \frac{1}{315} l^3 z_2^3 \right).$$

The third term of the expression (5.10.54) for the kinetic energy is (expression (5.10.65))

$$\frac{1}{2} \rho^{(3)} b \int_0^l \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(3)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right)^T \underset{(3\times3)}{\left[\widetilde{D}^{(3)}\right]} \left(\underset{(3\times3)}{\left[\widetilde{\partial}^{(3)}\right]} \frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}} \right) dx =$$

$$= \frac{1}{2} \underset{(1\times10)}{\left\{\dot{d}\right\}} \underset{(10\times10)}{\left[m^{(3)}\right]} \underset{(10\times1)}{\left\{\dot{d}\right\}},$$

where components of the matrix $\left[m^{(3)}\right]$ are

$$m_{11}^{(3)} = -\frac{1}{35}\rho^{(3)}b\frac{-14z_4^3 + 14z_3^3 - 13l^2 z_4 + 13l^2 z_3}{l},$$

$$m_{12}^{(3)} = \frac{1}{210}\rho^{(3)}b\left(z_4 - z_3\right)\left(7z_4^2 + 7z_3z_4 + 11l^2 + 7z_3^2\right),$$

$$m_{13}^{(3)} = \frac{1}{70}\rho^{(3)}b\left(z_4 - z_3\right)\frac{-28z_4^2 - 28z_3z_4 + 9l^2 - 28z_3^2}{l},$$

$$m_{14}^{(3)} = -\frac{1}{420}\rho^{(3)}b\left(z_4 - z_3\right)\left(-14z_4^2 - 14z_3z_4 + 13l^2 - 14z_3^2\right),$$

$$m_{15}^{(3)} = \frac{1}{2}\rho^{(3)}bz_3\left(z_4 - z_3\right)\left(z_4 + z_3\right),$$

$$m_{16}^{(3)} = \frac{1}{2}\rho^{(3)}bz_3\left(z_4 - z_3\right)\left(z_4 + z_3\right),$$

$$m_{17}^{(3)} = \frac{1}{70}\rho^{(3)}bz_3\left(z_4 - z_3\right)\frac{28z_4^2 + 7z_3z_4 + 26l^2 + 7z_3^2}{l},$$

$$m_{18}^{(3)} = \frac{1}{840}\rho^{(3)}bz_3\left(z_4 - z_3\right)\left(28z_4^2 + 7z_3z_4 + 44l^2 + 7z_3^2\right),$$

$$m_{19}^{(3)} = -\frac{1}{70}\rho^{(3)}bz_3\frac{-21z_3z_4^2 - 7z_3^3 + 28z_4^3 - 9l^2 z_4 + 9l^2 z_3}{l},$$

$$m_{1,10}^{(3)} = -\frac{1}{840}\rho^{(3)}bz_3\left(z_4 - z_3\right)\left(-28z_4^2 - 7z_3z_4 + 26l^2 - 7z_3^2\right),$$

$$m_{22}^{(3)} = \frac{1}{315}\rho^{(3)}bl\left(z_4 - z_3\right)\left(14z_4^2 + 14z_3z_4 + 3l^2 + 14z_3^2\right),$$

$$m_{23}^{(3)} = \frac{1}{420}\rho^{(3)}b\left(z_4 - z_3\right)\left(-14z_4^2 - 14z_3z_4 + 13l^2 - 14z_3^2\right),$$

$$m_{24}^{(3)} = -\frac{1}{1260}\rho^{(3)}bl\left(z_4 - z_3\right)\left(14z_4^2 + 14z_3z_4 + 9l^2 + 14z_3^2\right),$$

$$m_{25}^{(3)} = -\frac{1}{12}\rho^{(3)} b l z_3 \left(z_4 - z_3\right)\left(z_4 + z_3\right),$$

$$m_{26}^{(3)} = \frac{1}{12}\rho^{(3)} b z_3 l \left(z_4 - z_3\right)\left(z_4 + z_3\right),$$

$$m_{27}^{(3)} = \frac{1}{840}\rho^{(3)} b z_3 \left(z_4 - z_3\right)\left(28z_4^2 + 7z_3z_4 + 44l^2 + 7z_3^2\right),$$

$$m_{28}^{(3)} = \frac{1}{630}\rho^{(3)} b z_3 l \left(z_4 - z_3\right)\left(28z_4^2 + 7z_3z_4 + 6l^2 + 7z_3^2\right),$$

$$m_{29}^{(3)} = -\frac{1}{840}\rho^{(3)} b z_3 \left(z_4 - z_3\right)\left(28z_4^2 + 7z_3z_4 - 26l^2 + 7z_3^2\right),$$

$$m_{2,10}^{(3)} = -\frac{1}{2520}\rho^{(3)} b z_3 l \left(z_4 - z_3\right)\left(28z_4^2 + 7z_3z_4 + 18l^2 + 7z_3^2\right),$$

$$m_{33}^{(3)} = \frac{1}{35}\rho^{(3)} b \left(z_4 - z_3\right)\frac{14z_4^2 + 14z_3z_4 + 13l^2 + 14z_3^2}{l},$$

$$m_{34}^{(3)} = -\frac{1}{210}\rho^{(3)} b \left(z_4 - z_3\right)\left(7z_4^2 + 7z_3z_4 + 11l^2 + 7z_3^2\right)$$

$$m_{35}^{(3)} = -\frac{1}{2}\rho^{(3)} b z_3 \left(z_4 - z_3\right)\left(z_4 + z_3\right),$$

$$m_{36}^{(3)} = -\frac{1}{2}\rho^{(3)} b z_3 \left(z_4 - z_3\right)\left(z_4 + z_3\right),$$

$$m_{37}^{(3)} = -\frac{1}{70}\rho^{(3)} b z_3 \left(z_4 - z_3\right)\frac{28z_4^2 + 7z_3z_4 - 9l^2 + 7z_3^2}{l},$$

$$m_{38}^{(3)} = -\frac{1}{840}\rho^{(3)} b z_3 \left(-z_4 + z_3\right)\left(-7z_3^2 - 7z_4z_3 - 28z_4^2 + 26l^2\right),$$

$$m_{39}^{(3)} = -\frac{1}{70}\rho^{(3)} b z_3 \left(-z_4 + z_3\right)\frac{7z_3^2 + 7z_4z_3 + 28z_4^2 + 26l^2}{l},$$

$$m_{3,10}^{(3)} = \frac{1}{840}\rho^{(3)} b z_3 \left(-z_4 + z_3\right)\left(7z_3^2 + 7z_4z_3 + 28z_4^2 + 44l^2\right),$$

$$m_{44}^{(3)} = -\frac{1}{315}\rho^{(3)}bl\left(-z_4+z_3\right)\left(14z_3^2+14z_4z_3+3l^2+14z_4^2\right),$$

$$m_{45}^{(3)} = -\frac{1}{12}\rho^{(3)}blz_3\left(-z_4+z_3\right)\left(z_3+z_4\right),$$

$$m_{46}^{(3)} = \frac{1}{12}\rho^{(3)}blz_3\left(-z_4+z_3\right)\left(z_3+z_4\right),$$

$$m_{47}^{(3)} = \frac{1}{840}\rho^{(3)}bz_3\left(-z_4+z_3\right)\left(-7z_3^2-7z_4z_3-28z_4^2+26l^2\right),$$

$$m_{48}^{(3)} = \frac{1}{2520}\rho^{(3)}blz_3\left(-z_4+z_3\right)\left(7z_3^2+7z_4z_3+28z_4^2+18l^2\right),$$

$$m_{49}^{(3)} = \frac{1}{840}\rho^{(3)}bz_3\left(-z_4+z_3\right)\left(7z_3^2+7z_4z_3+28z_4^2+44l^2\right),$$

$$m_{4,10}^{(3)} = -\frac{1}{630}\rho^{(3)}blz_3\left(-z_4+z_3\right)\left(7z_3^2+7z_4z_3+28z_4^2+6l^2\right),$$

$$m_{55}^{(3)} = -\frac{4}{3}\rho^{(3)}blz_3^2\left(-z_4+z_3\right),$$

$$m_{56}^{(3)} = -\frac{2}{3}\rho^{(3)}blz_3^2\left(-z_4+z_3\right),$$

$$m_{57}^{(3)} = -\frac{1}{2}\rho^{(3)}bz_3^2z_4\left(-z_4+z_3\right),$$

$$m_{58}^{(3)} = \frac{1}{12}\rho^{(3)}blz_3^2z_4\left(-z_4+z_3\right),$$

$$m_{59}^{(3)} = \frac{1}{2}\rho^{(3)}bz_3^2z_4\left(-z_4+z_3\right),$$

$$m_{5,10}^{(3)} = -\frac{1}{12}\rho^{(3)}blz_3^2z_4\left(-z_4+z_3\right),$$

$$m_{66}^{(3)} = -\frac{4}{3}\rho^{(3)} b l z_3^2 \left(-z_4 + z_3\right),$$

$$m_{67}^{(3)} = -\frac{1}{2}\rho^{(3)} b z_3^2 z_4 \left(-z_4 + z_3\right),$$

$$m_{68}^{(3)} = -\frac{1}{12}\rho^{(3)} b l z_3^2 z_4 \left(-z_4 + z_3\right),$$

$$m_{69}^{(3)} = \frac{1}{2}\rho^{(3)} b z_3^2 z_4 \left(-z_4 + z_3\right),$$

$$m_{6,10}^{(3)} = \frac{1}{12}\rho^{(3)} b l z_3^2 z_4 \left(-z_4 + z_3\right),$$

$$m_{77}^{(3)} = -\frac{1}{70}\rho^{(3)} b z_3^2 \left(-z_4 + z_3\right) \frac{7z_3^2 - 14z_4z_3 + 26l^2 + 28z_4^2}{l},$$

$$m_{78}^{(3)} = -\frac{1}{840}\rho^{(3)} b z_3^2 \left(-z_4 + z_3\right)\left(7z_3^2 - 14z_4z_3 + 28z_4^2 + 44l^2\right),$$

$$m_{79}^{(3)} = -\frac{1}{70}\rho^{(3)} b z_3^2 \left(-z_4 + z_3\right) \frac{-7z_3^2 + 14z_4z_3 + 9l^2 - 28z_4^2}{l},$$

$$m_{7,10}^{(3)} = \frac{1}{840}\rho^{(3)} b z_3^2 \left(-z_4 + z_3\right)\left(-7z_3^2 + 14z_4z_3 + 26l^2 - 28z_4^2\right),$$

$$m_{88}^{(3)} = -\frac{1}{630}\rho^{(3)} b l z_3^2 \left(-z_4 + z_3\right)\left(7z_3^2 - 14z_4z_3 + 28z_4^2 + 6l^2\right),$$

$$m_{89}^{(3)} = -\frac{1}{840}\rho^{(3)} b z_3^2 \left(-z_4 + z_3\right)\left(-7z_3^2 + 14z_4z_3 + 26l^2 - 28z_4^2\right),$$

$$m_{8,10}^{(3)} = \frac{1}{2520}\rho^{(3)} b l z_3^2 \left(-z_4 + z_3\right)\left(7z_3^2 - 14z_4z_3 + 28z_4^2 + 18l^2\right),$$

$$m_{99}^{(3)} = -\frac{1}{70}\rho^{(3)} b z_3^2 \left(-z_4 + z_3\right) \frac{7z_3^2 - 14z_4z_3 + 26l^2 + 28z_4^2}{l},$$

$$m_{9,10}^{(3)} = \frac{1}{840}\rho^{(3)} b z_3^2 \left(-z_4 + z_3\right) \left(7z_3^2 - 14z_4z_3 + 44l^2 + 28z_4^2\right),$$

$$m_{10,10}^{(3)} = -\frac{1}{630}\rho^{(3)} b l z_3^2 \left(-z_4 + z_3\right) \left(7z_3^2 - 14z_4z_3 + 6l^2 + 28z_4^2\right).$$

If the upper surface of a finite element is completely covered by the cargo, and the weight of the cargo is evenly distributed over the length of the finite element ($\mu = const$), then the fourth term of the expression (5.10.54) for the kinetic energy is (expression 5.10.67)

$$\frac{1}{2} b\mu \int_0^l \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}}\right)^T \underset{(3\times3)}{\left[\tilde{D}_c\right]} \left(\frac{\partial}{\partial t} \underset{(3\times1)}{\{f\}}\right) dx =$$

$$= \frac{1}{2} \underset{(1\times10)}{\left\{\dot{d}\right\}}^T \underset{(10\times10)}{\left[m^{(c)}\right]} \underset{(10\times1)}{\left\{\dot{d}\right\}},$$

where

$$
\left[m^{(c)}\right] = b\mu \begin{bmatrix}
\frac{13}{35}l & \frac{11}{210}l^2 & \frac{9}{70}l & -\frac{13}{420}l^2 & 0 & 0 & \frac{13}{35}lz_3 & \frac{11}{210}l^2z_3 & \frac{9}{70}lz_3 & -\frac{13}{420}l^2z_3 \\
\frac{11}{210}l^2 & \frac{1}{105}l^3 & \frac{13}{420}l^2 & -\frac{1}{140}l^3 & 0 & 0 & \frac{11}{210}l^2z_3 & \frac{1}{105}l^3z_3 & \frac{13}{420}l^2z_3 & -\frac{1}{140}l^3z_3 \\
\frac{9}{70}l & \frac{13}{420}l^2 & \frac{13}{35}l & -\frac{11}{210}l^2 & 0 & 0 & \frac{9}{70}lz_3 & \frac{13}{420}l^2z_3 & \frac{13}{35}lz_3 & -\frac{11}{210}l^2z_3 \\
-\frac{13}{420}l^2 & -\frac{1}{140}l^3 & -\frac{11}{210}l^2 & \frac{1}{105}l^3 & 0 & 0 & -\frac{13}{420}l^2z_3 & -\frac{1}{140}l^3z_3 & -\frac{11}{210}l^2z_3 & \frac{1}{105}l^3z_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{13}{35}lz_3 & \frac{11}{210}l^2z_3 & \frac{9}{70}lz_3 & -\frac{13}{420}l^2z_3 & 0 & 0 & \frac{13}{35}lz_3^2 & \frac{11}{210}l^2z_3^2 & \frac{9}{70}lz_3^2 & -\frac{13}{420}l^2z_3^2 \\
\frac{11}{210}l^2z_3 & \frac{1}{105}l^3z_3 & \frac{13}{420}l^2z_3 & -\frac{1}{140}l^3z_3 & 0 & 0 & \frac{11}{210}l^2z_3^2 & \frac{1}{105}l^3z_3^2 & \frac{13}{420}l^2z_3^2 & -\frac{1}{140}l^3z_3^2 \\
\frac{9}{70}lz_3 & \frac{13}{420}l^2z_3 & \frac{13}{35}lz_3 & -\frac{11}{210}l^2z_3 & 0 & 0 & \frac{9}{70}lz_3^2 & \frac{13}{420}l^2z_3^2 & \frac{13}{35}lz_3^2 & -\frac{11}{210}l^2z_3^2 \\
-\frac{13}{420}l^2z_3 & -\frac{1}{140}l^3z_3 & -\frac{11}{210}l^2z_3 & \frac{1}{105}l^3z_3 & 0 & 0 & -\frac{13}{420}l^2z_3^2 & -\frac{1}{140}l^3z_3^2 & -\frac{11}{210}l^2z_3^2 & \frac{1}{105}l^3z_3^2
\end{bmatrix}
$$

The mass matrix of a finite element is

$$
[m] = \left[m^{(1)}\right] + \left[m^{(2)}\right] + \left[m^{(3)}\right] + \left[m^{(c)}\right]
$$

5.17 Appendix 5-C. Expressions for the 1-st component of the nonlinear part of the internal force vector

In this appendix, the first component of the nonlinear part of the internal force vector $\frac{\partial U_{nl}}{\partial\{\theta\}}$, that enters into the equation of motion (5.10.85) of a finite element, is written explicitely in terms of the nodal parameters θ_i and the material characteristics of the sandwich plate. The other components of the vector $\frac{\partial U_{nl}}{\partial\{\theta\}}$ are not written here due to the limitation on the size of the dissertation. The expression presented in this appendix was derived by the program for symbolic computation MAPLE, and it was transformed automatically into the FORTRAN format. The quantities s_1, s_2,etc. are the auxiliary quantities that allow to break up a very lengthy expression for $\frac{\partial U_{nl}}{\partial\theta_1}$ into a number of shorter expressions.

So, the expression for the first component of the nonlinear internal force vector $q_1 = \frac{\partial U_{nl}}{\partial\theta_1}$ in FORTRAN format is:

```
s3 = 1/l**2*(1008*theta4**2*l-12*theta10**2*l**3-1008*l*theta9**2+
#12*theta5**2*l**3-24*theta5*l**2*theta9-24*theta4*l**2*theta10+192
#*theta9*l**2*theta10+192*theta4*l**2*theta5)*D2HAT_42/3360
s4 = 1/l**2*(1008*theta4*theta1*l-216*theta10*l**2*theta1+216*thet
#a5*l**2*theta1+1008*theta9*theta1*l+192*theta9*theta2*l**2-12*thet
#a10*l**3*theta7-36*theta10*l**3*theta2-24*theta9*theta7*l**2+12*th
#eta5*l**3*theta2+216*theta10*l**2*theta6+36*theta5*l**3*theta7-216
#*theta5*l**2*theta6-1008*theta4*l*theta6+192*theta4*theta7*l**2-10
#08*theta9*l*theta6-24*theta4*theta2*l**2)*D2HAT_41/3360+(1/l**2*(-
#504*theta4*theta10+168*theta10**2*l-168*theta5**2*l-504*theta9*the
#ta5+504*theta4*theta5+504*theta9*theta10)/3360+1/l**2*(1008*theta4
#*theta10-336*theta10**2*l+336*theta5**2*l+1008*theta9*theta5-1008*
#theta4*theta5-1008*theta9*theta10)/3360)*D2HAT_32
s2 = s3+s4
s1 = s2+1/l**2*(504*theta4*theta7-504*theta9*theta7+504*theta9*the
#ta2+1008*theta10*theta6-1008*theta5*theta6-504*theta4*theta2+336*t
#heta5*l*theta7+168*theta10*l*theta7-168*theta5*l*theta2+1008*theta
#5*theta1-336*theta10*l*theta2-1008*theta10*theta1)*D2HAT_31/3360+1
#/l**2*(1008*theta4**2*l-12*theta10**2*l**3-1008*l*theta9**2+12*the
```

```
#ta5**2*l**3-24*theta5*l**2*theta9-24*theta4*l**2*theta10+192*theta
#9*l**2*theta10+192*theta4*l**2*theta5)*D2HAT_24/3360+(1/l**2*(-504
#*theta4*theta10+168*theta10**2*l-168*theta5**2*l-504*theta9*theta5
#+504*theta4*theta5+504*theta9*theta10)/3360+1/l**2*(1008*theta4*th
#eta10-336*theta10**2*l+336*theta5**2*l+1008*theta9*theta5-1008*the
#ta4*theta5-1008*theta9*theta10)/3360)*D2HAT_23
s3 = s1+1/l**2*(-1008*theta4*theta7+1008*theta9*theta7-1008*theta9
#*theta2-2016*theta10*theta6+2016*theta5*theta6+1008*theta4*theta2-
#4032*theta3*theta4+4032*theta8*theta4-4032*theta8*theta9+4032*thet
#a3*theta9-672*theta5*l*theta7-336*theta10*l*theta7+336*theta5*l*th
#eta2-2016*theta5*theta1-336*theta3*l*theta10+336*theta8*l*theta5+3
#36*theta8*l*theta10+672*theta10*l*theta2-336*theta3*l*theta5+2016*
#theta10*theta1)*D2HAT_22/1680
s2 = s3+1/l**2*(-336*theta3*l*theta7-4032*theta3*theta1-336*theta3
#*l*theta2+336*theta8*l*theta7+4032*theta3*theta6+4032*theta8*theta
#1-4032*theta8*theta6+336*theta8*l*theta2)*D2HAT_21/3360+1/l**2*(10
#08*theta4*theta1*l-216*theta10*l**2*theta1+216*theta5*l**2*theta1+
#1008*theta9*theta1*l+192*theta9*theta2*l**2-12*theta10*l**3*theta7
#-36*theta10*l**3*theta2-24*theta9*theta7*l**2+12*theta5*l**3*theta
#2+216*theta10*l**2*theta6+36*theta5*l**3*theta7-216*theta5*l**2*th
#eta6-1008*theta4*l*theta6+192*theta4*theta7*l**2-1008*theta9*l*the
#ta6-24*theta4*theta2*l**2)*D2HAT_14/3360
s3 = s2+1/l**2*(504*theta4*theta7-504*theta9*theta7+504*theta9*the
#ta2+1008*theta10*theta6-1008*theta5*theta6-504*theta4*theta2+336*t
#heta5*l*theta7+168*theta10*l*theta7-168*theta5*l*theta2+1008*theta
#5*theta1-336*theta10*l*theta2-1008*theta10*theta1)*D2HAT_13/3360
s4 = s3+1/l**2*(-336*theta3*l*theta7-4032*theta3*theta1-336*theta3
#*l*theta2+336*theta8*l*theta7+4032*theta3*theta6+4032*theta8*theta
#1-4032*theta8*theta6+336*theta8*l*theta2)*D2HAT_12/3360
s5 = s4
s8 = 1/l**2/60
s11 = 6*theta2*l*theta5*z3**2+12*theta5*z3**2*l*theta8-12*theta5*z
```

```
#3**2*l*theta3-12*theta2*l*z3*theta3+12*theta2*l*z3*theta8+12*theta
#10*z3**2*l*theta8-12*theta7*l*theta5*z3**2-6*theta7*l*z3**2*theta1
#0+12*theta7*l*z3*theta8-12*theta7*l*z3*theta3-12*theta10*z3**2*l*t
#heta3-18*theta2*z3**2*theta9+18*theta9*z3**3*theta5+18*theta4*z3**
#3*theta10-18*theta4*z3**3*theta5+18*theta2*z3**2*theta4+144*theta9
#*z3**2*theta3
s10 = s11+6*theta5**2*z3**3*l+144*theta6*z3*theta3-6*theta10**2*z3
#**3*l+36*theta6*theta5*z3**2-36*theta6*z3**2*theta10+18*theta7*z3*
#*2*theta9-18*theta7*z3**2*theta4-18*theta9*z3**3*theta10+144*theta
#4*z3**2*theta8-144*theta4*z3**2*theta3-144*theta9*z3**2*theta8-144
#*theta6*z3*theta8+12*theta2*l*z3**2*theta10-144*theta1*z3*theta3+1
#44*theta1*z3*theta8-36*theta1*theta5*z3**2+36*theta1*z3**2*theta10
s11 = D3_11
s9 = s10*s11
s7 = s8*s9
s9 = 1/l**2/60
s12 = 36*theta1*z2**2*theta10+144*theta6*z2*theta3-144*theta6*z2*t
#heta8+144*theta1*z2*theta8-6*theta10**2*z2**3*l-144*theta9*z2**2*t
#heta8+144*theta9*z2**2*theta3-18*theta2*z2**2*theta9+6*theta5**2*z
#2**3*l-18*theta9*z2**3*theta10-144*theta1*z2*theta3+144*theta4*z2*
#*2*theta8-36*theta1*theta5*z2**2-144*theta4*z2**2*theta3-18*theta4
#*z2**3*theta5-36*theta6*z2**2*theta10+36*theta6*theta5*z2**2
s11 = s12+18*theta4*z2**3*theta10+18*theta2*z2**2*theta4-18*theta7
#*z2**2*theta4+18*theta9*z2**3*theta5+18*theta7*z2**2*theta9-12*the
#ta7*l*theta5*z2**2+12*theta5*z2**2*l*theta8+12*theta2*l*z2*theta8-
#12*theta5*z2**2*l*theta3+12*theta2*l*z2**2*theta10+12*theta10*z2**
#2*l*theta8-12*theta10*z2**2*l*theta3-6*theta7*l*z2**2*theta10+6*th
#eta2*l*theta5*z2**2-12*theta7*l*z2*theta3+12*theta7*l*z2*theta8-12
#*theta2*l*z2*theta3
s12 = D1_11
s10 = s11*s12
s8 = s9*s10
```

s6 = s7+s8

q1 = s5+s6

5.18 Appendix 5-D. Location of the error-minimal points for computation of spacial derivatives of the field variables

In order to calculate the stresses, there is a need for accurate estimates of the derivatives of the field variables w_0, $\varepsilon_{xz}^{(2)}$, $\varepsilon_{zz}^{(2)}$. In the finite element formulation, the functions w_0 and $\varepsilon_{zz}^{(2)}$ are approximated by the Hermit interpolation polynomials of the third degree, and the function $\varepsilon_{xz}^{(2)}$ is approximated by the Lagrange polynomial of the first degree. The derivatives $\frac{\partial w_0}{\partial x}$, $\frac{\partial^2 w_0}{\partial x^2}$, $\frac{\partial^3 w_0}{\partial x^3}$, $\frac{\partial \varepsilon_{zz}^{(2)}}{\partial x}$, $\frac{\partial^2 \varepsilon_{zz}^{(2)}}{\partial x^2}$, $\frac{\partial^3 \varepsilon_{zz}^{(2)}}{\partial x^3}$, $\frac{\partial \varepsilon_{xz}^{(2)}}{\partial x}$ will be computed as the derivatives of the interpolation polynomials that were used in the finite element formulation. In this appendix we will discuss a question of location in a finite element of optimal points that give the most accurate estimates of these derivatives. In this discussion, the ideas of Akin (1987) will be used.

The values of primary variables (those variables that are involved in specification of the essential boundary conditions, and whose values at the nodes are used as the nodal parameters in the finite element formulation) are most accurate at the nodal points, in some problems even exact (Reddy, 1993, page 206). In our problem, the nodal parametrs are $w_0(x,t)$, $\frac{\partial w_0(x,t)}{\partial x}$, $\varepsilon_{zz}^{(2)}$, $\frac{\partial \varepsilon_{zz}^{(2)}}{\partial x}$ and $\varepsilon_{xz}^{(2)}$. Therefore, the values of $\frac{\partial w_0}{\partial x}$, $\varepsilon_{zz}^{(2)}$, $\frac{\partial \varepsilon_{zz}^{(2)}}{\partial x}$ and $\varepsilon_{xz}^{(2)}$, that enter into the expressions for the stresses, must be computed at the nodes and can be taken directly from the finite element solution.

Now, let us consider the **computation of the second derivative** $\frac{\partial^2 w_0}{\partial x^2}$. In the finite element formulation, displacement w_0 is approximated by a polynomial of the third degree. If the exact solution for w_0 is a polynomial of the same or lower degree (or if the exact solution can be best approximated by a polynomial of the third or lower degree), then the finite element solution for w_0 will be exact at each point of the finite element (or very close to exact, if the exact solution can be best approximated by a polynomial of the third or lower degree). In this case the second derivative (with respect to x) of the interpolation polynomial for w_0 will coincide with $\frac{\partial^2 w_0}{\partial x^2}$ obtained from the exact solution, at each point of the finite element. But such situations occur very rarely. Now, let us consider a situation, when the exact solution for w_0 is a polynomial of the fourth degree, and let the superscripts e and f denote the exact and the finite element solution, respectively:

$$w_0^{(f)} = a_0 + a_1\overline{x} + a_2\overline{x}^2 + a_3\overline{x}^3, \tag{5-D.1}$$

$$w_0^{(e)} = b_0 + b_1\overline{x} + b_2\overline{x}^2 + b_3\overline{x}^3 + b_4\overline{x}^4. \tag{5-D.2}$$

As it was mentioned previously, the computed values of the primary variables are most accurate at the nodal points. For simplicity we will assume that the values of the primary variables at the nodes are exact:

$$w_0^{(f)}(0) = w_0^{(e)}(0), \tag{5-D.3}$$

$$\frac{\partial w_0^{(f)}}{\partial x}(0) = \frac{\partial w_0^{(e)}}{\partial x}(0), \tag{5-D.4}$$

$$w_0^{(f)}(l) = w_0^{(e)}(l), \tag{5-D.5}$$

$$\frac{\partial w_0^{(f)}}{\partial x}(l) = \frac{\partial w_0^{(e)}}{\partial x}(l). \tag{5-D.6}$$

If we substitute equations (5-D.1) and (5-D.2) into equations (5-D.3)-(5-D.6) we obtain, respectively

$$a_0 = b_0, \tag{5-D.7}$$

$$a_1 = b_1, \tag{5-D.8}$$

$$a_0 + la_1 + l^2a_2 + l^3a_3 = b_0 + lb_1 + l^2b_2 + l^3b_3 + l^4b_4, \tag{5-D.9}$$

$$a_1 + 2la_2 + 3l^2a_3 = b_1 + 2lb_2 + 3l^2b_3 + 4l^3b_4. \tag{5-D.10}$$

Let x_0 be an optimal point for computation of $\frac{\partial^2 w_0}{\partial x^2}$, i.e.

$$\frac{\partial^2 w_0^{(f)}}{\partial x^2}(x_0) = \frac{\partial^2 w_0^{(e)}}{\partial x^2}(x_0). \tag{5-D.11}$$

Substitution of equations (5-D.1) and (5-D.2) into equation (5-D.11) yields:

$$2a_2 + 6x_0a_3 = 2b_2 + 6x_0b_3 + 12x_0^2b_4. \tag{5-D.12}$$

Solving equations (5-D.7)–(5-D.10) and (5-D.12) simultaneously for a_0, a_1, a_2, a_3 and x_0, we obtain:

$$a_0 = b_0,\ a_1 = b_1,\ a_3 = b_3 + 2lb_4,\ a_2 = -l^2b_4 + b_2,\ x_0 = \left(\frac{1}{2} \pm \frac{1}{6}\sqrt{3}\right)l, \tag{5-D.13}$$

So, if the exact solution for w_0 is a polynomial of the fourth degree, then the coordinates of the optimal points for computing $\frac{\partial^2 w_0}{\partial x^2}$ are

$$x_0^{(1)} = \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right) l = 0.788\,68\ l \ \text{ and } x_0^{(2)} = \left(\frac{1}{2} - \frac{1}{6}\sqrt{3}\right) l = 0.211\,32\ l. \tag{5-D.14}$$

These are the Gauss points of the third-degree polynomial.

Now, let us consider a situation, when the exact solution for w_0 is a polynomial of the fifth degree:

$$w_0^{(e)} = b_0 + b_1\overline{x} + b_2\overline{x}^2 + b_3\overline{x}^3 + b_4\overline{x}^4 + b_5\overline{x}^5 \tag{5-D.15}$$

Then equations (5-D.3)–(5-D.6) and (5-D.11) lead to the following equations:

$$\left.\begin{aligned} a_0 &= b_0 \\ a_1 &= b_1 \\ a_0 + la_1 + l^2a_2 + l^3a_3 &= b_0 + lb_1 + l^2b_2 + l^3b_3 + l^4b_4 + l^5b_5 \\ a_1 + 2la_2 + 3l^2a_3 &= b_1 + 2lb_2 + 3l^2b_3 + 4l^3b_4 + 5l^4b_5 \\ 2a_2 + 6x_0a_3 &= 2b_2 + 6x_0b_3 + 12x_0^2b_4 + 20x_0^3b_5 \end{aligned}\right\} \tag{5-D.16}$$

From the first four equations of the system (5-D.16) we obtain:

$$a_2 = -l^2b_4 - 2l^3b_5 + b_2, \qquad a_3 = b_3 + 2lb_4 + 3l^2b_5. \tag{5-D.17}$$

If we substitute expressions for a_2 and a_3 into the last equation of the system (5-D.16), we obtain

$$\left(12x_0l - 2l^2 - 12x_0^2\right) b_4 + \left(-4l^3 + 18l^2x_0 - 20x_0^3\right) b_5 = 0. \tag{5-D.18}$$

Equation (5-D.18) can be satisfied for arbitrary b_4 and b_5 if coefficients of b_4 and b_5 are equal to zero. This leads to the following two equations for the coordinate x_0 of the optimum point:

$$12x_0l - 2l^2 - 12x_0^2 = 0, \tag{5-D.19}$$

$$-4l^3 + 18l^2x_0 - 20x_0^3 = 0. \tag{5-D.20}$$

The solutions of equation (5-D.19) are

$$x_0^{(1)} = \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right) l = 0.788\,68\ l \ \text{ and } \ x_0^{(2)} = \left(\frac{1}{2} - \frac{1}{6}\sqrt{3}\right) l = 0.211\,32\ l \tag{5-D.21}$$

The solutions of equation (5-D.20) in the element's domain $0 \le \overline{x} \le l$ are

$$x_0^{(3)} = 0.8077\, l, \;\; x_0^{(4)} = 0.23702\, l\,. \tag{5-D.22}$$

So, if the exact solution for w_0 is a polynomial of the fifth degree, then the coordinates of the optimal points for computing $\frac{\partial^2 w_0}{\partial x^2}$ are

$$x_0^{(1)} = \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right) l = 0.78868\, l\,, \;\; x_0^{(2)} = \left(\frac{1}{2} - \frac{1}{6}\sqrt{3}\right) l = 0.21132\, l,$$

$$x_0^{(3)} = 0.8077\, l, \;\; x_0^{(4)} = 0.23702\, l. \tag{5-D.23}$$

Coordinates $x_0^{(1)}$ and $x_0^{(2)}$ are the Gauss points.

If the exact solution for w_0 is a polynomial of the sixth degree, then, in a similar manner, we find the following coordinates of the optimal points for computation of $\frac{\partial^2 w_0}{\partial x^2}$:

$$x_0^{(1)} = \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right) l = 0.78868\, l\,, \;\; x_0^{(2)} = \left(\frac{1}{2} - \frac{1}{6}\sqrt{3}\right) l = 0.21132\, l,$$

$$x_0^{(3)} = 0.8077\, l, \;\; x_0^{(4)} = 0.23702\, l,$$

$$x_0^{(5)} = 0.82274\, l, \;\; x_0^{(6)} = 0.25531\, l. \tag{5-D.24}$$

So, regardless of the degree of a polynomial of exact solution, the Gauss points $x_0^{(1)} = \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right) l = 0.78868\, l$ and $x_0^{(2)} = \left(\frac{1}{2} - \frac{1}{6}\sqrt{3}\right) l = 0.21132\, l$ are the coordinates of the optimal points for computation of $\frac{\partial^2 w_0}{\partial x^2}$ (but there may exist other optimal points, in addition to the Gauss points).

Now, let us consider **computation of the third derivative** $\frac{\partial^3 w_0}{\partial x^3}$. Let us consider a situation, when the exact solution for w_0 is a polynomial of the fourth degree (equation (2)). Let x_0 be a point where the finite element solution and the exact solution for $\frac{\partial^3 w_0}{\partial x^3}$ are the same. Then

$$\frac{\partial^3 w_0^{(f)}}{\partial x^3}(x_0) = \frac{\partial^3 w_0^{(e)}}{\partial x^3}(x_0)\,. \tag{5-D.25}$$

If we substitute equations (5-D.1) and (5-D.2) into equations (5-D.25), we receive

$$6a_3 = 6b_3 + 24x_0 b_4. \tag{5-D.26}$$

Equations (5-D.7)-(5-D.10) and equation (5-D.26) make the following system:

$$\left.\begin{aligned} a_0 &= b_0,\\ a_1 &= b_1,\\ a_0 + la_1 + l^2a_2 + l^3a_3 &= b_0 + lb_1 + l^2b_2 + l^3b_3 + l^4b_4,\\ a_1 + 2la_2 + 3l^2a_3 &= b_1 + 2lb_2 + 3l^2b_3 + 4l^3b_4,\\ 6a_3 &= 6b_3 + 24x_0b_4. \end{aligned}\right\} \tag{5-D.27}$$

If we solve this system of equations for a_0, a_1, a_2, a_3, x_0, we receive

$$a_0 = b_0,\ a_1 = b_1,\ a_2 = -l^2b_4 + b_2,\ a_3 = b_3 + 2lb_4,\ x_0 = \frac{1}{2}l. \tag{5-D.28}$$

So, the if the exact solution is a polynomial of the fourth degree (or if it is best approximated by the polynomial of the fourth degree), then the finite element solution for $\frac{\partial^3 w_0}{\partial x^3}$ is equal to the exact solution (or is the closest to the exact solution) in the middle of the element, at the point $x_0 = \frac{l}{2}$.

Let us consider a situation, when the exact solution for w_0 is a polynomial of the fifth degree (equation (5-D.15)). Then equations (5-D.3)-(5-D.6) and (5-D.25) lead to the following equations:

$$\left.\begin{aligned} a_0 &= b_0,\\ a_1 &= b_1,\\ a_0 + la_1 + l^2a_2 + l^3a_3 &= b_0 + lb_1 + l^2b_2 + l^3b_3 + l^4b_4 + l^5b_5,\\ a_1 + 2la_2 + 3l^2a_3 &= b_1 + 2lb_2 + 3l^2b_3 + 4l^3b_4 + 5l^4b_5,\\ 6a_3 &= 6b_3 + 24x_0b_4 + 60x_0^2b_5. \end{aligned}\right\} \tag{5-D.29}$$

From the first four equations of the system (5-D.29) we obtain:

$$a_2 = -l^2b_4 - 2l^3b_5 + b_2,\ a_3 = b_3 + 2lb_4 + 3l^2b_5. \tag{5-D.30}$$

If we substitute expressions (5-D.30) for a_2 and a_3 into the last equation of the system (5-D.29), we obtain the following equation:

$$(12l - 24x_0)\, b_4 + \left(18l^2 - 60x_0^2\right) b_5 = 0. \tag{5-D.31}$$

Equation (5-D.31) can be satisfied for arbitrary b_4 and b_5 if coefficients of b_4 and b_5 are equal to zero. This leads to the following two equations for the coordinate x_0, at which the finite element and the exact solution for $\frac{\partial^3 w_0}{\partial x^3}$ coincide:

$$12l - 24x_0 = 0, \tag{5-D.32}$$

$$18l^2 - 60x_0^2 = 0. \tag{5-D.33}$$

The solutions of these equations in the element's domain $0 \le \overline{x} \le l$ are

$$x_0^{(1)} = \frac{l}{2},\ x_0^{(2)} = 0.54772\ l \tag{5-D.34}$$

So, if the exact solution for w_0 is a polynomial of the fifth degree, then the finite element solution for $\frac{\partial^3 w_0}{\partial x^3}$ is equal to the exact solution at the points $x_0^{(1)} = \frac{l}{2}$, $x_0^{(2)} = 0.54772\ l$.

In a similar manner it can be shown that if the exact solution for w_0 is a polynomial of any degree higher than three, then the finite element solution for $\frac{\partial^3 w_0}{\partial x^3}$ is equal to the exact solution for $\frac{\partial^3 w_0}{\partial x^3}$ at the point $x_0^{(1)} = \frac{l}{2}$ (and at some other points, if the exact solution is a polynomial of a degree higher than four).

The interpolation polynomial for $\varepsilon_{zz}^{(2)}$ is the same as for the w_0. Therefore, all conclusions regarding computation of spatial derivatives of w_0 are also valid for the computation of spatial derivatives of $\varepsilon_{zz}^{(2)}$.

Now, let us consider location of error-minimal points for computation of $\varepsilon_{xz}^{(2)}$ and $\frac{\partial \varepsilon_{xz}^{(2)}}{\partial x}$. Strain $\varepsilon_{xz}^{(2)}$ is one of the primary variables of the problem, and in the finite element formulation $\varepsilon_{xz}^{(2)}$ is approximated by the Lagrange polynomial of the first degree:

$$\left(\varepsilon_{xz}^{(2)}\right)^{(f)} = a_0 + a_1\overline{x}. \tag{5-D.35}$$

Therefore, **the most accurate values of $\varepsilon_{xz}^{(2)}$ are at the nodes.**

Let us consider a situation, when an exact solution is a polynomial of the second degree:

$$\left(\varepsilon_{xz}^{(2)}\right)^{(e)} = b_0 + b_1\overline{x} + b_2\overline{x}^2, \tag{5-D.36}$$

and let x_0 be a point, where the finite element and the exact solution for $\frac{\partial \varepsilon_{xz}^{(2)}}{\partial x}$ coincide:

$$\frac{\partial \left(\varepsilon_{xz}^{(2)}\right)^{(f)}}{\partial x}(x_0) = \frac{\partial \left(\varepsilon_{xz}^{(2)}\right)^{(e)}}{\partial x}(x_0). \tag{5-D.37}$$

From equation (5-D.35) – (5-D.37) we obtain:

$$a_1 = b_1 + 2b_2x_0. \tag{5-D.38}$$

Since the most accurate values of $\varepsilon_{xz}^{(2)}$ are at the nodes, at points $\overline{x} = 0$ and $\overline{x} = l$, we can write

$$\left.\left(\varepsilon_{xz}^{(2)}\right)^{(f)}\right|_{\overline{x}=0} = \left.\left(\varepsilon_{xz}^{(2)}\right)^{(e)}\right|_{\overline{x}=0}, \quad \left.\left(\varepsilon_{xz}^{(2)}\right)^{(f)}\right|_{\overline{x}=l} = \left.\left(\varepsilon_{xz}^{(2)}\right)^{(e)}\right|_{\overline{x}=l}. \tag{5-D.39}$$

Substitution of equations (5-D.35) and (5-D.36) into equations (5-D.39) gives the following equations:

$$a_0 = b_0 \tag{5-D.40}$$

$$a_0 + a_1 l = b_0 + b_1 l + b_2 l^2 \tag{5-D.41}$$

The solution of equations (5-D.38), (5-D.40) and (5-D.41) with respect to a_0, a_1, x_0 is:

$$a_0 = b_0,\ a_1 = b_1 + b_2 l,\ x_0 = \frac{1}{2} l. \tag{5-D.42}$$

Therefore, if the exact solution is a polynomial of the second degree, then the finite element solution for $\frac{\partial \varepsilon_{xz}^{(2)}}{\partial x}$ coincides with exact solution in the middle of the element, at the point $x_0 = \frac{1}{2} l$.

Let us consider a situation, when an exact solution is a polynomial of the third degree:

$$\left(\varepsilon_{xz}^{(2)}\right)^{(e)} = b_0 + b_1 \overline{x} + b_2 \overline{x}^2 + b_3 \overline{x}^3. \tag{5-D.43}$$

Then, from equations (5-D.35), (5-D.43) and (5-D.37) we receive

$$a_1 = b_1 + 2 b_2 x_0 + 3 b_3 x_0^2 \tag{5-D.44}$$

From equations (5-D.40) and (5-D.41) we obtain

$$a_1 = b_1 + b_2 l \tag{5-D.45}$$

Substitution of equation (5-D.45) into equation (5-D.44) yields:

$$b_2 \left(l - 2 x_0\right) - 34 b_3 x_0^2 = 0, \tag{5-D.46}$$

from where we find

$$x_0^{(1)} = \frac{l}{2},\ x_0^{(2)} = 0, \tag{5-D.47}$$

i.e. if the exact solution is a polynomial of the third degree, then the finite element solution for $\frac{\partial \varepsilon_{xz}^{(2)}}{\partial x}$ coincides with exact solution in the middle of the element, at the point $x_0^{(1)} = \frac{1}{2} l$, and at the left end of the element, at the point $x_0^{(2)} = 0$.

In a similar manner it can be shown that if the exact solution for $\varepsilon_{xz}^{(2)}$ is a polynomial of any degree higher than one, then the finite element solution for $\frac{\partial \varepsilon_{xz}^{(2)}}{\partial x}$ is equal to the exact solution for $\frac{\partial \varepsilon_{xz}^{(2)}}{\partial x}$ at the point $x_0^{(1)} = \frac{l}{2}$ (and at some other points, if the exact solution is a polynomial of a degree higher than two).

5.19 Appendix 5-E. Verification problem for finite element program: exact analysis for vibration of simply-supported homogeneous isotropic plate in cylindrical bending

An exact analysis for vibration of a simply-supported rectangular plate was performed by Srinivas, Joga Rao and Rao (1970). In this chapter we will find natural frequencies for the simply-supported plate in cylindrical bending, following the method of Srinivas, Joga Rao and Rao. The plate is considered to be homogeneous and isotropic. Besides, we will find transient response of such plate, dropped on the simple supports. The solution, that we obtain in this chapter, has the form of the infinite series, and it is exact in the sense that

1) each term of the series for the displacements satisfies the equations of motion of linear elasticity, written in terms of displacements, with **no additional assumptions** about through-the-thickness variation of displacements, strains or stresses;

2) each term of the series for the displacements satisfies boundary conditions of a simply supported plate.

The displacements of the solution, represented by the finite number of terms in the series, satisfy the initial conditions approximately, but with any desired accuracy, that is achieved by taking sufficient number of terms in the expansion. In other words, the series that represent the displacements and their time derivatives at the initial moment of time, converge to the initial displacements and initial velocities.

Let us write equations of motion for a plate in cylindrical bending in terms of displacements:

$$\frac{\partial^2 u(x,z,t)}{\partial x^2}+\frac{\partial^2 u(x,z,t)}{\partial z^2}+\frac{1}{1-2\nu}\frac{\partial^2 u(x,z,t)}{\partial x^2}+\frac{1}{1-2\nu}\frac{\partial^2 w(x,z,t)}{\partial x\partial z}=\frac{\rho}{G}\frac{\partial^2 u(x,z,t)}{\partial t^2}\ , \tag{5-E.1}$$

$$\frac{\partial^2 w(x,z,t)}{\partial x^2}+\frac{\partial^2 w(x,z,t)}{\partial z^2}+\frac{1}{1-2\nu}\frac{\partial^2 u(x,z,t)}{\partial x\partial z}+\frac{1}{1-2\nu}\frac{\partial^2 w(x,z,t)}{\partial z^2}=\frac{\rho}{G}\frac{\partial^2 w(x,z,t)}{\partial t^2}. \tag{5-E.2}$$

To separate variables we seek solution in the form:

$$u\,(x,z,t)=U\,(x,z)\,T\,(t)\ , \tag{5-E.3}$$

$$w\,(x,z,t)=W\,(x,z)\,T\,(t)\ . \tag{5-E.4}$$

Substitution of equations (5-E.3) and (5-E.4) into equations (5-E.1) and (5-E.2) yields:

$$\left[\frac{\partial^2 U(x,z)}{\partial x^2} + \frac{\partial^2 U(x,z)}{\partial z^2} + \frac{1}{1-2\nu}\frac{\partial^2 U(x,z)}{\partial x^2} + \frac{1}{1-2\nu}\frac{\partial^2 W(x,z)}{\partial x \partial z}\right]\frac{1}{U(x,z)} =$$

$$= \frac{\rho}{G}\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = -\frac{\rho}{G}\Omega^2, \tag{5-E.5}$$

$$\left[\frac{\partial^2 W(x,z)}{\partial x^2} + \frac{\partial^2 W(x,z)}{\partial z^2} + \frac{1}{1-2\nu}\frac{\partial^2 U(x,z)}{\partial x \partial z} + \frac{1}{1-2\nu}\frac{\partial^2 W(x,z)}{\partial z^2}\right]\frac{1}{W(x,z)} =$$

$$= \frac{\rho}{G}\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = -\frac{\rho}{G}\Omega^2. \tag{5-E.6}$$

Therefore, we have the following differential equations for the functions $U(x,z)$, $W(x,z)$ and $T(t)$:

$$2\frac{1-\nu}{1-2\nu}\frac{\partial^2 U(x,z)}{\partial x^2} + \frac{\partial^2 U(x,z)}{\partial z^2} + \frac{1}{1-2\nu}\frac{\partial^2 W(x,z)}{\partial x \partial z} + \frac{\rho}{G}\Omega^2 U(x,z) = 0, \tag{5-E.7}$$

$$2\frac{1-\nu}{1-2\nu}\frac{\partial^2 W(x,z)}{\partial z^2} + \frac{\partial^2 W(x,z)}{\partial x^2} + \frac{1}{1-2\nu}\frac{\partial^2 U(x,z)}{\partial x \partial z} + \frac{\rho}{G}\Omega^2 W(x,z) = 0, \tag{5-E.8}$$

$$\frac{d^2 T(t)}{dt^2} + \Omega^2 T(t) = 0. \tag{5-E.9}$$

The solution of equation (5-E.9) is the following

$$T(t) = Q\cos\Omega t + R\sin\Omega t, \tag{5-E.10}$$

where Q and R are constants of integration.

The boundary conditions for a simply supported plate are

$$w(x,z,t) = 0 \text{ and } \sigma_{xx} = 0 \qquad \text{at } x = 0 \text{ and } x = L. \tag{5-E.11}$$

But, according to Hooke's law

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial u}{\partial x} + \nu\frac{\partial w}{\partial z}\right]. \tag{5-E.12}$$

Therefore, the boundary conditions for the functions $U(x,z)$ and $W(x,z)$ are

$$W(x,z) = 0 \text{ and } (1-\nu)\frac{\partial U(x,z)}{\partial x} + \nu\frac{\partial W(x,z)}{\partial z} = 0 \text{ at } x = 0 \text{ and } x = L. \tag{5-E.13}$$

These boundary conditions are identically satisfied by setting

$$U(x,z)=\sum_{m=1}^{\infty}\phi_m(z)\cos\left(\frac{m\pi x}{L}\right), \tag{5-E.14}$$

$$W(x,z)=\sum_{m=1}^{\infty}\chi_m(z)\sin\left(\frac{m\pi x}{L}\right). \tag{5-E.15}$$

Substitution of equations (5-E.14) and (5-E.15) into equations (5-E.7) and (5-E.8) yields:

$$\sum_{m=1}^{\infty}\left[-2\frac{1-\nu}{1-2\nu}\left(\frac{m\pi}{L}\right)^2\phi_m(z)+\frac{d^2\phi_m(z)}{dz^2}+\frac{1}{1-2\nu}\frac{m\pi}{L}\frac{d\chi_m(z)}{dz}+\frac{\rho}{G}\Omega^2\phi_m(z)\right]\cos\left(\frac{m\pi x}{L}\right)=0, \tag{5-E.16}$$

$$\sum_{m=1}^{\infty}\left[2\frac{1-\nu}{1-2\nu}\frac{d^2\chi_m(z)}{dz^2}-m^2\frac{\pi^2}{L^2}\chi_m(z)-\frac{1}{1-2\nu}m\frac{\pi}{L}\frac{d\phi_m(z)}{dz}+\frac{\rho}{G}\Omega^2\chi_m(z)\right]\sin\left(\frac{m\pi x}{L}\right)=0. \tag{5-E.17}$$

Equating to zero the coefficients of $\cos\left(\frac{m\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ in equations (5-E.16) and (5-E.17), we obtain the following differential equations for the functions $\phi_m(z)$ and $\chi_m(z)$:

$$\frac{d^2\phi_m(z)}{dz^2}+\frac{1}{1-2\nu}\frac{m\pi}{L}\frac{d\chi_m(z)}{dz}+\left[\frac{\rho}{G}\Omega^2-2\frac{1-\nu}{1-2\nu}\left(\frac{m\pi}{L}\right)^2\right]\phi_m(z)=0, \tag{5-E.18}$$

$$2\frac{1-\nu}{1-2\nu}\frac{d^2\chi_m(z)}{dz^2}-\frac{1}{1-2\nu}\frac{m\pi}{L}\frac{d\phi_m(z)}{dz}+\left[\frac{\rho}{G}\Omega^2-\left(\frac{m\pi}{L}\right)^2\right]\chi_m(z)=0. \tag{5-E.19}$$

The non-trivial solution of these differential equations is the following:

$$\begin{aligned}\phi_m(z)=&\sqrt{\left(\frac{m\pi}{L}\right)^2-\Omega^2\frac{\rho}{G}}\frac{m\pi}{L}\left[A\exp\left(\sqrt{\left(\frac{m\pi}{L}\right)^2-\Omega^2\frac{\rho}{G}}z\right)-K\exp\left(-\sqrt{\left(\frac{m\pi}{L}\right)^2-\Omega^2\frac{\rho}{G}}z\right)\right]+\\&+\frac{m\pi}{L}\left[C\exp\left(\sqrt{\left(\frac{m\pi}{L}\right)^2-\Omega^2\frac{\rho}{G}\frac{1-2\nu}{2(1-\nu)}}z\right)+S\exp\left(-\sqrt{\left(\frac{m\pi}{L}\right)^2-\Omega^2\frac{\rho}{G}\frac{1-2\nu}{2(1-\nu)}}z\right)\right]\end{aligned} \tag{5-E.20}$$

$$\chi_m(z) = \left(\frac{m\pi}{L}\right)^2 \left(A\exp\left(\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2\frac{\rho}{G}}z\right) + K\exp\left(-\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2\frac{\rho}{G}}z\right)\right) +$$
$$+\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2\frac{\rho}{G}\frac{1-2\nu}{2(1-\nu)}}\times$$
$$\times\left(C\exp\left(\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2\frac{\rho}{G}\frac{1-2\nu}{2(1-\nu)}}z\right) - S\exp\left(-\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2\frac{\rho}{G}\frac{1-2\nu}{2(1-\nu)}}z\right)\right), \tag{5-E.21}$$

where A, K, C, S are the constants of integration.

Now, let us write the stress-displacement relations:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} E\frac{\nu-1}{2\nu^2+\nu-1} & -E\frac{\nu}{2\nu^2+\nu-1} & -E\frac{\nu}{2\nu^2+\nu-1} & 0 & 0 & 0 \\ -E\frac{\nu}{2\nu^2+\nu-1} & E\frac{\nu-1}{2\nu^2+\nu-1} & -E\frac{\nu}{2\nu^2+\nu-1} & 0 & 0 & 0 \\ -E\frac{\nu}{2\nu^2+\nu-1} & -E\frac{\nu}{2\nu^2+\nu-1} & E\frac{\nu-1}{2\nu^2+\nu-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2(1+\nu)}E & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2(1+\nu)}E & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2(1+\nu)}E \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \tag{5-E.22}$$

In case of cylindrical bending, when $v = 0$ and derivatives with respect to y are equal to zero, the stress-displacement relations take the form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} E\frac{\nu-1}{2\nu^2+\nu-1} & -E\frac{\nu}{2\nu^2+\nu-1} & 0 \\ -E\frac{\nu}{2\nu^2+\nu-1} & -E\frac{\nu}{2\nu^2+\nu-1} & 0 \\ -E\frac{\nu}{2\nu^2+\nu-1} & E\frac{\nu-1}{2\nu^2+\nu-1} & 0 \\ 0 & 0 & \frac{1}{2(1+\nu)}E \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{Bmatrix} \tag{5-E.23}$$

$$\sigma_{yz} = 0, \quad \sigma_{xy} = 0 \tag{5-E.24}$$

Substitution of expressions for displacements

$$u(x,z,t) = U(x,z)\,T(t) = \sum_{m=1}^{\infty} \phi_m(z)\cos\left(\frac{m\pi x}{L}\right)T(t), \tag{5-E.25}$$

$$w(x,z,t) = W(x,z)\,T(t) = \sum_{m=1}^{\infty} \chi_m(z)\sin\left(\frac{m\pi x}{L}\right)T(t) \tag{5-E.26}$$

into the stress-displacement relations (5-E.23) yields

$$\sigma_{xx} = \frac{E}{(1+\nu)(2\nu-1)}\left[(\nu-1)\frac{\partial u}{\partial x} - \nu\frac{\partial w}{\partial z}\right] =$$

$$= \frac{E}{(1+\nu)(2\nu-1)}\sum_{m=1}^{\infty}\left[\left((1-\nu)\phi_m(z)\frac{m\pi}{L} - \nu\frac{d\chi_m(z)}{dz}\right)\sin\left(\frac{m\pi x}{L}\right)T(t)\right], \tag{5-E.27}$$

where $\phi_m(z)$ is defined by equation (5-E.20), and, according to equation (5-E.21),

$$\frac{d\chi_m(z)}{dz} = \left(\frac{m\pi}{L}\right)^2\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2\frac{\rho}{G}}\times$$

$$\times\left[A\exp\left(\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2\frac{\rho}{G}}z\right) - K\exp\left(-\sqrt{\left(m^2\frac{\pi^2}{L^2} - \Omega^2\frac{\rho}{G}\right)}\,z\right)\right] +$$

$$+\left(\left(\frac{m\pi}{L}\right)^2 - \Omega^2\frac{\rho}{G}\frac{1-2\nu}{2(1-\nu)}\right)\times$$

$$\times\left[C\exp\left(\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2\frac{\rho}{G}\frac{1-2\nu}{2(1-\nu)}}\,z\right) + S\exp\left(-\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2\frac{\rho}{G}\frac{1-2\nu}{2(1-\nu)}}\,z\right)\right], \tag{5-E.28}$$

$$\sigma_{yy} = E\frac{\nu}{(\nu+1)(1-2\nu)}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) =$$

$$= E\frac{\nu}{(1+\nu)(1-2\nu)}\sum_{m=1}^{\infty}\left[\left(-\phi_m(z)\frac{m\pi}{L} + \frac{d\chi_m(z)}{dz}\right)\sin\left(\frac{m\pi x}{L}\right)T(t)\right], \tag{5-E.29}$$

$$\sigma_{zz} = \frac{E}{(\nu+1)(1-2\nu)}\left[(1-\nu)\frac{\partial w}{\partial z} + \nu\frac{\partial u}{\partial x}\right] =$$

$$= \frac{E}{(\nu+1)(1-2\nu)}\sum_{m=1}^{\infty}\left[\left((1-\nu)\frac{d\chi_m(z)}{dz} - \nu\frac{m\pi}{L}\phi_m(z)\right)\sin\left(\frac{m\pi x}{L}\right)T(t)\right], \tag{5-E.30}$$

$$\sigma_{xz} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) =$$

$$= \frac{E}{2(1+\nu)} \sum_{m=1}^{\infty} \left[\left(\frac{d\phi_m(z)}{dz} + \chi_m(z) \frac{m\pi}{L} \right) \cos\left(\frac{m\pi x}{L} \right) T(t) \right], \tag{5-E.31}$$

where

$$\frac{d\phi_m(z)}{dz} = \left(\left(\frac{m\pi}{L} \right)^2 - \Omega^2 \frac{\rho}{G} \right) \frac{m\pi}{L} \times$$

$$\times \left[A \exp\left(z\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2 \frac{\rho}{G}} \right) + K \exp\left(-z\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2 \frac{\rho}{G}} \right) \right] +$$

$$+ \frac{m\pi}{L} \sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2 \frac{\rho}{G} \frac{1-2\nu}{2(1-\nu)}} \times$$

$$\times \left[C \exp\left(z\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2 \frac{\rho}{G} \frac{1-2\nu}{2(1-\nu)}} \right) - S \exp\left(-z\sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2 \frac{\rho}{G} \frac{1-2\nu}{2(1-\nu)}} \right) \right] \tag{5-E.32}$$

To simplify the subsequent derivations let us introduce the following notations:

$$M = \frac{m\pi}{L}, \tag{5-E.33}$$

$$r = \sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2 \frac{\rho}{G}}, \quad s = \sqrt{\left(\frac{m\pi}{L}\right)^2 - \Omega^2 \frac{\rho}{G} \frac{1-2\nu}{2(1-\nu)}}. \tag{5-E.34}$$

In order to simplify computation of natural frequencies, we will write formulas (5-E.34) in the form

$$r = \sqrt{\left(\frac{m\pi}{L}\right)^2 - \lambda^2}, \quad s = \sqrt{\left(\frac{m\pi}{L}\right)^2 - \lambda^2 \frac{1-2\nu}{2(1-\nu)}}, \tag{5-E.35}$$

where

$$\lambda = \Omega\sqrt{\frac{\rho}{G}}. \tag{5-E.36}$$

Then expressions for stresses take the form:

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)} \sum_{m=1}^{\infty} \left[-\left((1-\nu)\phi_m(z) M + \nu \frac{d\chi_m(z)}{dz} \right) \sin(Mx) T(t) \right], \tag{5-E.37}$$

$$\sigma_{yy} = E \frac{\nu}{(1+\nu)(1-2\nu)} \sum_{m=1}^{\infty} \left[\left(-\phi_m(z) M + \frac{d\chi_m(z)}{dz} \right) \sin(Mx) T(t) \right], \tag{5-E.38}$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)} \sum_{m=1}^{\infty} \left[\left((1-\nu) \frac{d\chi_m(z)}{dz} - \nu M \phi_m(z) \right) \sin(Mx) T(t) \right], \quad (5\text{-E.39})$$

$$\sigma_{xz} = \frac{E}{2(1+\nu)} \sum_{m=1}^{\infty} \left[\left(\frac{d\phi_m(z)}{dz} + \chi_m(z) M \right) \cos(Mx) T(t) \right], \quad (5\text{-E.40})$$

where

$$\phi_m(z) = rM[A\exp(rz) - K\exp(-rz)] + M[C\exp(sz) + S\exp(-sz)], \quad (5\text{-E.41})$$

$$\chi_m(z) = M^2[A\exp(rz) + K\exp(-rz)] + s[C\exp(sz) - S\exp(-sz)], \quad (5\text{-E.42})$$

$$\frac{d\phi_m(z)}{dz} = r^2 M[A\exp(zr) + K\exp(-zr)] + Ms[C\exp(zs) - S\exp(-zs)], \quad (5\text{-E.43})$$

$$\frac{d\chi_m(z)}{dz} = M^2 r[A\exp(rz) - K\exp(-rz)] + s^2[C\exp(sz) + S\exp(-sz)]. \quad (5\text{-E.44})$$

For stress-free upper and lower surfaces the boundary conditions are:

$$\sigma_{zz} = \sigma_{xz} = 0 \text{ at } z = 0 \text{ and } z = h. \quad (5\text{-E.45})$$

Substitution of equation (5-E.39) for stress σ_{zz} into the boundary condition $\sigma_{zz}\Big|_{z=0} = 0$ gives the following equation:

$$Mr\frac{1-2\nu}{\nu}A - Mr\frac{1-2\nu}{\nu}K + \frac{s^2(1-\nu) - \nu M^2}{\nu M}C + \frac{s^2(1-\nu) - \nu M^2}{\nu M}S = 0. \quad (5\text{-E.46})$$

Substitution of equation (5-E.39) for stress σ_{zz} into the boundary condition $\sigma_{zz}\Big|_{z=h} = 0$ gives equation

$$\frac{1-2\nu}{\nu}rMe^{rh}A - \frac{1-2\nu}{\nu}rMe^{-rh}K + \left[-M + \frac{(1-\nu)s^2}{\nu M}\right]e^{sh}C + \left[-M + \frac{(1-\nu)s^2}{\nu M}\right]e^{-sh}S = 0. \quad (5\text{-E.47})$$

From boundary condition $\sigma_{xz}\Big|_{z=0} = 0$ we obtain:

$$\left(r^2 + M^2\right)A + \left(r^2 + M^2\right)K + 2sC - 2sS = 0. \quad (5\text{-E.48})$$

The boundary condition $\sigma_{xz}\Big|_{z=h} = 0$ gives equation

$$\left(r^2 + M^2\right)e^{rh}A + \left(r^2 + M^2\right)e^{-rh}K + 2se^{sh}C - 2se^{-sh}S = 0. \quad (5\text{-E.49})$$

Equations (5-E.46)-(5-E.49), written in matrix from, are

$$\begin{bmatrix} Mr\frac{1-2\nu}{\nu} & -Mr\frac{1-2\nu}{\nu} & \frac{s^2(1-\nu)-\nu M^2}{\nu M} & \frac{s^2(1-\nu)-\nu M^2}{\nu M} \\ \frac{1-2\nu}{\nu}rMe^{rh} & -\frac{1-2\nu}{\nu}rMe^{-rh} & \left(-M+\frac{(1-\nu)s^2}{\nu M}\right)e^{sh} & \left(-M+\frac{(1-\nu)s^2}{\nu M}\right)e^{-sh} \\ \left(r^2+M^2\right) & \left(r^2+M^2\right) & 2s & -2s \\ \left(r^2+M^2\right)e^{rh} & \left(r^2+M^2\right)e^{-rh} & 2se^{sh} & -2se^{-sh} \end{bmatrix} \begin{Bmatrix} A \\ K \\ C \\ S \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{5-E.50}$$

For a non-trivial solution of this problem the determinant of equation (5-E.50) must be equal to zero, and this yields the characteristic equation, the solution of which for each value of m ($m = 1, 2, ...\infty$) yields an infinite sequence of eigenvalues.

Let $E = 114.8 \times 10^9 \frac{N}{m^3}$, $\nu = 0.3$, $\rho = 1614 \frac{kg}{m^3}$, $L = 1m$, $h = 0.06m$. The MAPLE session that computes the second period of vibration, corresponding to $m = 2$, is shown below:

```
>m:=2: pi:=3.14159:
>Young:=114.8e9: nu:=0.3: rho:=1614: L:=1: h:=0.06: G:=Young/2/(1+nu):
>M:=m*pi/L: r:=(M^2-lambda^2)^(1/2):
>s:=(M^2-lambda^2*(1-2*nu)/2/(1-nu))^(1/2):
>a11:=M*r*(1-2*nu)/nu: a12:=-M*r*(1-2*nu)/nu:
>a13:=(s^2*(1-nu)-nu*M^2)/nu/M: a14:=a13: a21:=(1-2*nu)/nu*r*M*exp(r*h):
>a22:=-(1-2*nu)/nu*r*M*exp(-r*h): a23:=a14*exp(s*h): a24:=a14*exp(-s*h):
>a31:=r^2+M^2: a32:=a31: a33:=2*s: a34:=-2*s:

a41:=(r^2+M^2)*exp(r*h):
>a42:=(r^2+M^2)*exp(-r*h): a43:=2*s*exp(s*h): a44:=-2*s*exp(-s*h):
>Young:=114.8e9: nu:=0.3: rho:=1614: L:=1: h:=0.1: G:=Young/2/(1+nu):
>pi:=3.14159: m:=1:
>M:=m*pi/L: r:=(M^2-lambda^2)^(1/2): s:=(M^2-lambda^2*(1-2*nu)/2/(1-nu))^(1/2):
>a11:=M*r*(1-2*nu)/nu: a12:=-M*r*(1-2*nu)/nu: a13:=(s^2*(1-nu)-nu*M^2)/nu/M:
>a14:=a13: a21:=(1-2*nu)/nu*r*M*exp(r*h):
>a22:=-(1-2*nu)/nu*r*M*exp(-r*h): a23:=a14*exp(s*h): a24:=a14*exp(-s*h):
>a31:=r^2+M^2: a32:=a31: a33:=2*s: a34:=-2*s:
>a41:=(r^2+M^2)*exp(r*h): a42:=(r^2+M^2)*exp(-r*h): a43:=2*s*exp(s*h):
>a44:=-2*s*exp(-s*h):
```

```
>ar:=array([[a11,a12,a13,a14],[a21,a22,a23,a24],[a31,a32,a33,a34],[a41,a42,a43,a44]]):
>with(linalg):
>f:=det(ar):
>plot(f, lambda=0...0.5);
>lamb:=fsolve(f=0,lambda=0..1);
>T:=2.*pi/lamb*(rho/G)^(0.5);
>ar:=array([[a11,a12,a13,a14],[a21,a22,a23,a24],[a31,a32,a33,a34],[a41,a42,
>a43,a44]]):
>with(linalg):
>f:=det(ar):
>plot(f, lambda=0...0.5);
>lamb:=fsolve(f=0,lambda=0..1);
>T:=2.*pi/lamb*(rho/G)^(0.5);
>classical_period:=2*pi/M^2/h/(1/12*Y/(1-nu^2)/rho)^0.5;
```

The last line of this MAPLE session is meant to compute the periods from the classical plate theory, based on Kirchhoff-Love assumptions. The results of computation are shown in the table:

m	Periods from elasticity solution (s)	Periods from classical plate theory (s)
1	0.00450462	0.00415740
2	0.0010649	0.00103935
3	0.000486968	0.000461936
4	0.000284078	0.000259837
5	0.000189665	0.000166296
6	0.000137940	0.000115484
7	0.000106384	0.0000848449
8	0.0000856016	0.0000649594
9	0.0000711066	0.0000513258
10	0.0000605364	0.0000415740

By equating the determinant of the system of equations (5-E.50) to zero, we make the number of independent equations in the system (5-E.50) one less. So, the system (5-E.50) of four equations

is reduced to the system of three equations with four unknowns A_m, K_m, C_m, S_m for each m:

$$\begin{bmatrix} Mr\frac{1-2\nu}{\nu} & -Mr\frac{1-2\nu}{\nu} & \frac{s^2(1-\nu)-\nu M^2}{\nu M} & \frac{s^2(1-\nu)-\nu M^2}{\nu M} \\ \frac{1-2\nu}{\nu}rMe^{rh} & -\frac{1-2\nu}{\nu}rMe^{-rh} & \left(-M+\frac{(1-\nu)s^2}{\nu M}\right)e^{sh} & \left(-M+\frac{(1-\nu)s^2}{\nu M}\right)e^{-sh} \\ (r^2+M^2) & (r^2+M^2) & 2s & -2s \end{bmatrix} \begin{Bmatrix} A_m \\ K_m \\ C_m \\ S_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{5-E.51}$$

or

$$\begin{bmatrix} Mr\frac{1-2\nu}{\nu} & -Mr\frac{1-2\nu}{\nu} & \frac{s^2(1-\nu)-\nu M^2}{\nu M} \\ \frac{1-2\nu}{\nu}rMe^{rh} & -\frac{1-2\nu}{\nu}rMe^{-rh} & \left(-M+\frac{(1-\nu)s^2}{\nu M}\right)e^{sh} \\ (r^2+M^2) & (r^2+M^2) & 2s \end{bmatrix} \begin{Bmatrix} A_m \\ K_m \\ C_m \end{Bmatrix} = \begin{Bmatrix} \frac{\nu M^2-s^2(1-\nu)}{\nu M} \\ \left(M-\frac{(1-\nu)s^2}{\nu M}\right)e^{-sh} \\ 2s \end{Bmatrix} S_m \tag{5-E.52}$$

For each value of m we can express coefficients A_m, K_m, C_m in terms of the unknown coefficient S_m :

$$\begin{Bmatrix} A_m \\ K_m \\ C_m \end{Bmatrix} = \begin{bmatrix} Mr\frac{1-2\nu}{\nu} & -Mr\frac{1-2\nu}{\nu} & \frac{s^2(1-\nu)-\nu M^2}{\nu M} \\ \frac{1-2\nu}{\nu}rMe^{rh} & -\frac{1-2\nu}{\nu}rMe^{-rh} & \left(-M+\frac{(1-\nu)s^2}{\nu M}\right)e^{sh} \\ (r^2+M^2) & (r^2+M^2) & 2s \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\nu M^2-s^2(1-\nu)}{\nu M} \\ \left(M-\frac{(1-\nu)s^2}{\nu M}\right)e^{-sh} \\ 2s \end{Bmatrix} S_m. \tag{5-E.53}$$

For example, for $m = 2$, i.e. for $\Omega = \Omega_2 = \frac{2\pi}{T_2} = \frac{2\pi}{1.271088781\times10^{-3}} = 4943.2$ we find

$$\begin{Bmatrix} A_2 \\ K_2 \\ C_2 \end{Bmatrix} = \begin{bmatrix} Mr\frac{1-2\nu}{\nu} & -Mr\frac{1-2\nu}{\nu} & \frac{s^2(1-\nu)-\nu M^2}{\nu M} \\ \frac{1-2\nu}{\nu}rMe^{rh} & -\frac{1-2\nu}{\nu}rMe^{-rh} & \left(-M+\frac{(1-\nu)s^2}{\nu M}\right)e^{sh} \\ (r^2+M^2) & (r^2+M^2) & 2s \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\nu M^2-s^2(1-\nu)}{\nu M} \\ \left(M-\frac{(1-\nu)s^2}{\nu M}\right)e^{-sh} \\ 2s \end{Bmatrix} S_2 =$$

$$= \begin{Bmatrix} .11754 \\ .16028 \\ -.73144 \end{Bmatrix} S_2.$$

In general, coefficients A_m, K_m, C_m can be presented in the form

$$\begin{Bmatrix} A_m \\ K_m \\ C_m \end{Bmatrix} = \begin{Bmatrix} \alpha_m \\ \beta_m \\ \gamma_m \end{Bmatrix} S_m, \tag{5-E.54}$$

where $\alpha_m, \beta_m, \gamma_m$ are known coefficients that depend on natural frequencies Ω_m.

According to equation (5-E.25),

$$u(x,z,t) = \sum_{m=1}^{\infty} \phi_m(z) \cos\left(\frac{m\pi x}{L}\right) T_m(t), \tag{5-E.55}$$

where

$$\phi_m(z) = r_m \frac{m\pi}{L}\left[A_m \exp(r_m z) - K_m \exp(-r_m z)\right] + \frac{m\pi}{L}\left[C_m \exp(s_m z) + S_m \exp(-s_m z)\right], \tag{5-E.56}$$

$$T_m(t) = Q_m \cos \Omega_m t + R_m \sin \Omega_m t. \tag{5-E.57}$$

Substitution of equation (5-E.54) into equation (5-E.56), yields

$$\phi_m(z) = \left\{ r_m \left[\alpha_m \exp(r_m z) - \beta_m \exp(-r_m z)\right] + \left[\gamma_m \exp(s_m z) + \exp(-s_m z)\right]\right\} \frac{m\pi}{L} S_m. \tag{5-E.58}$$

If we substitute equations (5-E.57) and (5-E.58) into equation (5-E.55), we receive

$$u(x,z,t) = \sum_{m=1}^{\infty} \left\{ r_m \left[\alpha_m \exp(r_m z) - \beta_m \exp(-r_m z)\right] + \left[\gamma_m \exp(s_m z) + \exp(-s_m z)\right]\right\} \times$$

$$\times \frac{m\pi}{L} \cos\left(\frac{m\pi x}{L}\right)\left(Q_m \cos \Omega_m t + R_m \sin \Omega_m t\right). \tag{5-E.59}$$

In formula (5-E.59) the unknown coefficient S_m has been absorbed by the unknown coefficients Q_m and R_m. These coefficients will be found from initial conditions.

According to equation (5-E.26),

$$w(x,z,t) = \sum_{m=1}^{\infty} \chi_m(z) \sin\left(\frac{m\pi x}{L}\right) T_m(t), \tag{5-E.60}$$

where

$$\chi_m(z) = \left(\frac{m\pi}{L}\right)^2 \left[A_m \exp(r_m z) + K_m \exp(-r_m z)\right] +$$

$$+ s_m \left[C_m \exp(s_m z) - S_m \exp(-s_m z)\right], \tag{5-E.61}$$

and $T_m(t)$ is defined by formula (5-E57). Substitution of equation (5-E5.4) into equation (5-E.61) yields

$$\chi_m(z) = \left\{ \left(\frac{m\pi}{L}\right)^2 \left[\alpha_m \exp(r_m z) + \beta_m \exp(-r_m z)\right] + \right.$$

$$\left. + s_m \left[\gamma_m \exp(s_m z) - \exp(-s_m z)\right] \right\} S_m, \tag{5-E.62}$$

If we substitute equations (5-E.62) and (5-E.57) into equation (5E.60), we receive

$$w(x,z,t) = \sum_{m=1}^{\infty} \left\{ \left(\frac{m\pi}{L}\right)^2 \left[\alpha_m \exp(r_m z) + \beta_m \exp(-r_m z)\right] + \right.$$

$$\left. + s_m \left[\gamma_m \exp(s_m z) - \exp(-s_m z)\right] \right\} \sin\left(\frac{m\pi x}{L}\right) (Q_m \cos\Omega_m t + R_m \sin\Omega_m t). \tag{5-E.63}$$

In equation (5-E.63) the unknown coefficient S_m has been absorbed by the unknown coefficients Q_m and R_m. These coefficients will be found from the initial conditions.

Vibrations of a plate in cylindrical bending dropped on simple supports

In this case we have the following initial conditions for $w(x,z,t)$, i.e. conditions at moment $t=0$, when the plate touches the simple supports:

$$w(x,z,0) = 0 \quad (0 \le x \le L,\ 0 \le z \le h), \tag{5-E.64}$$

$$\frac{\partial w}{\partial t}(x,z,0) = const(x,z) \quad (0 \le x \le L,\ 0 \le z \le h) \tag{5-E.65}$$

We will satisfy initial condition (5-E.65) approximately, i.e. instead of the initial condition (5-E.65) we will use initial condition

$$\frac{\partial w}{\partial t}\left(x, \frac{h}{2}, 0\right) \equiv v_0 = const(x) \quad (0 \le x \le L). \tag{5-E.66}$$

From equation (5-E.63) and initial condition (5-E.64) we receive equation

$$0 = w(x,z,0) = \sum_{m=1}^{\infty} \left\{ \left(\frac{m\pi}{L}\right)^2 \left[\alpha_m \exp(r_m z) + \beta_m \exp(-r_m z)\right] + \right.$$

$$+s_m\left[\gamma_m \exp\left(s_m z\right) - \exp\left(-s_m z\right)\right]\Big\} Q_m, \tag{5-E.67}$$

from which it follows that

$$Q_m = 0. \tag{5-E.68}$$

Now equation (5-E.63) for $w(x,z,t)$ takes the form:

$$w(x,z,t) = \sum_{m=1}^{\infty}\left\{\left(\frac{m\pi}{L}\right)^2\left[\alpha_m \exp\left(r_m z\right) + \beta_m \exp\left(-r_m z\right)\right] + \right.$$

$$\left. + s_m\left[\gamma_m \exp\left(s_m z\right) - \exp\left(-s_m z\right)\right]\right\} \sin\left(\frac{m\pi x}{L}\right) R_m \sin\left(\Omega_m t\right). \tag{5-E.69}$$

From equation (5-E.69) and initial condition (5-E.66) we receive the following equation

$$\sum_{m=1}^{\infty}\left\{\left(\frac{m\pi}{L}\right)^2\left[\alpha_m \exp\left(r_m \frac{h}{2}\right) + \beta_m \exp\left(-r_m \frac{h}{2}\right)\right] + \right.$$

$$\left. + s_m\left[\gamma_m \exp\left(s_m \frac{h}{2}\right) - \exp\left(-s_m \frac{h}{2}\right)\right]\right\} \sin\left(\frac{m\pi x}{L}\right) R_m \Omega_m = v_0 \tag{5-E.70}$$

The constant initial velocity v_0 can be expanded into Fourier series as follows:

$$v_0 = v_0 \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1-(-1)^m}{m}\sin\left(\frac{m\pi x}{L}\right). \tag{5-E.71}$$

If we substitute equation (5-E.71) into equation (5-E.70) and equate the coefficients of $\sin\left(\frac{m\pi x}{L}\right)$, we can express the constant of integration R_m in terms of known quantities:

$$R_m = \frac{2\left[1-(-1)^m\right] v_0}{\left[\left(\frac{m\pi}{L}\right)^2\left(\alpha_m e^{r_m h/2} + \beta_m e^{-r_m h/2}\right) + s_m\left(\gamma_m e^{s_m h/2} - e^{-s_m h/2}\right)\right] m\pi\Omega_m}. \tag{5-E.72}$$

For even values of m the constants R_m are equal to zero. Therefore, in the series representations of displacements and stresses, only terms with odd values of m will be present. In view of this, the solution of the problem can be rewritten as follows:

$$u(x,z,t) = \sum_{k=1}^{\infty}\frac{(2k-1)\pi}{L} R_{(2k-1)}\left\{ r_{(2k-1)}\left[\alpha_{(2k-1)} \exp\left(r_{(2k-1)} z\right) - \beta_{(2k-1)} \exp\left(-r_{(2k-1)} z\right)\right]\right.$$

$$+\left[\gamma_{(2k-1)}\exp\left(s_{(2k-1)}z\right)+\exp\left(-s_{(2k-1)}z\right)\right]\Bigg\}\times$$

$$\times\cos\left(\frac{(2k-1)\,\pi x}{L}\right)\sin\left(\Omega_{(2k-1)}t\right), \tag{5-E.73}$$

$$w\left(x,z,t\right)=\sum_{k=1}^{\infty}R_{(2k-1)}\left\{\left(\frac{(2k-1)\,\pi}{L}\right)^2\left[\alpha_{(2k-1)}\exp\left(r_{(2k-1)}z\right)+\beta_{(2k-1)}\exp\left(-r_{(2k-1)}z\right)\right]+\right.$$

$$\left.+s_{(2k-1)}\left[\gamma_{(2k-1)}\exp\left(s_{(2k-1)}z\right)-\exp\left(-s_{(2k-1)}z\right)\right]\right\}\sin\left(\frac{(2k-1)\,\pi x}{L}\right)\sin\left(\Omega_{(2k-1)}t\right), \tag{5-E.74}$$

$$\sigma_{xx}=-\frac{E\left(1-\nu\right)}{\left(1+\nu\right)\left(1-2\nu\right)}\times$$

$$\times\sum_{k=1}^{\infty}\left(\frac{(2k-1)\,\pi}{L}\right)^2R_{(2k-1)}\left\{r_{(2k-1)}\left[\alpha_{(2k-1)}\exp\left(r_{(2k-1)}z\right)-\beta_{(2k-1)}\exp\left(-r_{(2k-1)}z\right)\right]+\right.$$

$$\left.+\left[\gamma_{(2k-1)}\exp\left(s_{(2k-1)}z\right)+\exp\left(-s_{(2k-1)}z\right)\right]\right\}\times$$

$$\times\sin\left(\frac{(2k-1)\,\pi x}{L}\right)\sin\left(\Omega_{(2k-1)}t\right)+$$

$$+\frac{E\nu}{\left(1+\nu\right)\left(1-2\nu\right)}\times$$

$$\times\sum_{k=1}^{\infty}R_{(2k-1)}\left\{\left(\frac{(2k-1)\,\pi}{L}\right)^2\left[\alpha_{(2k-1)}r_{(2k-1)}\exp\left(r_{(2k-1)}z\right)-\beta_{(2k-1)}r_{(2k-1)}\exp\left(-r_{(2k-1)}z\right)\right]+\right.$$

$$\left.+s_{(2k-1)}\left[\gamma_{(2k-1)}s_{(2k-1)}\exp\left(s_{(2k-1)}z\right)+s_{(2k-1)}\exp\left(-s_{(2k-1)}z\right)\right]\right\}\sin\left(\frac{(2k-1)\,\pi x}{L}\right)\sin\left(\Omega_{(2k-1)}t\right), \tag{5-E.75}$$

$$\sigma_{yy} = E\frac{\nu}{(1+\nu)(1-2\nu)}\sum_{k=1}^{\infty}R_{(2k-1)}\left[\gamma_{(2k-1)}\exp\left(s_{(2k-1)}z\right)+\exp\left(-s_{(2k-1)}z\right)\right]\times$$

$$\times\left[s^2_{(2k-1)}-\left(\frac{(2k-1)\pi}{L}\right)^2\right]\sin\left(\frac{(2k-1)\pi x}{L}\right)\sin\left(\Omega_{(2k-1)}t\right), \tag{5-E.76}$$

$$\sigma_{xz} = \frac{E}{2(1+\nu)}\sum_{k=1}^{\infty}R_{(2k-1)}\frac{(2k-1)\pi}{L}\times$$

$$\times\left(r^2_{(2k-1)}\alpha_{(2k-1)}e^{r_{(2k-1)}z}+r^2_{(2k-1)}\frac{\beta_{(2k-1)}}{e^{r_{(2k-1)}z}}+2s_{(2k-1)}\gamma_{(2k-1)}e^{s_{(2k-1)}z}-2\frac{s_{(2k-1)}}{e^{s_{(2k-1)}z}}+\right.$$

$$\left.+(2k-1)^2\frac{\pi^2}{L^2}\alpha_{(2k-1)}e^{r_{(2k-1)}z}+(2k-1)^2\frac{\pi^2}{L^2}\frac{\beta_{(2k-1)}}{e^{r_{(2k-1)}z}}\right)\cos\left(\frac{(2k-1)\pi x}{L}\right)\sin\left(\Omega_{(2k-1)}t\right), \tag{5-E.77}$$

$$\sigma_{zz} = \frac{E\nu}{(1+\nu)(1-2\nu)}\times$$

$$\times\sum_{k=1}^{\infty}\left(\frac{(2k-1)\pi}{L}\right)^2R_{(2k-1)}\times$$

$$\times\left\{r_{(2k-1)}\left[\alpha_{(2k-1)}\exp\left(r_{(2k-1)}z\right)-\beta_{(2k-1)}\exp\left(-r_{(2k-1)}z\right)\right]\right.$$

$$\left.+\left[\gamma_{(2k-1)}\exp\left(s_{(2k-1)}z\right)+\exp\left(-s_{(2k-1)}z\right)\right]\right\}\times$$

$$\times\sin\left(\frac{(2k-1)\pi x}{L}\right)\sin\left(\Omega_{(2k-1)}t\right)+$$

$$+\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\times$$

$$\times\sum_{k=1}^{\infty}R_{(2k-1)}\left\{\left(\frac{(2k-1)\pi}{L}\right)^{2}\left[\alpha_{(2k-1)}r_{(2k-1)}\exp\left(r_{(2k-1)}z\right)-\beta_{(2k-1)}r_{(2k-1)}\exp\left(-r_{(2k-1)}z\right)\right]+\right.$$

$$\left.+s_{(2k-1)}\left[\gamma_{(2k-1)}s_{(2k-1)}\exp\left(s_{(2k-1)}z\right)+s_{(2k-1)}\exp\left(-s_{(2k-1)}z\right)\right]\right\}\sin\left(\frac{(2k-1)\pi x}{L}\right)\sin\left(\Omega_{(2k-1)}t\right).$$

(5-E.78)

The circular frequencies Ω_m, that enter into the formulas (5-E.73)-(5-E.78), are computed for each value of m ($m = 1,2,3...$) as solutions of the nonlinear equation

$$\begin{vmatrix} M_m r_m \frac{1-2\nu}{\nu} & -M_m r_m \frac{1-2\nu}{\nu} & \frac{s_m^2(1-\nu)-\nu M_m^2}{\nu M_m} & \frac{s_m^2(1-\nu)-\nu M_m^2}{\nu M_m} \\ \frac{1-2\nu}{\nu} r_m M_m e^{rh} & -\frac{1-2\nu}{\nu} r_m M_m e^{-rh} & \left(-M_m+\frac{(1-\nu)s_m^2}{\nu M_m}\right)e^{s_m h} & \left(-M_m+\frac{(1-\nu)s_m^2}{\nu M_m}\right)e^{-s_m h} \\ \left(r_m^2+M_m^2\right) & \left(r_m^2+M_m^2\right) & 2s_m & -2s_m \\ \left(r_m^2+M_m^2\right)e^{rh} & \left(r_m^2+M_m^2\right)e^{-r_m h} & 2s_m e^{s_m h} & -2s_m e^{-s_m h} \end{vmatrix}=0,$$

(5-E.79)

where

$$M_m=\frac{m\pi}{L},\; r_m=\sqrt{\left(\frac{m\pi}{L}\right)^2-\Omega_m^2\frac{\rho}{G}},\; s_m=\sqrt{\left(\frac{m\pi}{L}\right)^2-\Omega_m^2\frac{\rho}{G}\frac{1-2\nu}{2(1-\nu)}}. \qquad (5\text{-E.80})$$

Quantities α_m, β_m, γ_m, are computed as follows:

$$\left\{\begin{array}{c}\alpha_m\\ \beta_m\\ \gamma_m\end{array}\right\}=\left[\begin{array}{ccc} Mr\frac{1-2\nu}{\nu} & -Mr\frac{1-2\nu}{\nu} & \frac{s^2(1-\nu)-\nu M^2}{\nu M} \\ \frac{1-2\nu}{\nu}rMe^{rh} & -\frac{1-2\nu}{\nu}rMe^{-rh} & \left(-M+\frac{(1-\nu)s^2}{\nu M}\right)e^{sh} \\ \left(r^2+M^2\right) & \left(r^2+M^2\right) & 2s \end{array}\right]^{-1}\left\{\begin{array}{c}\frac{\nu M^2-s^2(1-\nu)}{\nu M}\\ \left(M-\frac{(1-\nu)s^2}{\nu M}\right)e^{-sh}\\ 2s\end{array}\right\},$$

(5-E.81)

and coefficient R_m is computed by the formula

$$R_m=\frac{2\left[1-(-1)^m\right]v_0}{\left[\left(\frac{m\pi}{L}\right)^2\left(\alpha_m e^{r_m h/2}+\beta_m e^{-r_m h/2}\right)+s_m\left(\gamma_m e^{s_m h/2}-e^{-s_m h/2}\right)\right]m\pi\Omega_m}. \qquad (5\text{-E.82})$$

For verification of results of the finite element program, we will consider an example with the following numerical data:

$$E=114.8\times10^9\frac{N}{m^2},\nu=0.3,\rho=1614\frac{kg}{m^3},L=1m,h=0.06m,\; v_0=-10\frac{m}{s}. \qquad (5\text{-E.83})$$

The MAPLE session that is used to compute the values of the constants Ω_m, r_m, s_m , α_m, β_m, γ_m, R_m for these numerical values of material constants, geometric dimensions and for $m = 3$ is shown below:

```
>Digits:=10:
>m:=3: Young:=114.8e9: nu:=0.3: rho:=1614: L:=1: h:=0.06: G:=Young/2/(1+nu): w_dot:=-
10:
>pi:=3.141592654:
>M:=m*pi/L: r:=(M^2-lambda^2)^(1/2): s:=(M^2-lambda^2*(1-2*nu)/2/(1-nu))^(1/2):
>a11:=M*r*(1-2*nu)/nu: a12:=-M*r*(1-2*nu)/nu: a13:=(s^2*(1-nu)-nu*M^2)/nu/M: a14:=a13:
>a21:=(1-2*nu)/nu*r*M*exp(r*h): a22:=-(1-2*nu)/nu*r*M*exp(-r*h): a23:=a14*exp(s*h):
>a24:=a14*exp(-s*h): a31:=r^2+M^2: a32:=a31: a33:=2*s: a34:=-2*s:
> a41:=(r^2+M^2)*exp(r*h):
>a42:=(r^2+M^2)*exp(-r*h): a43:=2*s*exp(s*h): a44:=-2*s*exp(-s*h):
>ar:=array([[a11,a12,a13,a14],[a21,a22,a23,a24],[a31,a32,a33,a34],[a41,a42,a43,a44]]):
>with(linalg):
>f:=det(ar):
>lamb_classical:=M^2*h*(Young/(1-nu^2)/12./G)^0.5;
>lamb:=fsolve(f=0,lambda=48..60);
>Digits:=6: T:=2.*pi/lamb*(rho/G)^(0.5): Omega:=2*pi/T:
>M:=m*pi/L: r:=(M^2-lamb^2)^(1/2): s:=(M^2-lamb^2*(1-2*nu)/2/(1-nu))^(1/2):
>a11:=M*r*(1-2*nu)/nu: a12:=-M*r*(1-2*nu)/nu: a13:=(s^2*(1-nu)-nu*M^2)/nu/M: a14:=a13:
>
>inv:=inverse(array([[a11,a12,a13],[a21,a22,a23],[a31,a32,a33]])):
>b11:=(nu*M^2-s^2*(1-nu))/nu/M: b21:=(M-(1-nu)*s^2/nu/M)*exp(-s*h): b31:=2*s:
>bi1:=array([[b11],[b21],[b31]]):
>alpha_beta_gama:=multiply(inv, bi1):
>alpha:=alpha_beta_gama[1,1]: beta:=alpha_beta_gama[2,1]: gama:=alpha_beta_gama[3,1]:
>numerator:=2*(1-(-1)^m)*w_dot:
>denominator:=((m*pi/L)^2*(alpha*exp(r*h/2)+beta*exp(-r*h/2))+s*(gama*exp(s*h/2)-exp(-
s*h/2)))*
>m*pi*Omega: R:=numerator/denominator:
>Omega; r; s; alpha; beta; gama; R;
```

The values of constants Ω_m, r_m, s_m, α_m, β_m, γ_m, R_m for m ranging from 1 to 49, corresponding to the numerical data in equations (5-E.83), are listed in the following MAPLE session for computation of displacements:

```
>Omega[1]:=1395.05: r[1]:=3.13025: s[1]:=3.13835: alpha[1]:=0.257: beta[1]:=0.335474:
>gama[1]:=-0.81: R[1]:=-0.0604307:
>Omega[3]:=12902.7: r[3]:=9.09620: s[3]:=9.33207: alpha[3]:=0.06258: beta[3]:=0.108231:
>gama[3]:=-0.5706: R[3]:=-0.000630021:
>Omega[5]:=33127.8: r[5]:=14.3745: s[5]:=15.3388: alpha[5]:=0.028100:
>beta[5]:=0.0665388: gama[5]:=-0.3985: R[5]:=-0.0000389700:
>Omega[7]:=59061.4: r[7]:=18.8706: s[7]:=21.1466: alpha[7]:=0.015724: beta[7]:=0.0487995:
>gama[7]:=-0.28112: R[7]:=-0.705011e-5:
>Omega[9]:=88362.8: r[9]:=22.6721: s[9]:=26.7935: alpha[9]:=0.009998: beta[9]:=0.0389726:
>gama[9]:=-0.20036: R[9]:=-0.211367e-5:
>Omega[11]:=119567.0: r[11]:=25.9160: s[11]:=32.3251: alpha[11]:=0.0069103:
>beta[11]:=0.0327202: gama[11]:=-0.14378: R[11]:=-0.849163e-6:
>Omega[13]:=151811.0: r[13]:=28.7319: s[13]:=37.7791: alpha[13]:=0.0050624:
>beta[13]:=0.0283808: gama[13]:=-0.103648: R[13]:=-0.411960e-6:
>Omega[15]:=184592.0: r[15]:=31.2268: s[15]:=43.1832: alpha[15]:=0.0038677:
>beta[15]:=0.0251832: gama[15]:=-0.074947: R[15]:=-0.227776e-6:
>Omega[17]:=217620.0: r[17]:=33.4840: s[17]:=48.5562: alpha[17]:=0.0030471:
>beta[17]:=0.0227199: gama[17]:=-0.054291: R[17]:=-0.138566e-6:
>Omega[19]:=250724.0: r[19]:=35.5678: s[19]:=53.9109: alpha[19]:=0.0024567:
>beta[19]:=0.0207555: gama[19]:=-0.039376: R[19]:=-0.906374e-7:
>Omega[21]:=283800.0: r[21]:=37.5273: s[21]:=59.2561: alpha[21]:=0.0020147:
>beta[21]:=0.0191453: gama[21]:=-0.028570: R[21]:=-0.627758e-7:
>Omega[23]:=316802.0: r[23]:=39.3995: s[23]:=64.5974: alpha[23]:=0.0016738:
>beta[23]:=0.0177959: gama[23]:=-0.020738: R[23]:=-0.455232e-7:
>Omega[25]:=349694.0: r[25]:=41.2125: s[25]:=69.9382: alpha[25]:=0.00140390:
>beta[25]:=0.0166439: gama[25]:=-0.015051: R[25]:=-0.342883e-7:
>Omega[27]:=382460.0: r[27]:=42.9876: s[27]:=75.2808: alpha[27]:=0.00118645:
>beta[27]:=0.0156453: gama[27]:=-0.0109241: R[27]:=-0.266530e-7:
>Omega[29]:=415098.0: r[29]:=44.7415: s[29]:=80.6272: alpha[29]:=0.00100798:
```

```
>beta[29]:=0.0147686: gama[29]:=-0.0079256: R[29]:=-0.212841e-7:
>Omega[31]:=447610.0: r[31]:=46.4863: s[31]:=85.9778: alpha[31]:=0.00085998:
>beta[31]:=0.0139904: gama[31]:=-0.0057491: R[31]:=-0.173918e-7:
>Omega[33]:=479992.0: r[33]:=48.2306: s[33]:=91.3329: alpha[33]:=0.00073588:
>beta[33]:=0.0132939: gama[33]:=-0.0041686: R[33]:=-0.144991e-7:
>Omega[35]:=512252.0: r[35]:=49.9846: s[35]:=96.6942: alpha[35]:=0.00063115:
>beta[35]:=0.0126649: gama[35]:=-0.0030224: R[35]:=-0.123007e-7:
>Omega[37]:=544398.0: r[37]:=51.7494: s[37]:=102.060: alpha[37]:=0.00054211:
>beta[37]:=0.0120936: gama[37]:=-0.0021908: R[37]:=-0.105991e-7:
>Omega[39]:=576434.0: r[39]:=53.5313: s[39]:=107.430: alpha[39]:=0.00046613:
>beta[39]:=0.0115715: gama[39]:=-0.0015873: R[39]:=-0.925917e-8:
>Omega[41]:=608364.0: r[41]:=55.3335: s[41]:=112.806: alpha[41]:=0.00040105:
>beta[41]:=0.0110923: gama[41]:=-0.0011497: R[41]:=-0.818892e-8:
>Omega[43]:=640194.0: r[43]:=57.1586: s[43]:=118.188: alpha[43]:=0.00034503:
>beta[43]:=0.0106502: gama[43]:=-0.0008322: R[43]:=0.732472e-8:
>Omega[45]:=671938.0: r[45]:=59.0076: s[45]:=123.574: alpha[45]:=0.00029694:
>beta[45]:=0.0102407: gama[45]:=-0.0006024: R[45]:=-0.661695e-8:
>Omega[47]:=703586.0: r[47]:=60.8810: s[47]:=128.965: alpha[47]:=0.00025546:
>beta[47]:=0.00986046: gama[47]:=-0.00043587: R[47]:=-0.603312e-8:
>Omega[49]:=735156.0: r[49]:=62.7790: s[49]:=134.359: alpha[49]:=0.00021987:
>beta[49]:=0.00950587: gama[49]:=-0.00031551: R[49]:=-0.554492e-8:
>pi:=3.141592654:
>L:=1: h:=0.06: Y:=114.8e9: nu:=0.3:
>x:=L/2: z:=h/2: t:=0.002:
>m:=2*k-1:
>w:=sum(R[m]*((m*pi/L)^2*(alpha[m]*exp(r[m]*z)+beta[m]*exp(-r[m]*z))
>+s[m]*(gama[m]*exp(s[m]*z)
>-exp(-s[m]*z)))*sin(m*pi*x/L)*sin(Omega[m]*t),k=1..25):
```

The graphs of variation of transverse displacement at the middle surface $\left(\text{i.e. } w_0 \equiv w\Big|_{z=h/2}\right)$ as a function of x-coordinate at t=0.002s, and as a function of time at $x = \frac{L}{2}$ are shown in figure 5.2 and figure 5.3.

5.20 Appendix 5-F

Some considerations regarding comparison of displacements and stresses, obtained from geometrically linear and nonlinear models

Our finite element program is based on two models: geometrically linear model (small displacement gradients), with strain-displacement relations being

$$\varepsilon_{xx}^{(lin)} = u_{,x}, \;\; \varepsilon_{xz}^{(lin)} = \frac{1}{2}\left(u_{,z} + w_{,x}\right), \;\; \varepsilon_{zz}^{(lin)} = w_{,z}. \tag{5-F.1}$$

and geometrically nonlinear model, with von-Karman strain–displacement relations (moderately large displacement gradients)

$$\varepsilon_{xx}^{(K)} = u_{,x} + \frac{1}{2}\left(w_{,x}\right)^2, \quad \varepsilon_{xz}^{(K)} = \frac{1}{2}\left(u_{,z} + w_{,x}\right), \quad \varepsilon_{zz}^{(K)} = w_{,z}. \tag{5-F.2}$$

In both strain-displacement relations (5-F.1) and (5-F.2), derivatives are taken with respect to material coordinates, and the stress measure in both models, geometrically linear and nonlinear, is the second Piola-Kirchhoff stress tensor. Let us show that for both strain measures being used, equations (5-F.1) on the one hand and equations (5-F.2) on the other hand, the engineering elastic constants in the constitutive equations are the same.

Let us consider at first the Young's modulus E_x. It is defined as a ratio $\frac{\sigma_{xx}}{\varepsilon_{xx}}$, measured in a unidirectional tension test. In such a test the displacement gradient $\frac{\partial w}{\partial x}$ is equal to zero, therefore the components of the von-Karman and linear strain tensors are equal:

$$\varepsilon_{xx}^{(K)} = \varepsilon_{xx}^{(lin)} = \frac{\partial u}{\partial x}. \tag{5-F.3}$$

Therefore, the Young's modulus E_x that relates σ_{xx} and $\varepsilon_{xx}^{(K)}$ is equal to the Young's modulus that relates σ_{xx} and $\varepsilon_{xx}^{(lin)}$. The other elastic constants in our linear and nonlinear models are, obviously, equal too.

Besides, it can be shown that in deformations that involve moderately large rotations of line segments of a material (that is the case in our model), the von-Karman strain $\varepsilon_{xx}^{(K)}$ can be interpreted as a better approximation of a unit extension[10] $\frac{dS-ds}{ds}$, than the linear strain $\varepsilon_{xx}^{(lin)}$ (Cook, Malkus,

[10] Here dS denotes a length of a line segment after deformation, ds denotes the length of a line segment before deformation.

Plesha, 1989, page 430).

Therefore, the geometrically nonlinear model based on the von-Karman strains, in which the elastic constants are the same as in the geometrically linear model, can be compared in a meaningful manner with the linear model, and can be regarded as more accurate than the linear model.

In our opinion, the comparison of the model based on the fully nonlinear Green–Lagrange strain–displacement relations with the geometrically linear model would have been inappropriate. Indeed, the axial component of the Green–Lagrange strain tensor is

$$\varepsilon_{xx}^{(G)} = u_{,x} + \frac{1}{2}\left[(u_{,x})^2 + (v_{,x})^2 + (w_{,x})^2\right], \tag{5-F.4}$$

and in a unidirectional tension test for definition of the Young's modulus, this strain component takes the form:

$$\varepsilon_{xx}^{(G)} = u_{,x} + \frac{1}{2}(u_{,x})^2, \tag{5-F.5}$$

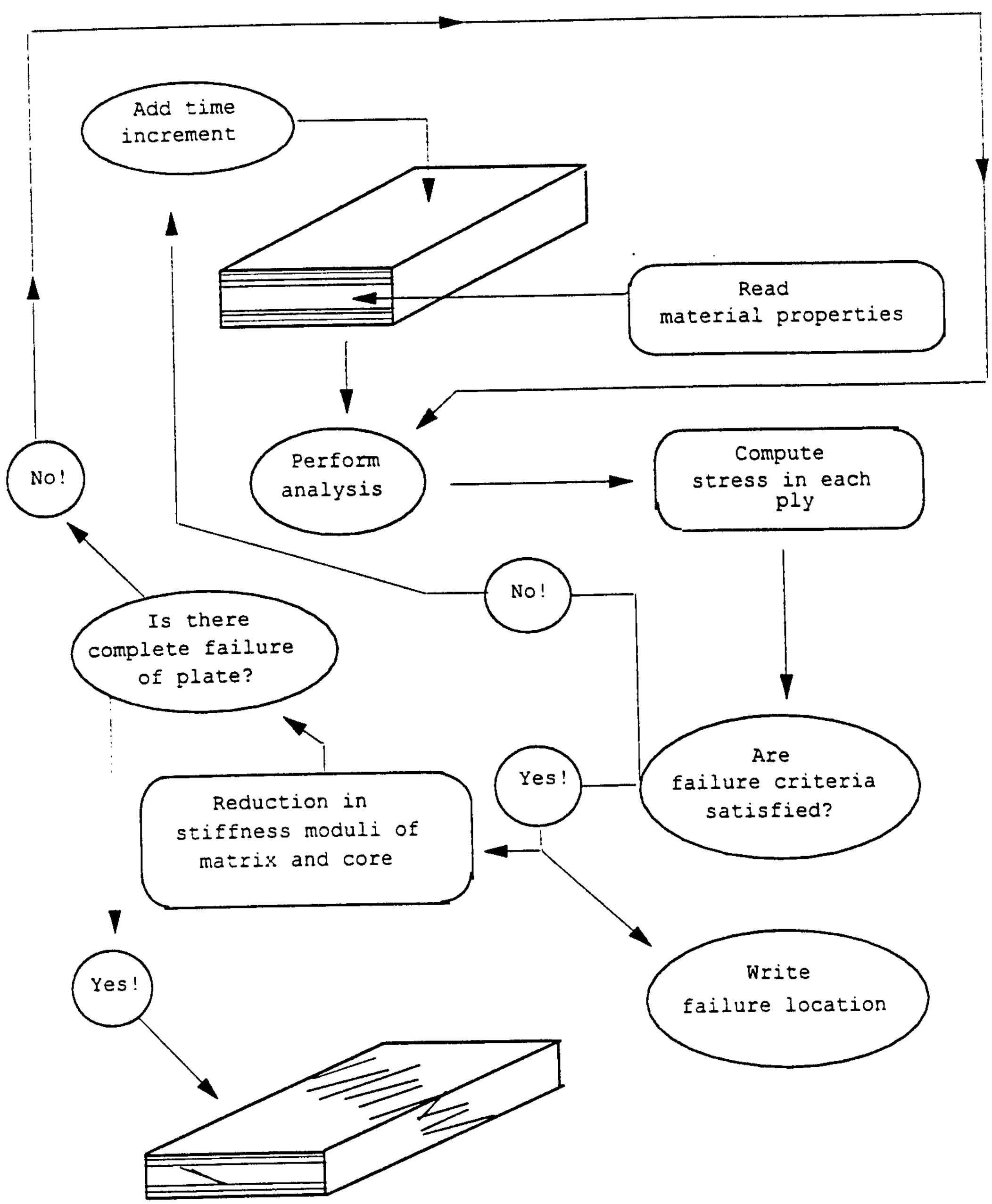
while the same component of the linear strain tensor is different[11]: $\varepsilon_{xx}^{(lin)} = u_{,x}$. Therefore, the Young's modulus that relates σ_{xx} and $\varepsilon_{xx}^{(G)}$ is not equal to the Young's modulus that relates σ_{xx} and $\varepsilon_{xx}^{(lin)}$. Moreover, in a material with linear dependence between σ_{xx} and $\varepsilon_{xx}^{(lin)} = u_{,x}$, there must be a nonlinear dependance between σ_{xx} and $\varepsilon_{xx}^{(G)} = u_{,x} + \frac{1}{2}(u_{,x})^2$.

But in our geometrically nonlinear model, based on the von-Karman strains, the comparison with the linear model is appropriate.

[11] Assuming that there is a uniform state of strain in the test sample, and denoting the length of the sample before deformation as l, and the length of the sample after deformation as L, we find that the axial component of the linear strain is $\varepsilon_{xx}^{(lin)} = u_{,x} = \frac{L-l}{l}$, because $u_{,x} = \frac{(dx+u_{,x}\,dx)-dx}{dx} = \frac{(dx+du)-dx}{dx} = \frac{dX-dx}{dx}$, where dX is a length of a small line segment after deformation. For the same component of the Green's strain in the test sample we receive: $\varepsilon_{xx}^{(G)} = u_{,x} + \frac{1}{2}(u_{,x})^2 = \frac{L^2-l^2}{2l^2}$, because $u_{,x} + \frac{1}{2}(u_{,x})^2 = \frac{(dx+u_{,x}\,dx)^2-(dx)^2}{2(dx)^2} = \frac{(dx+du)^2-(dx)^2}{2(dx)^2} = \frac{(dX)^2-(dx)^2}{2(dx)^2}$

Figure 5.1

Flow-chart of the damage progression algorithm



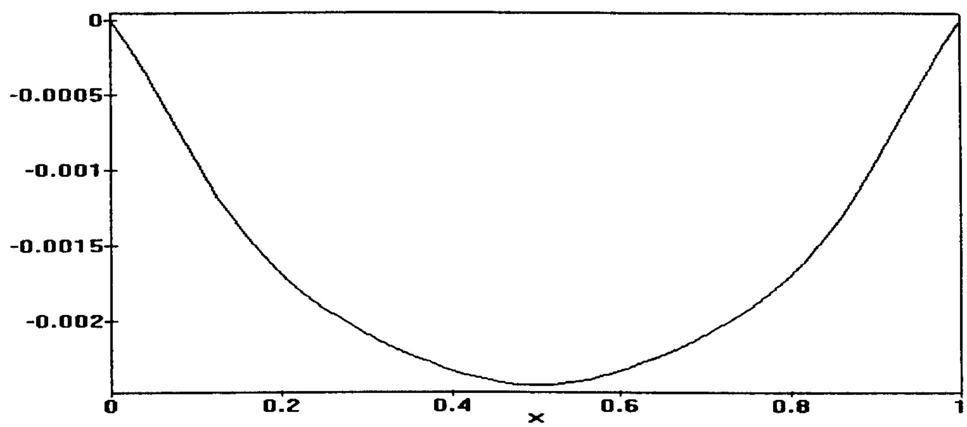


Figure 5.2

Transverse displacement w_0 as a function of x-coordinate (from exact elasticity solution) at a moment of time t=0.0002s for a wide beam dropped on simple supports.

In this example problem the material properties and geometric dimensions are: $E = 114.8 \times 10^9 \frac{N}{m^2}$, $\nu = 0.3$, $\rho = 1614 \frac{kg}{m^3}$, $L = 1m$, $h = 0.06m$, the initial velocity is $-10 \frac{m}{s}$

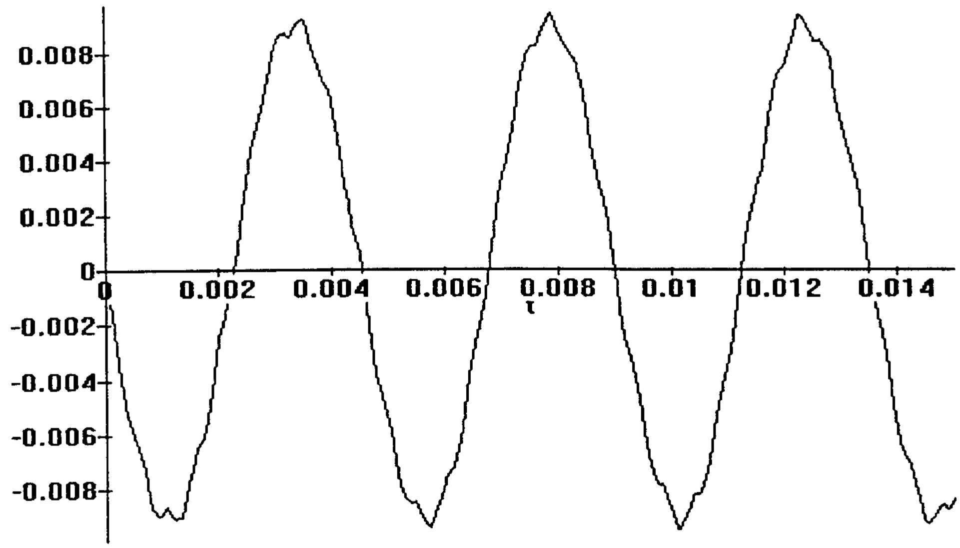


Figure 5.3

Transverse displacement w_0 as a function of time (from exact elasticity solution) at $x = \frac{L}{2}$ for a wide beam dropped on simple supports.

In this example problem the material properties and geometric dimensions are: $E = 114.8 \times 10^9 \frac{N}{m^2}$, $\nu = 0.3$, $\rho = 1614 \frac{kg}{m^3}$, $L = 1m$, $h = 0.06m$, the initial velocity is $-10 \frac{m}{s}$.

Figure 5.4. Exact and FE solutions for displacement of the middle surface of the plate, L=1m

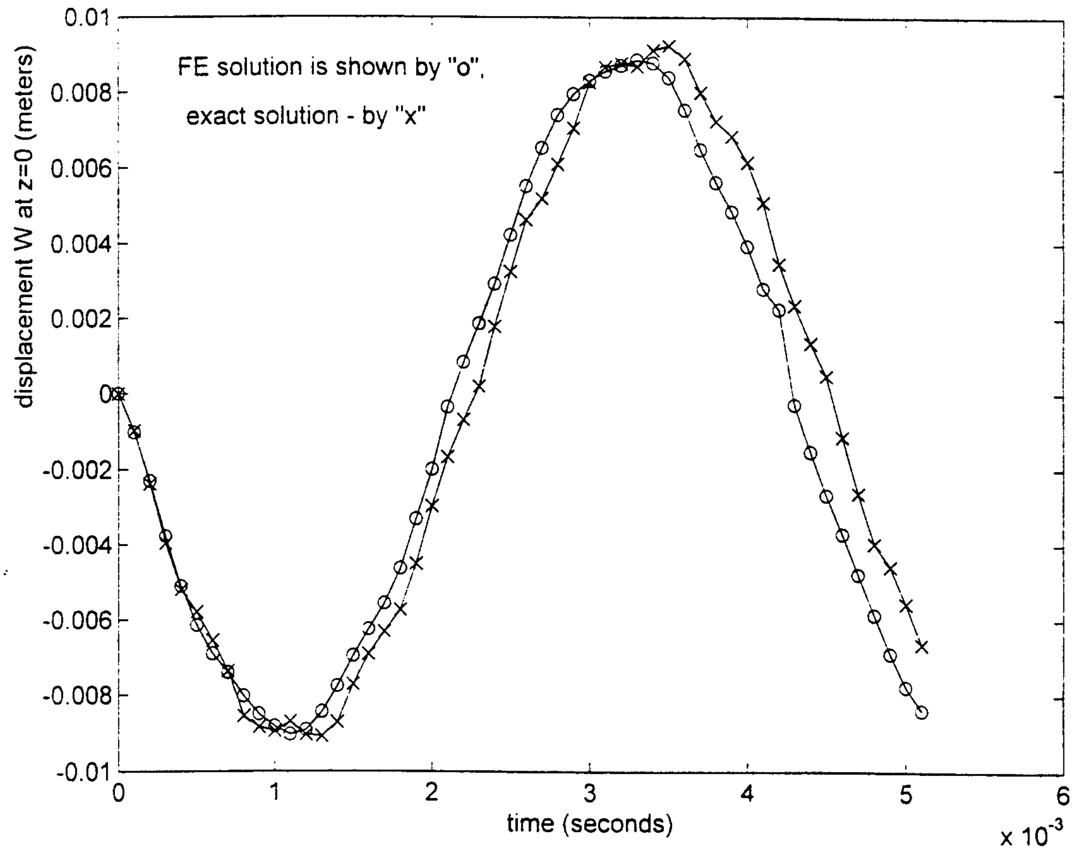


Figure 5.5. Displacement Wo as a function of x-coordinate at t=0.0001s

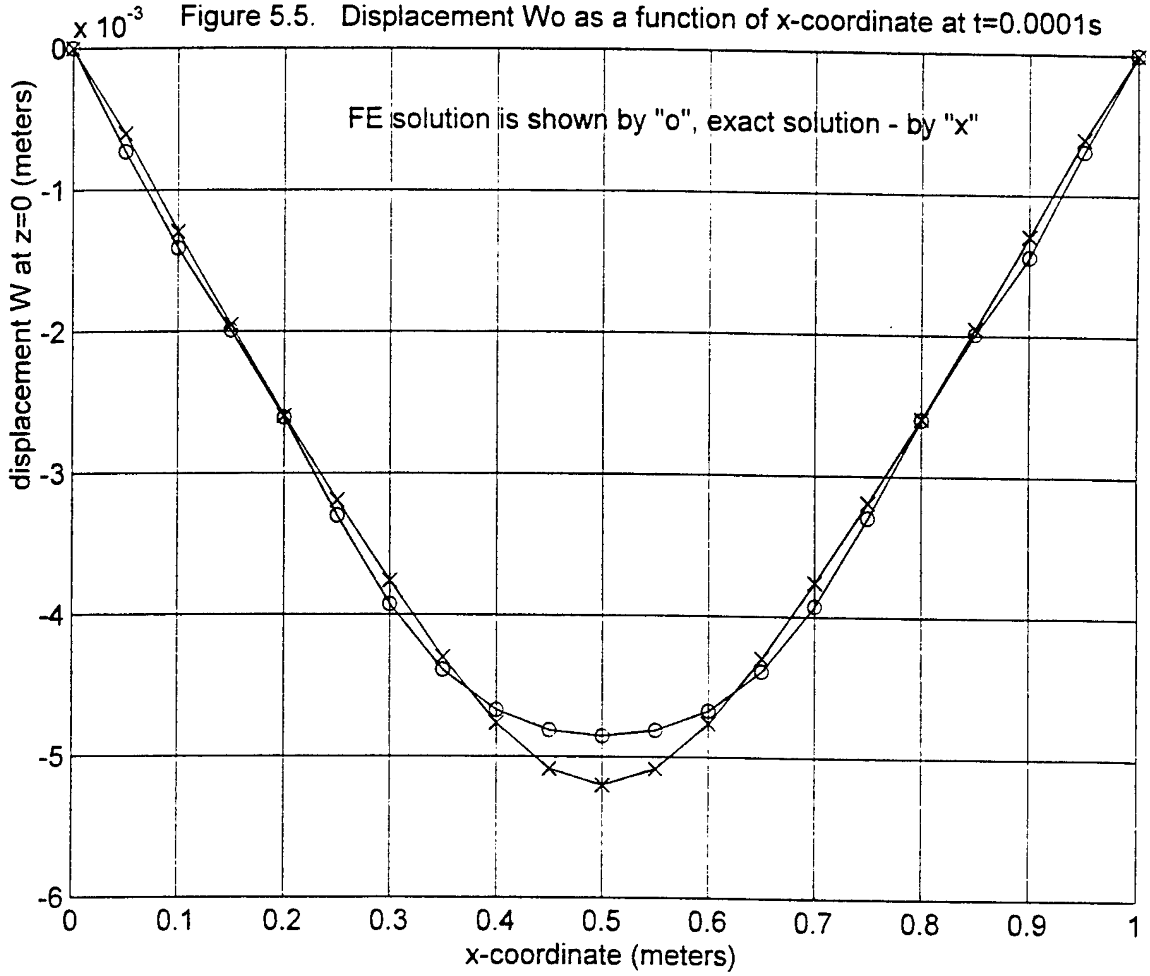
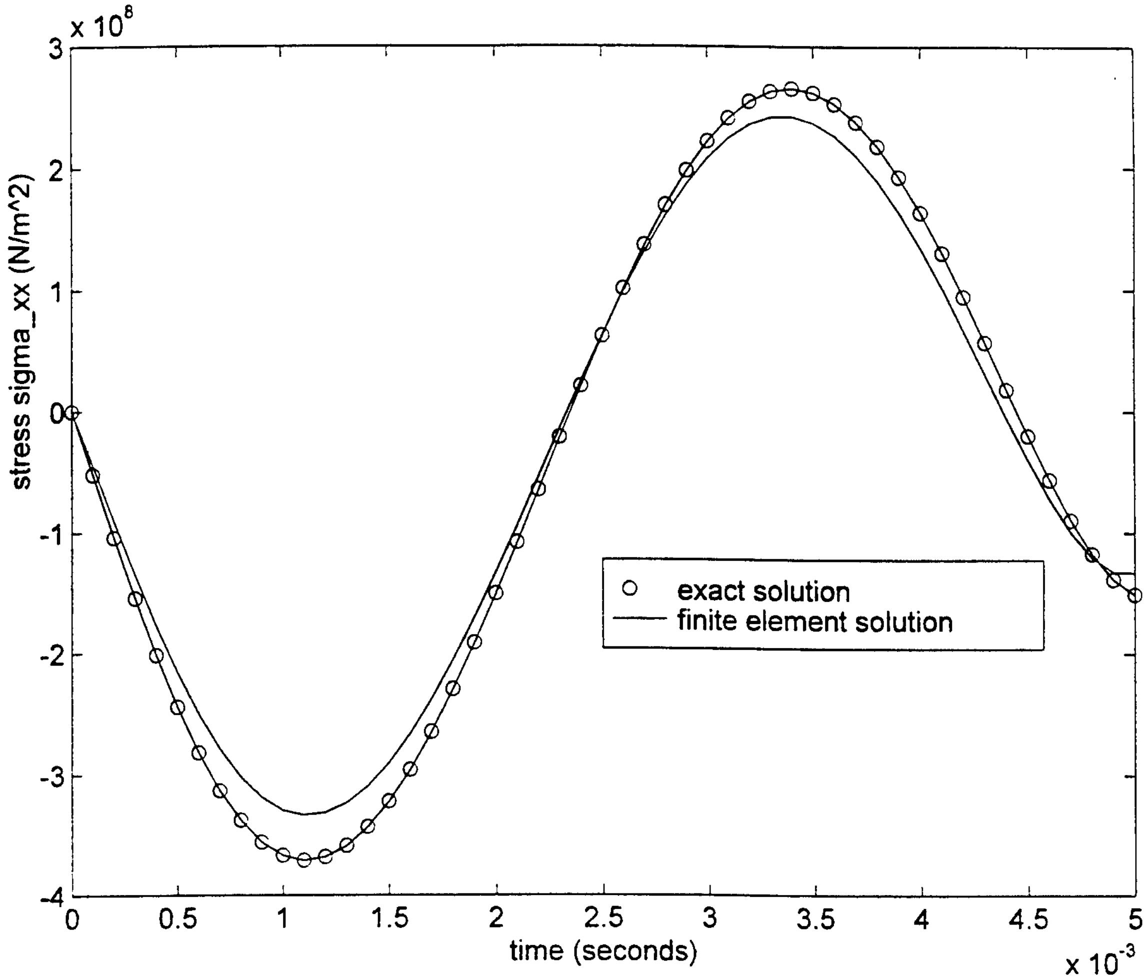


Figure 5.6

Least-square polynomial approximation of finite element

and analytical solutions for stress sigma_xx at x=L/2, z=h/2



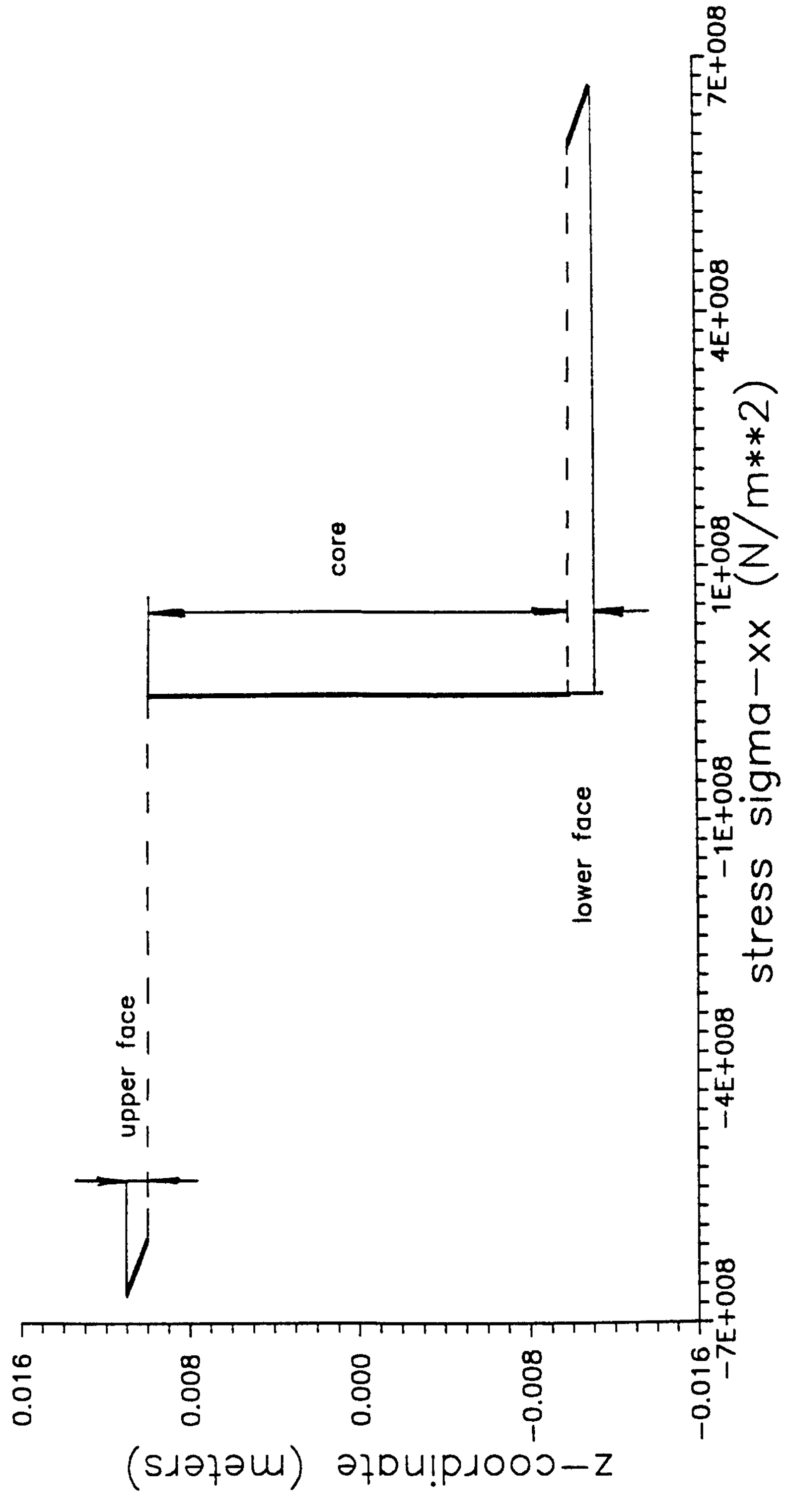


Figure 5.7
Comparison of exact elasticity solution and the finite element solution (based on the plate theory) for variation of stress sigma−xx in the thickness direction.
On this graph the exact and FE solutions visualy coincide.

Figure 5.8
Comparison of exact elasticity solution and the finite element solution (based on the plate theory) for variation of stress sigma−xz in the thickness direction.
The exact solution is shown by 'x',
the FE solution is shown by 'o'.

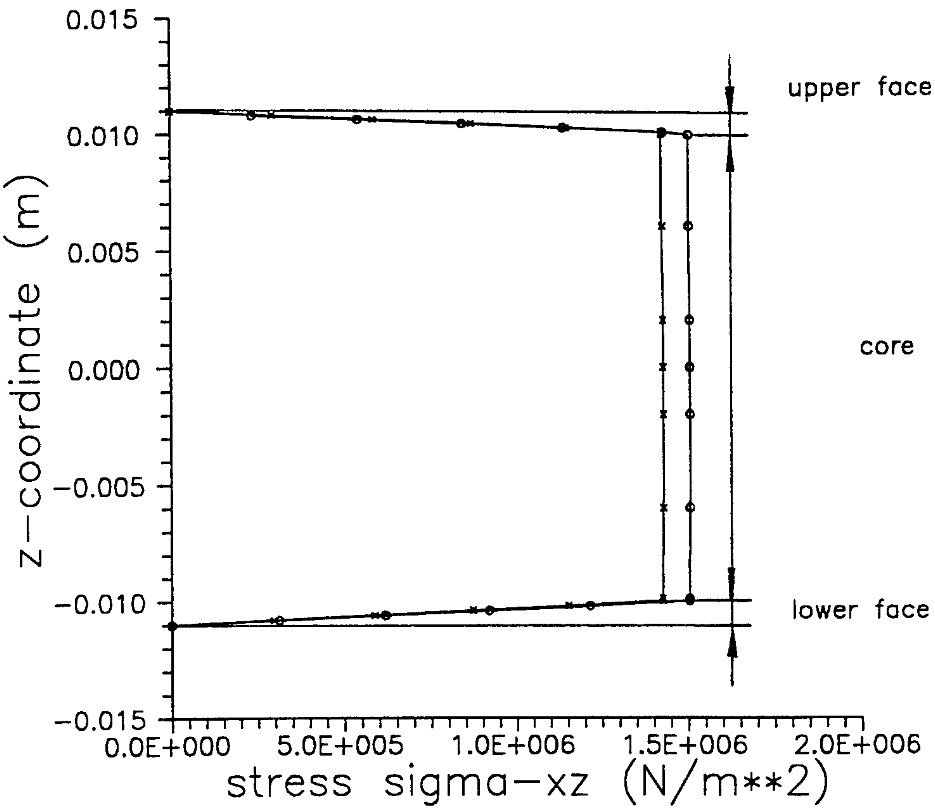


Figure 5.9
Comparison of exact elasticity solution and the finite element solution (based on the plate theory) for variation of stress sigma-zz in the thickness direction.
The exact solution is shown by solid line.

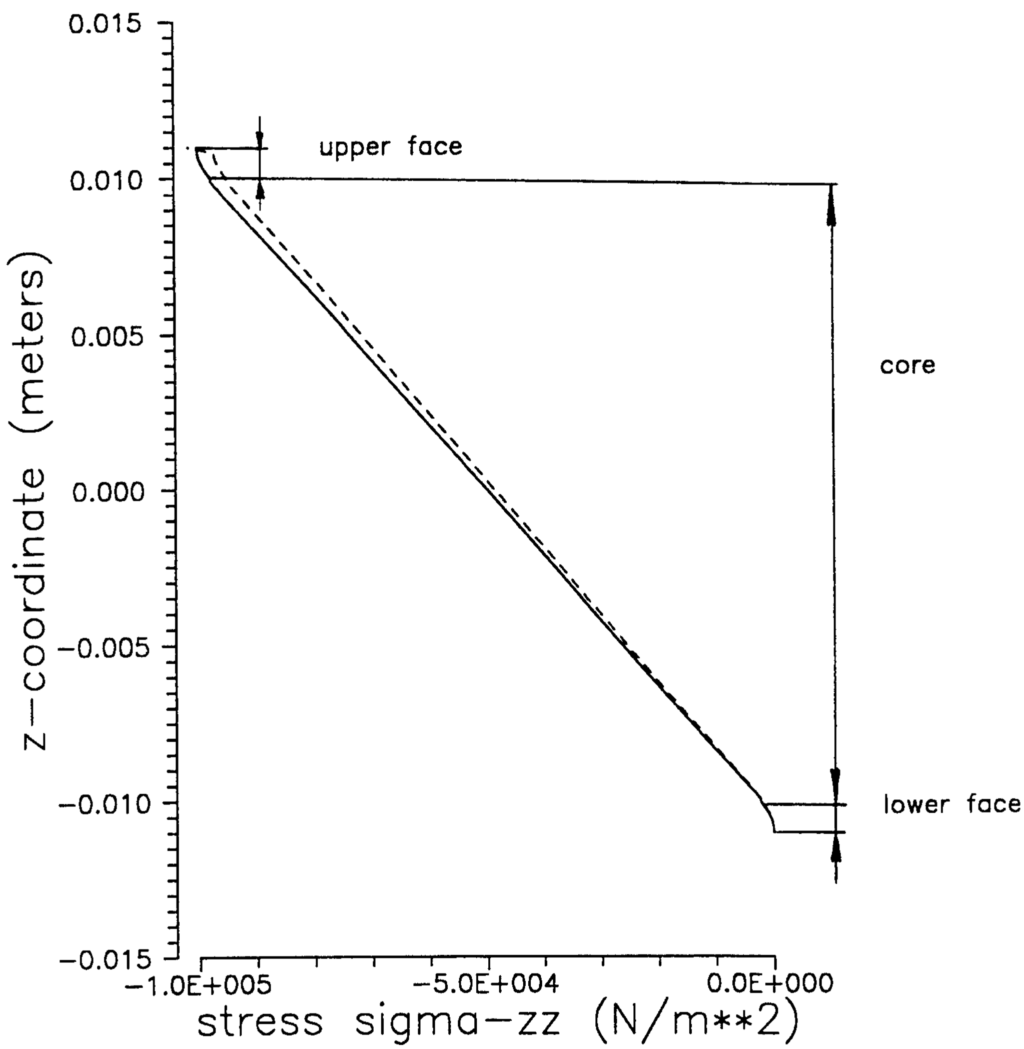


Figure 5.10

Stress sigma-xx (at x=L/2, z=-h/2) as a function of time in a sandwich platform dropped on elastic foundation with initial velocity -1m/s. The foundation modulus is 6.7864e7 Pa/m (sand). No damage occurs under this initial velocity, therefore the results of analyses with and without account of damage coincide.

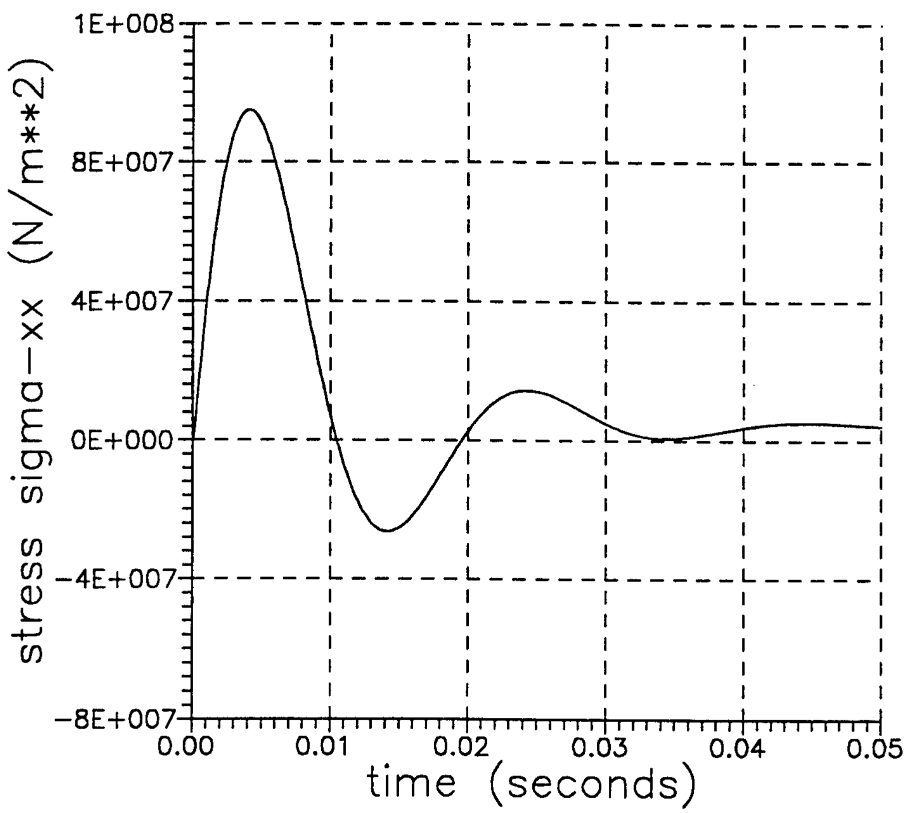


Figure 5.11

Stress sigma−zz (at x=L/2, z=−h/2) as a function of time in a sandwich platform, dropped on elastic foundation with initial velocity −1m/s. The foundation modulus is 6.7864e7 Pa/m (sand). No damage occurs under this initial velocity, therefore the results of analyses with and without account of damage coincide.

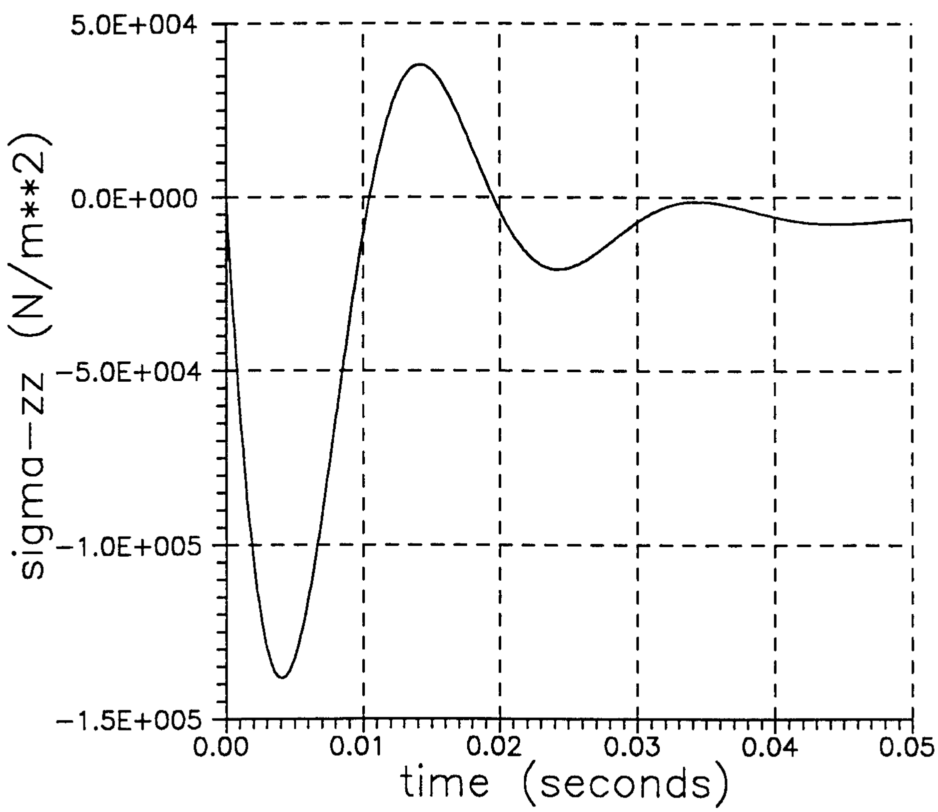


Figure 5.12

Stress–yy (at x=L/2, z=–h/2) as a function of time in a sandwich platform, dropped on elastic foundation with initial velocity –1m/s. The foundation modulus is 6.7864e7 Pa/m (sand). No damage occurs under this initial velocity, therefore the results of analyses with and without account of damage coincide.

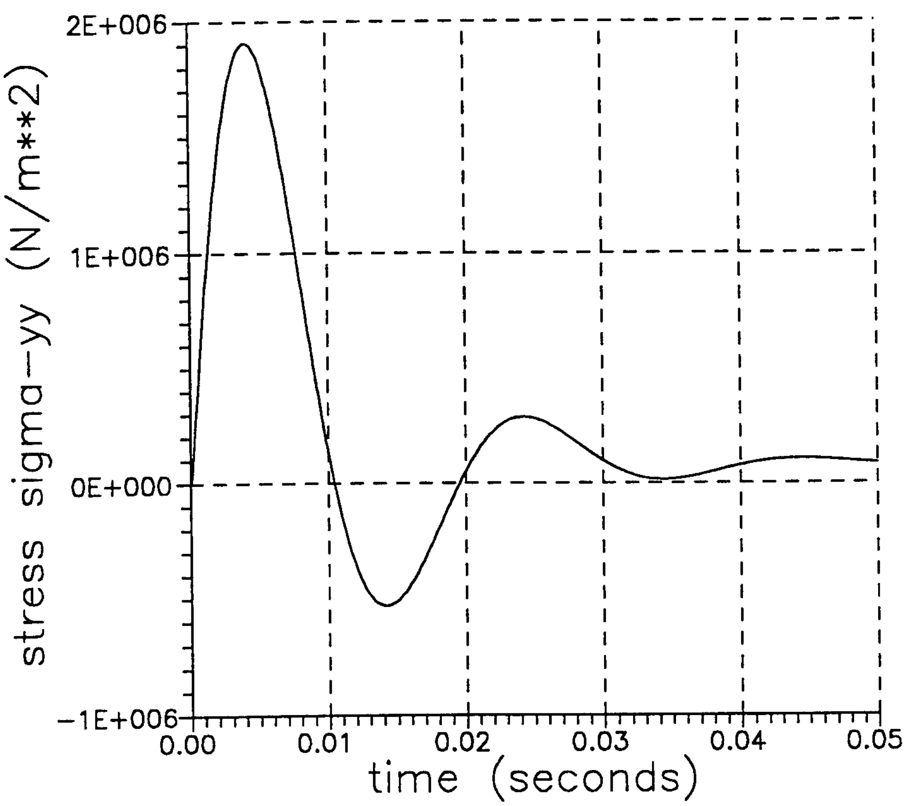


Figure 5.13

Transverse displacement (at x=L/2) as a function of time in a sandwich platform dropped on elastic foundation with initial velocity −1 m/s. The foundation modulus is 6.7864e7 Pa/m (sand). The solid line represents the displacement of the lower surface, the dashed line − displacement of the upper surface.

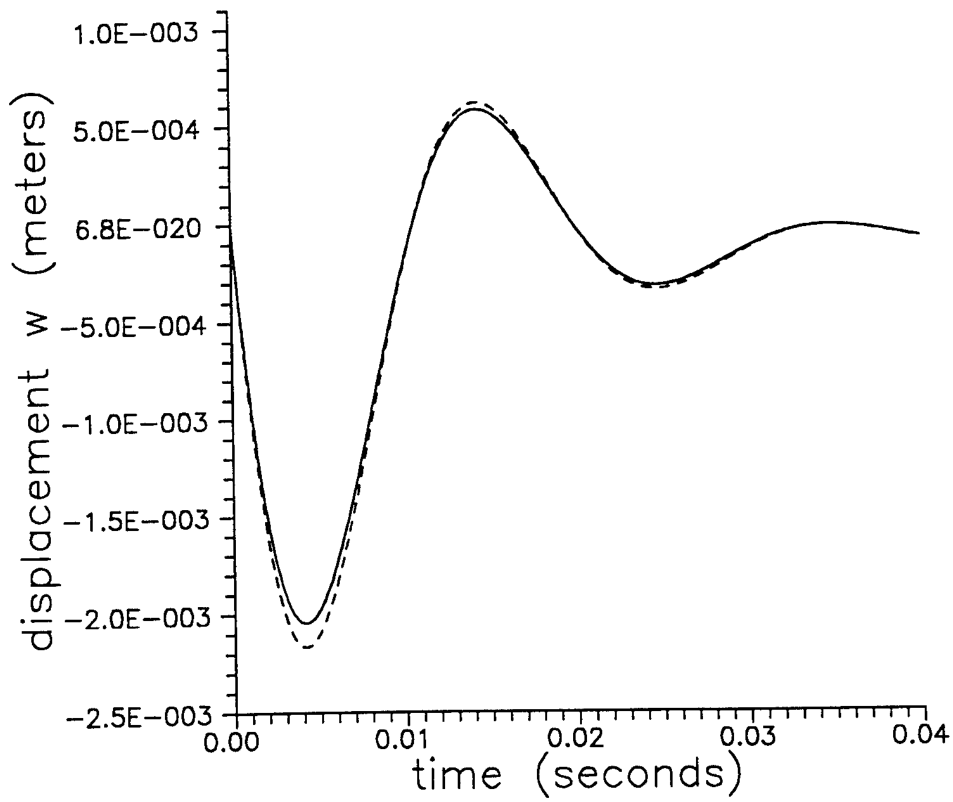


Figure 5.14

Transverse displacement (at t=0.005 s) as a function of x−coordinate in a sandwich platform dropped on elastic foundation with initial velocity −1 m/s.
The foundation modulus is 6.7864e7 Pa/m (sand).
The solid line represents the displacement of the lower surface, the dashed line − displacement of the upper surface.

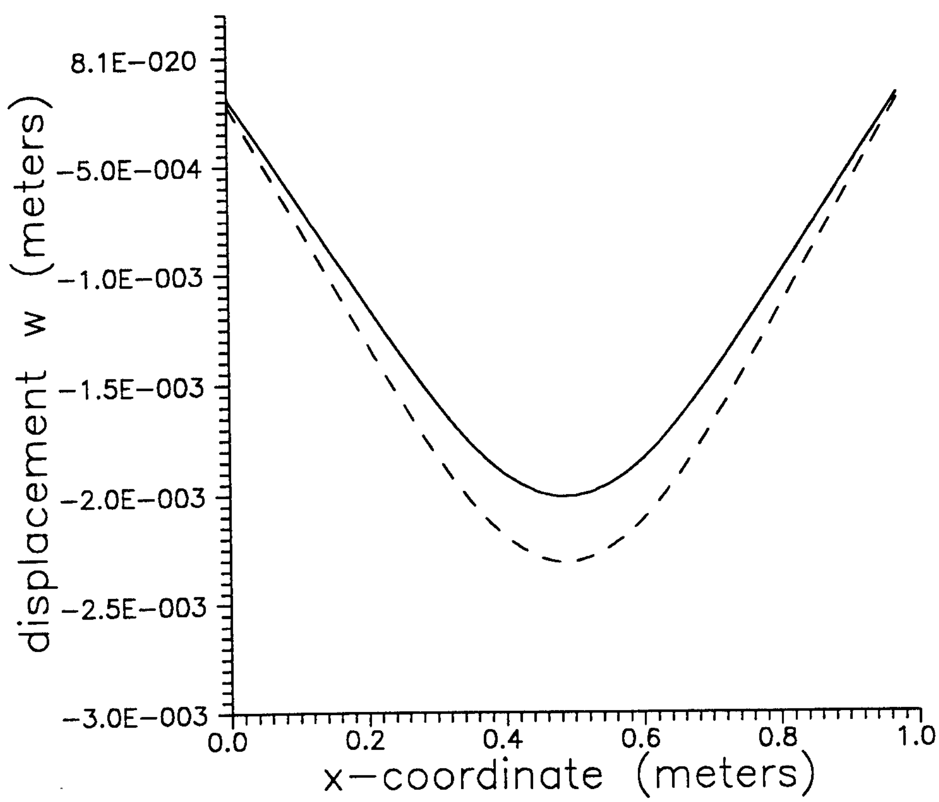


Figure 5.15

Stress sigma−xx (at x=L/2, z=−h/2) as a function of time in a sandwich platform, dropped on elastic foundation with initial velocity −30 m/s. The foundation modulus is 6.7864e7 Pa/m (sand). The dashed line represents results of analysis without account of damage, the solid line − with damage included.

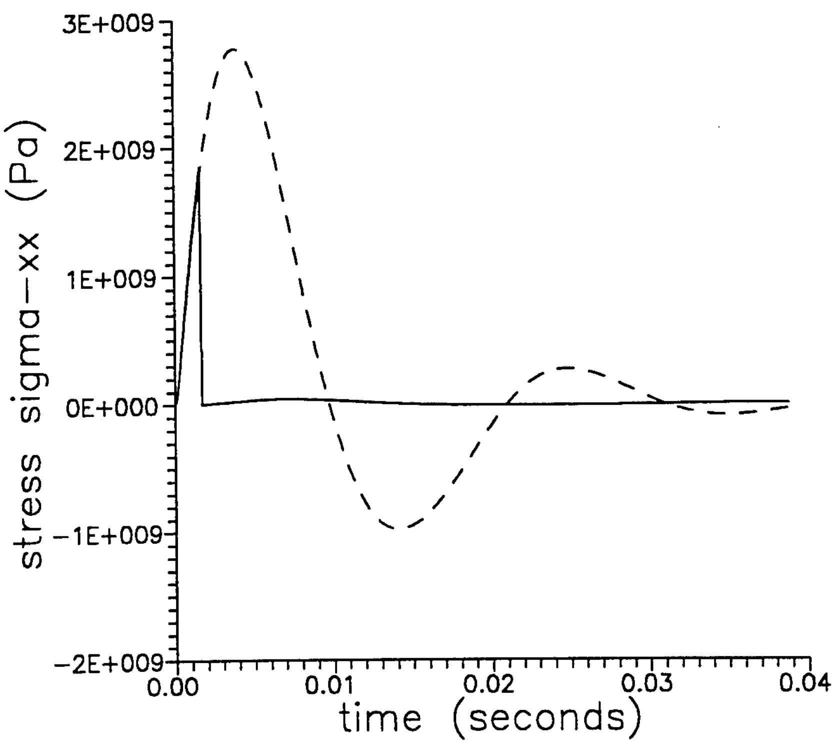


Figure 5.16

Stress sigma−zz (at x=L/2, z=−h/2) as a function of time in a sandwich platform dropped on elastic foundation with initial velocity −30 m/s. The foundation modulus is 6.7864e7 Pa/m (sand). The dashed line represents results of analysis without account of damage, the solid line − with damage included.

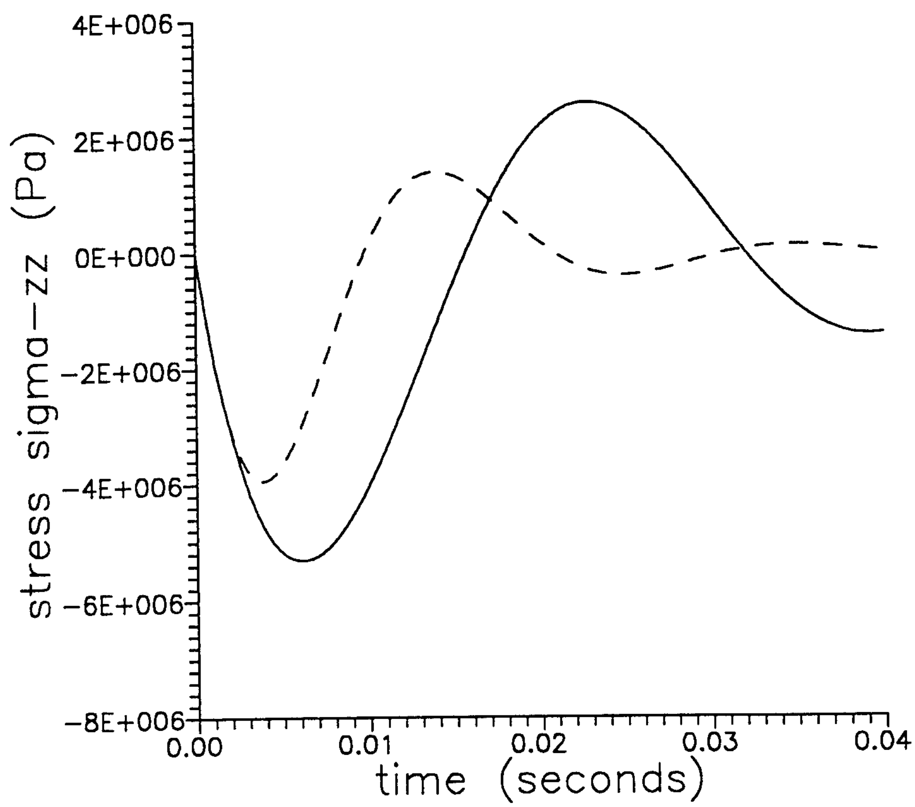


Figure 5.17

Stress sigma−yy (at x=L/2, z=−h/2) as a function of time in a sandwich platform dropped on elastic foundation with initial velocity −30 m/s. The foundation modulus is 6.7864e7 Pa/m (sand). The dashed line represents results of analysis without account of damage, the solid line − with damage included

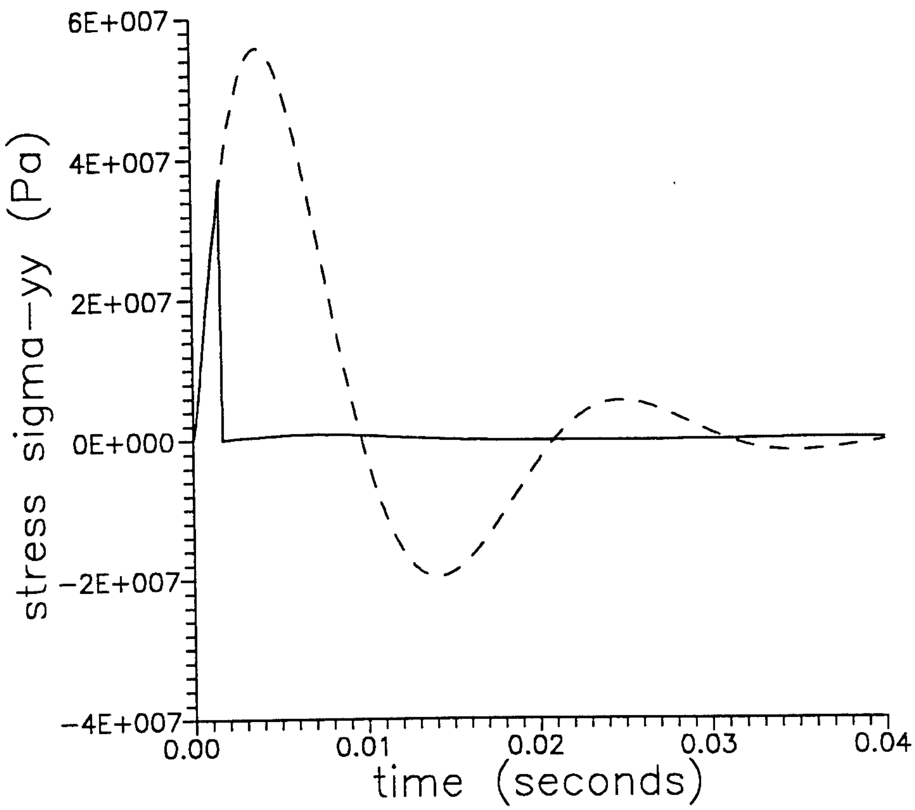


Figure 5.18

Transverse displacement w (at x=L/2, z=−h/2) as a function of time in a sandwich platform dropped on elastic foundation with initial velocity −30 m/s. The foundation modulus is 6.7864e7 Pa/m (sand). The dashed line represents results of analysis without account of damage, the solid line – with damage included.

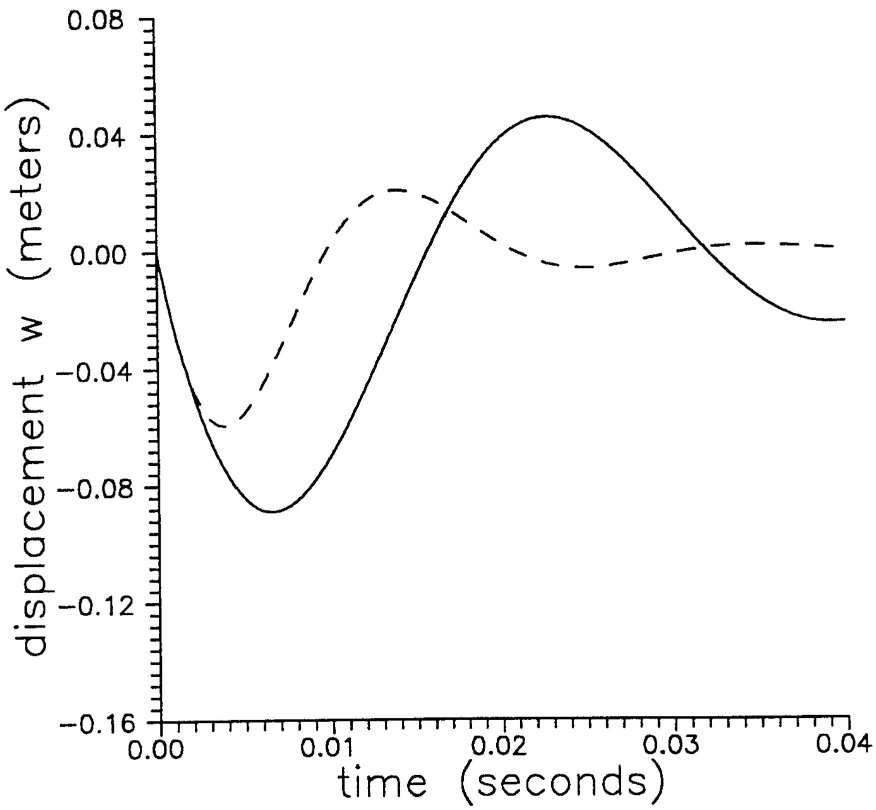


Figure 5.19

A sandwich plate in cylindrical bending divided into 20 elements.
Damage progression in the thickness direction will be shown in
Figures 5.20, 5.21, 5.26, 5.27 in the 11-th and 14-h elements, lined on this drawing.

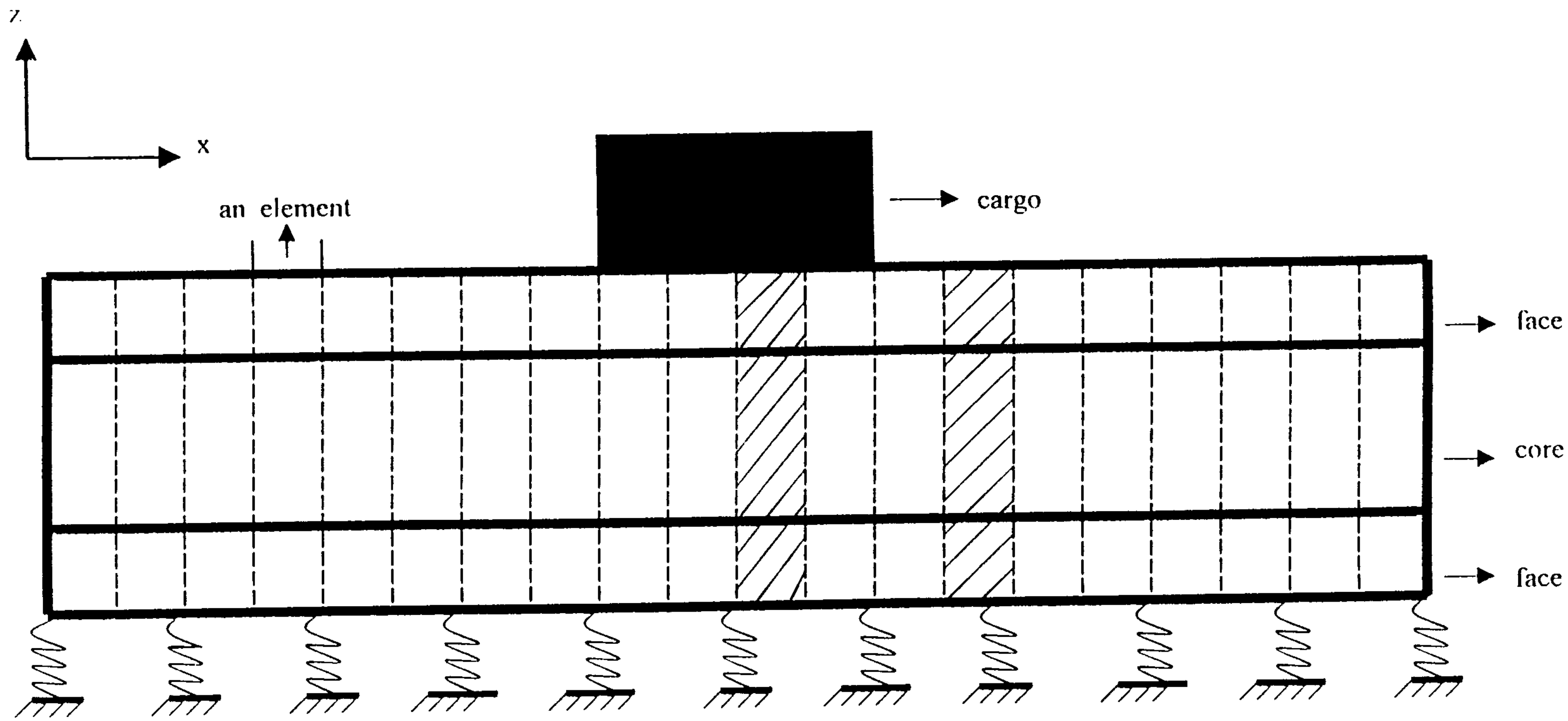


Figure 5.20

Damage progression in the thickness direction in the 11th element.

The elastic foundation modulus is $6.7864 \times 10^{7} \frac{Pa}{m}$

Z →

Lower face

Fiber orientation (degrees)	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0
Ply #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$t = 0.13 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.14 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4
$t = 0.15 \times 10^{-2}$ s	0	4	0	4	0	4	0	4	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4
$t = 0.16 \times 10^{-2}$ s	0	4	0	4	0	4	0	4	0	4	0	4	0	4	0	4	4	4	4	4	4	4	4	4	4
$t = 0.17 \times 10^{-2}$ s	0	4	0	4	0	4	0	4	0	4	0	4	0	4	0	4	4	4	4	4	4	4	4	4	4
$t = 0.18 \times 10^{-2}$ s	2	4	2	4	2	4	2	4	2	4	2	4	2	4	2	4	4	4	4	4	4	4	4	4	4
$t = 0.22 \times 10^{-2}$ s	2	4	2	4	2	4	2	4	2	4	2	4	2	4	2	4	4	4	4	4	4	4	4	4	4

Core

Fiber orientation (degrees)										
Ply #	26	27	28	29	30	31	32	33	34	35
$t = 0.13 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0
$t = 0.14 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.15 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.16 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.17 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.18 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.22 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5

Upper face

Fiber orientation (degrees)	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0
Ply #	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$t = 0.13 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.14 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.15 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.16 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$t = 0.17 \times 10^{-2}$ s	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$t = 0.18 \times 10^{-2}$ s	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$t = 0.22 \times 10^{-2}$ s	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1

Codes:

0 - no failure;

for face sheets:

1 - fiber failure in compression;
2 - fiber failure in tension;
3 - matrix failure in compression;
4 - matrix failure in tension

for the core:

5 - core failure in compression;

Figure 5.21

Damage progression in the thickness direction in the 14th element.

The elastic foundation modulus is $6.7864 \times 10^{7} \frac{Pa}{m}$

Z →

Lower face

Fiber orientation (degrees)	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0
Ply #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$t = 0.13 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.14 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.15 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.16 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.17 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.18 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.22 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Core

Fiber orientation (degrees)										
Ply #	26	27	28	29	30	31	32	33	34	35
$t = 0.13 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0
$t = 0.14 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.15 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.16 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.17 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.18 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.22 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5

Upper face

Fiber orientation (degrees)	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0
Ply #	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$t = 0.13 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.14 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.15 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.16 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.17 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.18 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.22 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Codes:

0 - no failure;

for face sheets:

1 - fiber failure in compression;
2 - fiber failure in tension;
3 - matrix failure in compression;
4 - matrix failure in tension

for the core:

5 - core failure in compression;

Figure 5.22

Stress sigma–xx (at x=L/2, z=–h/2) as a function of time in a sandwich platform, dropped on elastic foundation with initial velocity –30 m/s.
The foundation modulus is 6.7864e8 Pa/m (clay).
The dashed line represents results of analysis without account of damage, the solid line – with damage included.

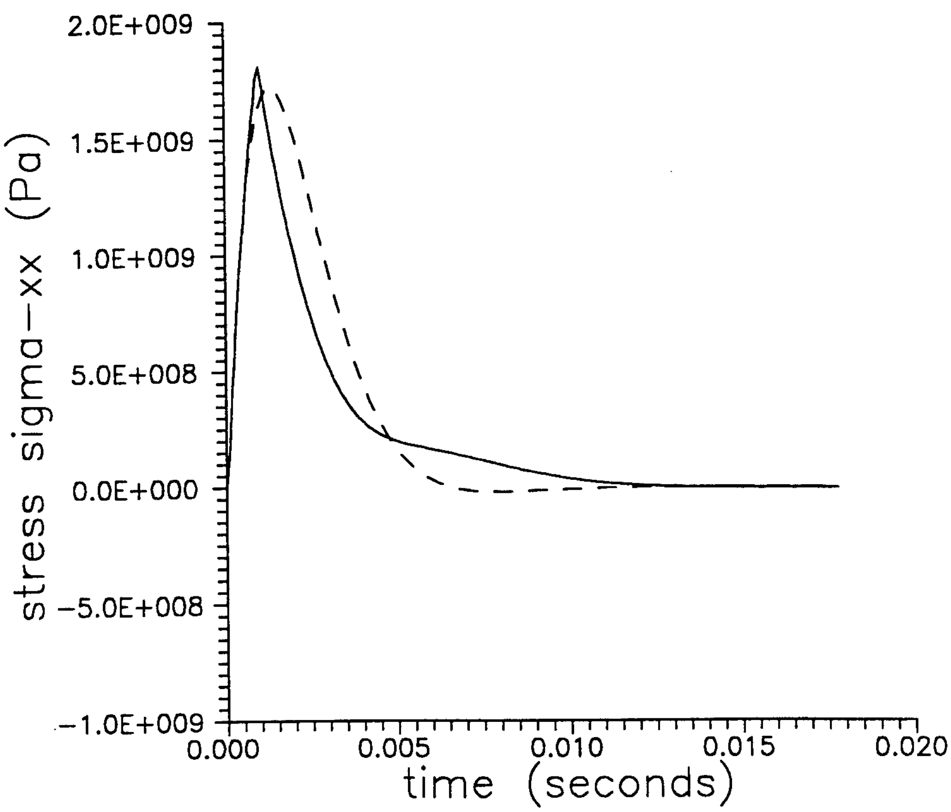


Figure 5.23

Stress sigma-zz (at x=L/2, z=-h/2) as a function of time in a sandwich platform dropped on elastic foundation with initial velocity -30 m/s.
The foundation modulus is 6.7864e8 Pa/m (clay).
The dashed line represents results of analysis without account of damage, the solid line - with damage included.

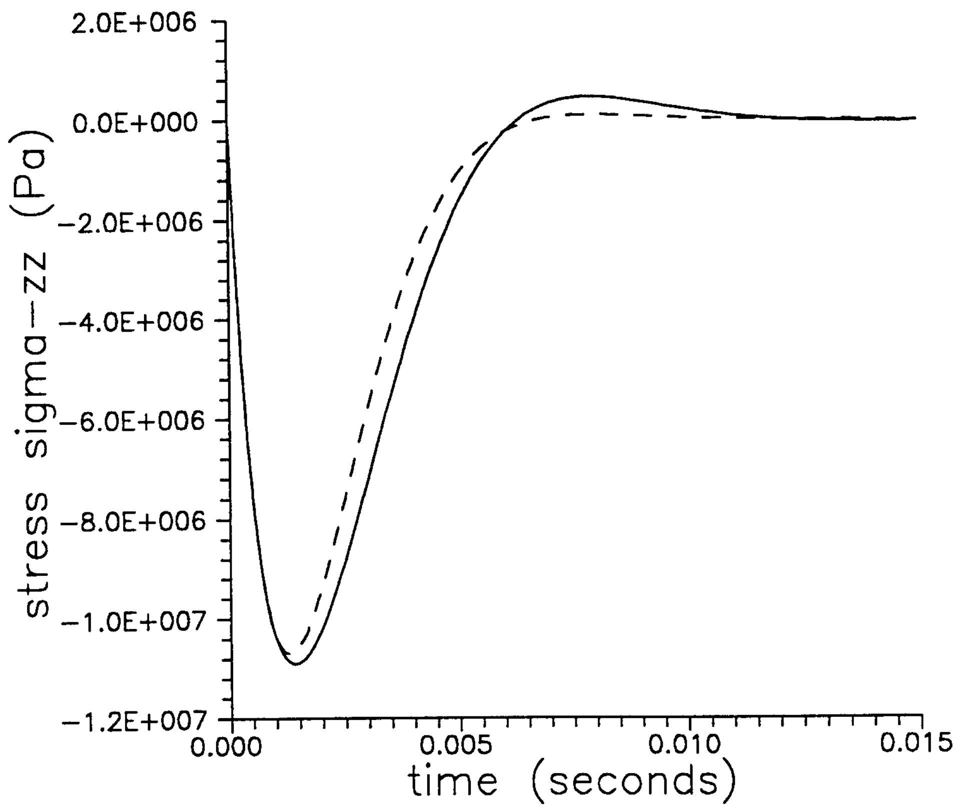


Figure 5.24

Stress sigma-yy (at x=L/2, z=-h/2) as a function of time in a sandwich platform dropped on elastic foundation with initial velocity -30 m/s.
The foundation modulus is 6.7864e8 Pa/m (clay).
The dashed line represents results of analysis without account of damage, the solid line - with damage included.

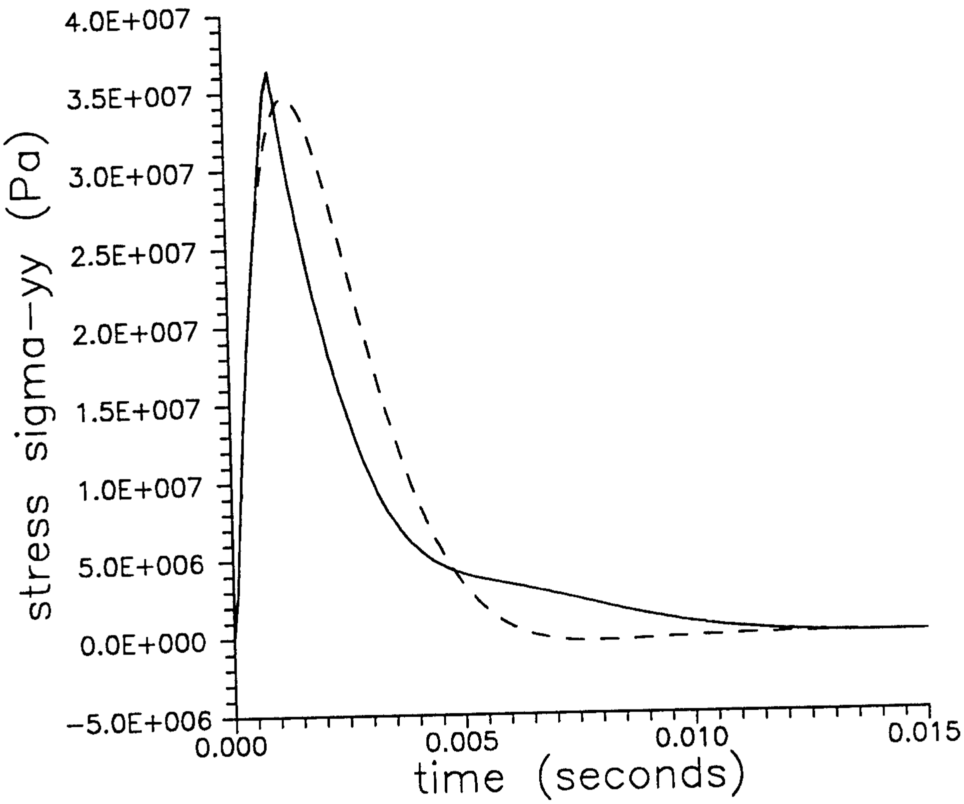


Figure 5.25

Displacement w (at $x=L/2$, $z=-h/2$) as a function of time in a sandwich platform dropped on elastic foundation with initial velocity −30 m/s.
The dashed line represents results of analysis without account of damage, the solid line — with damage included.

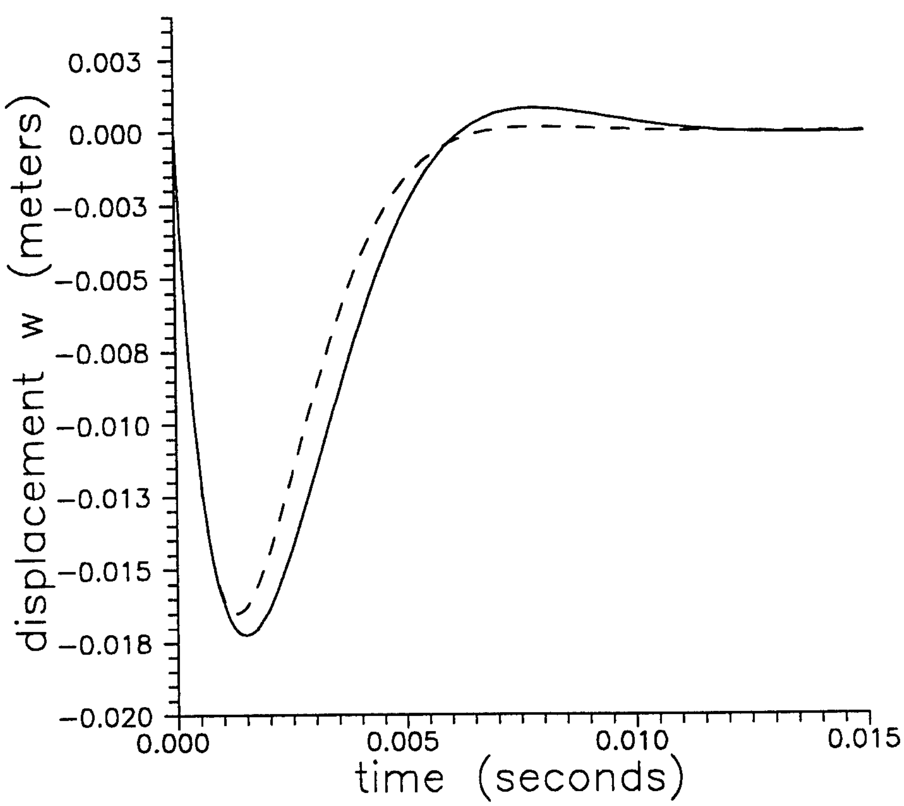


Figure 5.26

Damage progression in the thickness direction in the 11th element.

The elastic foundation modulus is $6.7864 \times 10^{8}\,\frac{\mathrm{Pa}}{\mathrm{m}}$

Z →

	Lower face																									Core										Upper face																								
Fiber orientation (degrees)	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0											0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0
Ply #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$t = 0.5 \times 10^{-3}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.6 \times 10^{-3}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	5	5	5	5	5	5	5	5	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.8 \times 10^{-3}$ s	0	4	0	4	0	4	0	0	0	4	0	0	0	0	0	0	0	0	4	0	4	0	4	0	4	5	5	5	5	5	5	5	5	5	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1
$t = 0.9 \times 10^{-3}$ s	0	4	0	4	0	4	0	4	0	4	0	4	0	4	4	0	4	0	4	0	4	0	4	0	4	5	5	5	5	5	5	5	5	5	5	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$t = 0.1 \times 10^{-2}$ s	2	4	2	4	2	4	2	4	2	4	2	4	2	4	4	4	4	4	4	4	4	4	4	4	4	5	5	5	5	5	5	5	5	5	5	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$t = 0.18 \times 10^{-2}$ s	2	4	2	4	2	4	2	4	2	4	2	4	2	4	4	4	4	4	4	4	4	4	4	4	4	5	5	5	5	5	5	5	5	5	5	1	0	1	0	1	0	1	0	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1
$t = 0.2 \times 10^{-2}$ s	2	4	2	4	2	4	2	4	2	4	2	4	2	4	4	4	4	4	4	4	4	4	4	4	4	5	5	5	5	5	5	5	5	5	5	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1	3	1

Codes:

0 - no failure;

for face sheets:

1 - fiber failure in compression;
2 - fiber failure in tension;
3 - matrix failure in compression;
4 - matrix failure in tension

for the core:

5 - core failure in compression;

Figure 5.27

Damage progression in the thickness direction in the 14th element.

The elastic foundation modulus is $6.7864 \times 10^{6} \frac{Pa}{m}$

Z $\longrightarrow$

Lower face

Fiber orientation (degrees)	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0
Ply #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$t = 0.5 \times 10^{-3}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.6 \times 10^{-3}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.8 \times 10^{-3}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.9 \times 10^{-3}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.1 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.18 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.2 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Core

Fiber orientation (degrees)										
Ply #	26	27	28	29	30	31	32	33	34	35
$t = 0.5 \times 10^{-3}$ s	0	0	0	0	0	0	0	0	0	0
$t = 0.6 \times 10^{-3}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.8 \times 10^{-3}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.9 \times 10^{-3}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.1 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.18 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5
$t = 0.2 \times 10^{-2}$ s	5	5	5	5	5	5	5	5	5	5

Upper face

Fiber orientation (degrees)	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0	90	0
Ply #	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$t = 0.5 \times 10^{-3}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.6 \times 10^{-3}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.8 \times 10^{-3}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.9 \times 10^{-3}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.1 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.18 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$t = 0.2 \times 10^{-2}$ s	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Codes:

0 - no failure;

for face sheets:

1 - fiber failure in compression;
2 - fiber failure in tension;
3 - matrix failure in compression;
4 - matrix failure in tension

for the core:

5 - core failure in compression;

Figure 5.28

Stress sigma−xx (at x=L/2, z=−h/2) as a function of time in a sandwich platform, dropped on elastic foundation with initial velocity −30 m/s.
The foundation modulus is 6.7864e7 Pa/m (sand).
The dashed line represents results of linear analysis with damage taken into account, the solid line − nonlinear analysis with damage.

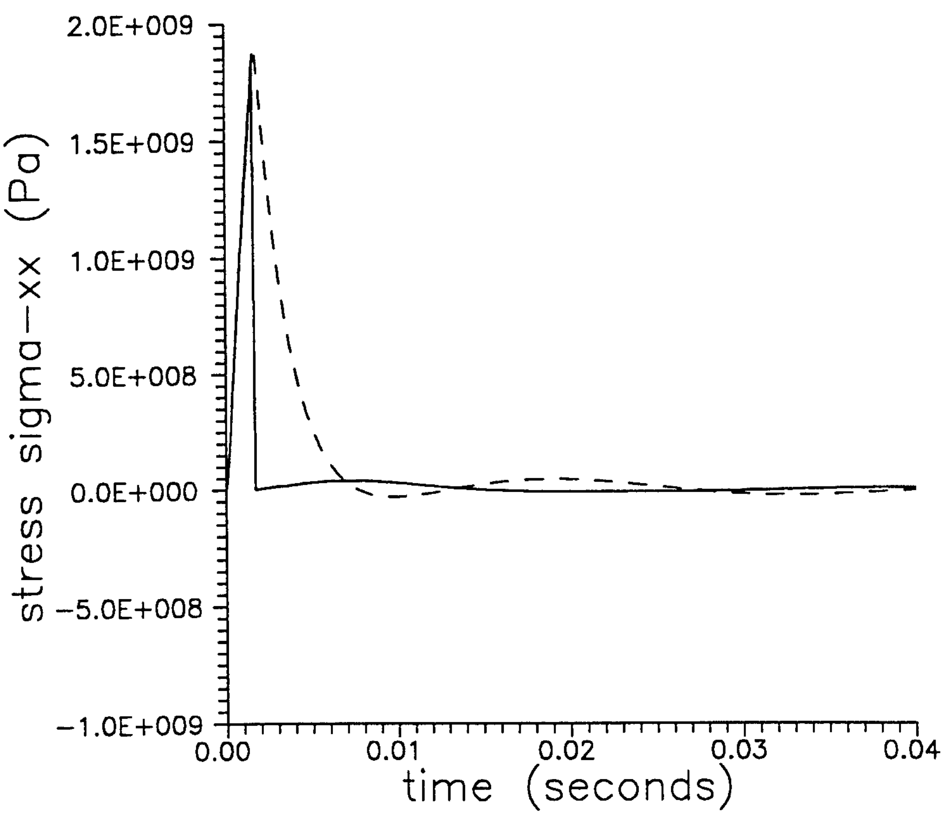


Figure 5.29

Stress sigma–zz (at x=L/2, z=–h/2) as a function of time in a sandwich platform, dropped on elastic foundation with initial velocity –30 m/s.
The foundation modulus is 6.7864e7 Pa/m (sand).
The dashed line represents results of linear analysis with damage taken into account, the solid line – nonlinear analysis with damage.

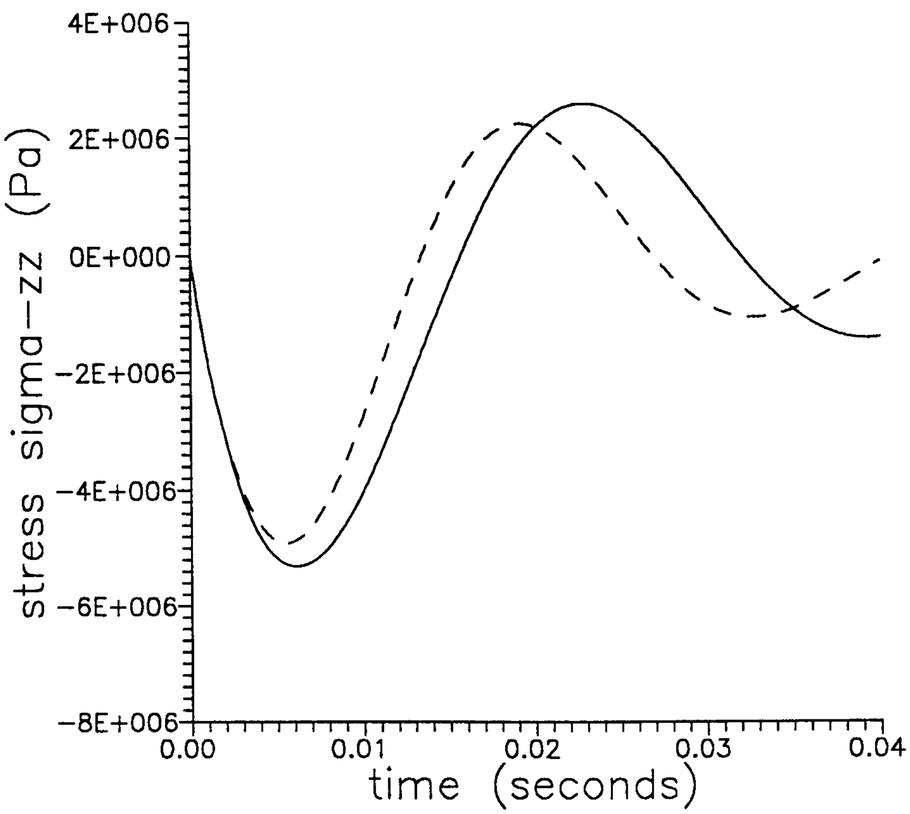


Figure 5.30

Stress sigma-yy (at x=L/2, z=-h/2) as a function of time in a sandwich platform, dropped on elastic foundation with initial velocity -30 m/s.
The foundation modulus is 6.7864e7 Pa/m (sand).
The dashed line represents results of linear analysis with damage taken into account, the solid line - nonlinear analysis with damage.

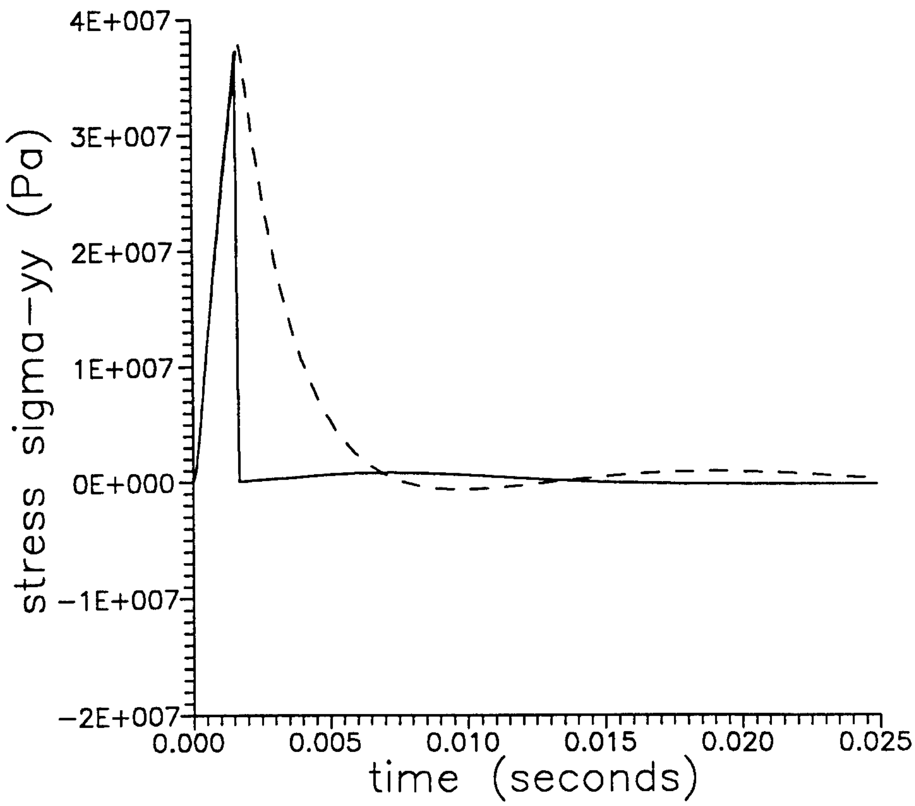
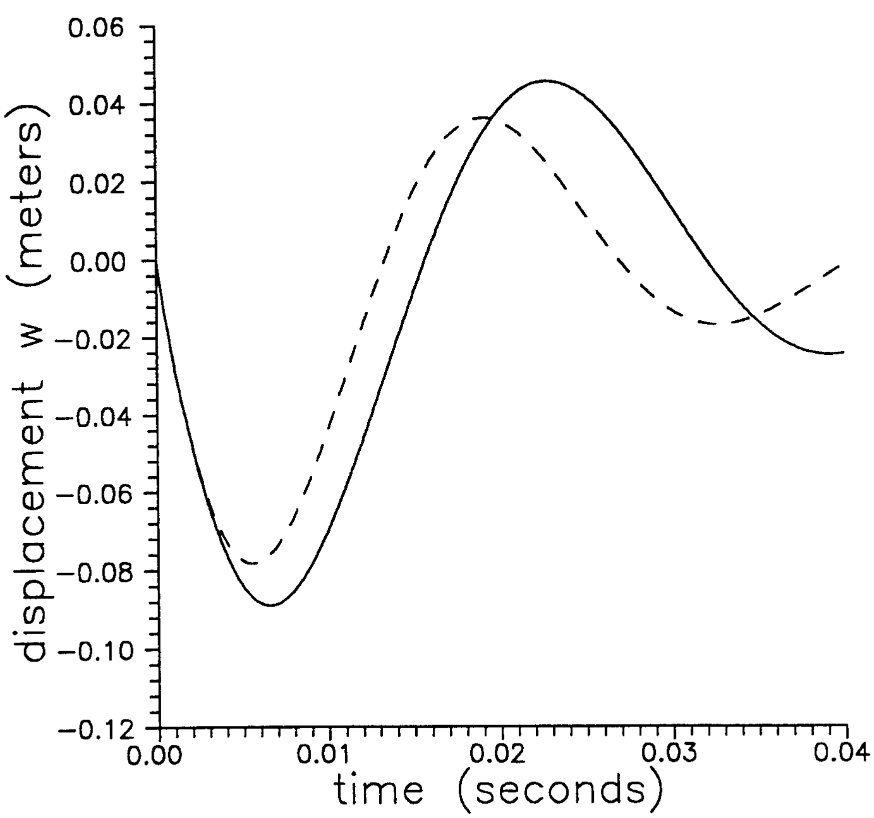


Figure 5.31

Displacement w (at $x=L/2$, $z=-h/2$) as a function of time in a sandwich platform, dropped on elastic foundation with initial velocity −30 m/s.
The founation modulus is 6.7864e7 Pa/m (sand).
The dashed line represents results of linear analysis with damage taken into account, the solid line − nonlinear analysis with damage.



Summary and Conclusions

In order to develop a dynamic two-dimensional finite element formulation for stress and progressive failure analysis of a thick sandwich plate with transversely compressible or extensible core and face sheets, a new layerwise geometrically nonlinear theory of the sandwich plate was developed by introducing assumptions on a variation of transverse strains in the thickness direction of the faces and the core of the sandwich plate. Displacements, obtained by integration of the strain-displacement relations, depend nonlinearly on a coordinate in the thickness direction, and are continuous at the boundaries between the face sheets and the core. The nonlinear von-Karman strain-displacement relations are used in order to provide more accurate representation of the moderately large rotations as compared with linear strain-displacement relations. The assumptions on the transverse strains, that lead to the layerwise theory, allow one to reduce a three-dimensional problem to a two-dimensional one and provide a proper method of averaging the material properties of the laminated composite face sheets and the core [1] over their thickness. The in-plane stresses are computed from the constitutive relations in each ply of the face sheets, using each ply's material properties (not the averaged through the thickness material properties). The transverse stresses are computed by substituting the in-plane stresses into the equations of motion and by integrating the equations of motion. Such a method of computation of the transverse stress components allows one to obtain accurate results, because this method leads to satisfaction of continuity conditions of the transverse stresses at the boundaries between the face sheets and the core, at the boundaries between the plies of the face sheets, and allows one to satisfy stress boundary conditions at both the upper and lower external

[1] material properties of the core vary in the thickness direction because of failure

surfaces. It was shown in chapter 2 that the transverse stresses, computed by integration of equations of motion, at both the upper and lower surfaces of the plate are equal to the external loads at these surfaces, despite the fact that the number of constants of integration is not sufficient to satisfy the stress boundary conditions at both the upper and lower surfaces. Thus, the adopted approach to the analysis of the sandwich plate allows one to compute accurately all six stress components, despite the reduction of the three-dimensional problem to the two-dimensional one.

A finite element formulation for the sandwich cargo platform, modelled as a plate in cylindrical bending, was done, and a finite element program was developed on the basis of this formulation with the capability of taking account of damage progression in time, that occurs in the platform during its interaction with the elastic foundation and the cargo on the upper surface. The stresses and displacements, computed by this program, are shown to be in good agreement with the known exact solutions of various static and dynamic problems. This finite element program for cylindrical bending is a necessary step in development of the finite element program based on the two-dimensional formulation, and it can be used by designers of the cargo platforms if the conditions of cylindrical bending are satisfied approximately. According to an estimate made in chapter 3 of the dissertation, the two-dimensional finite element program will allow one to compute all six stress components, needed for the progressive failure analysis, with a much smaller number of the degrees of freedom than a finite element model based on the three-dimensional finite elements.

Bibliography

[1] Abrate, S. "Impact on Laminated Composite Materials". *Applied Mechanical Reviews*, **44**(4), 155-190, 1991.

[2] Abrate, S., "Impact on Laminated Composites: Recent Advances". *Applied Mechanics Reviews*, **47**(11), pp. 517-544, 1994.

[3] Akin, J.E., *Finite Element Analysis for Undergraduates*, Academic Press, 1987.

[4] Al-Qarra, H. H. "Geometrically Nonlinear Finite Element Analysis of Sandwich Panels". *Aeronautical Journal*, **92**, pp. 356-364, 1988.

[5] Allen, H. G. *Analysis and Design of Structural Sandwich Panels.* Pergamon Press, Oxford, UK, 1969.

[6] Ambartsumyan, S. A. *Theory of Anisotropic Plates.* Technomic, Stamford, CT, 1969.

[7] Averill R. C. and Yip Y. C. "Thick Beam Theory and Finite Element Model with Zig-Zag Sublaminate Approximations". *AIAA Journal*, Vol. 34, No 8, August 1996, pp. 1627-1632.

[8] Azzi, V. D. and Tsai, S. W. "Anisotropic Strength of Composites." *Experimental Mechanics*, 1965, 5, pp. 283-288.

[9] Barbero, E. J. *Introduction to Composite Materials design.* Taylor and Francis, Inc., 1999.

[10] Barbero, E. J., Reddy, J. N. and Teply, J. L. "An Accurate Determination of Stresses in Thick Laminates Using a Generalized Plate Theory". *International Journal of Numerical Methods in Engineering*, 1990, 29, pp. 1-14.

[11] Barlow, J. "Optimal Stress Locations in Finite Element Models". *International Journal of Numerical Methods in Engineering*, 1976, 10 (2), pp. 243-251.

[12] Bathe, K.-J., *Finite Element Procedures*, Englewood Cliffs, N, J.: Prentice-Hall, Inc., 1995.

[13] Bogner, F. K., Fox, R. L., Schmidt, L. A. "The generation of Interelement-Compatible Stiffness and Mass Matrices by the Use of Interpolation Formulas". *Proceedings of the Conference on Matrix Methods in Structural Mechanics, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio*, pp. 397-443 (1965).

[14] Broutman, L.J., Krock, R.H. *Modern Composite Materials*, Reading, Mass., Addison- Wesley Pub. Co., 1967

[15] Byun, C. and Kapania R. K. "Prediction of Interlaminar Stresses in Laminated Plates Using Orthogonal Interpolation Polynomials". *Enhancing Analysis Techniques in Laminated Plates*, NDE-Vol. 10, ASME 1991.

[16] Chan, H. C., Foo, O., "Buckling of Multilayered Sandwich Plates by the Finite Strip Method". *International Journal of Mechanical Science*, **19**(8), pp. 447-456, 1977.

[17] Chang, F. K., Scott, R. A. and Springer, G. S. "Failure Strength of Nonlinearly Elastic Composite Laminates Containing a Pin Loaded Hole". *Journal of Composite Materials*, 1984, **18**, pp.464-477.

[18] Chang F. K. and Chang K. Y. "A Progressive Damage Model for Laminated Composites Containing Stress Concentrations". *Journal of Composite Materials*, 1987, **21**, pp. 834-855.

[19] Chaudhuri, R. A. "An Equilibrium Method for Prediction of Transverse Shear Stresses in a Thick Laminated Plate". *Computers and Structures*, 1986, Vol. 23, No. 2, pp. 139-146.

[20] Chaudhuri, R. A. and Seide, P. "An Approximate Semi-Analytical Method for Prediction of Interlaminar Shear Stress in an Arbitrary Laminated Thick Plate". *Computers and Structures*, 1987, Vol. 25, No. 4, pp. 627-636.

[21] Chaudhuri, R. A. and Seide, P. "Triangular Finite Element for Analysis of Thick Laminated Plates". *International Journal for Numerical Methods in Engineering*. 1987, 24, pp. 1203- 1224.

[22] Chou, P. C. and Carleone, J. "Transverse Shear in Laminated Plate Theories". *AIAA Journal*, 1973, pp. 1333-1336.

[23] Cook, R. D., Malkus, D. S., Plesha M. E. *Concepts and Applications of Finite Element Analysis*. 1989, John Wiley and Sons.

[24] Daniel , I. M. and Ishai, O. *Engineering Mechanics of Composite Materials*, 1994, Oxford University Press.

[25] Di Sciuva, M. "A Refined Transverse Shear Deformation Theory for Multilayered Anisotropic Plates". *Atti della Academia delle Scienze di Torino*, 1984, 118, pp. 269-295.

[26] Di Sciuva, M. "Bending, Vibration and Buckling of Simply Supported Thick Multilayered Orthotropic Plates: An Evaluation of a New Displacement Model". *Journal of Sound and Vibration*, 1986, pp. 425-442.

[27] Di Sciuva, M. "An Improved Shear-Deformation Theory for Moderately Thick Multilayered Anisotropic Shells and Plates". *Journal of Applied Mechanics*, 1987, 54(3), pp. 589-596.

[28] Durocher, L. L. and Solecki, R., "Bending and Vibration of Isotropic Two-Layer Plates". *AIAA Journal*, 1975, 13, pp.1522-1523.

[29] Eason, T. G. and Ochoa, O. O. "Modeling Progressive Damage in Composites: a Shear Deformable Element for ABAQUS". *Composite Structures*, 1996, 34, 119-128.

[30] Engelmann , B.E., "Dynamics of Structures: Solution Techniques". *Finite Element Analysis for Engineering Design*, edited by Reddy, J.N., Krishnamoorthy, C.S., Seetharamu K.N., Springer-Verlag, 1988

[31] Frostig, Y., Baruch, M., Vilanay, O., Sheinman, I., "High-order Theory for Sandwich Beam Behavior with Transversely Flexible Core". *Journal of Engineering Mechanics*, 1992, 118(5), pp. 1026-1043.

[32] Greer, J. M. and Palazotto A. N. "Non-linear Finite Element Analysis of Isotropic and Composite Shells by a Total Lagrangian Decomposition Scheme". *Mechanics of Composite Materials and Structures*, 1996, 3, pp. 241-271.

[33] Greszczuk, L. B. "Microbuckling of Lamina-Reinforced Composites", *Composite Materials: Testing and Design (3-rd Conf.) ASTM STP* N 546, Philadelphia (Pa.), 1974, pp.5-29.

[34] Guz, A.N. *Fracture Mechanics of Composite Materials in Compression*, 1989, Kiev, Naukova Dumka (in Russian).

[35] Gutierrez, A. J., Webber, J. P. H. "Flexural Wrinkling of Honeycomb Sandwich Beams with Laminated Faces". *International Journal of Solids and Structures*, 16(7), pp. 645-651, 1980.

[36] Hashin, Z.-H. "Failure Criteria for Unidirectional Fiber Composites". *Journal of Applied Mechanics*, 1980, 47, pp. 329-334.

[37] Hashin, Z.-H. "Analysis of Cracked Laminates: A Variational Approach". *Mechanics of Materials*, 1985, 4, pp. 121-136.

[38] Herrmann, L.R., Mason W.E., Chan S.T.K. "Response of Reinforcing Wires to Compressive State of Stress". *Journal of Composite Materials*, 1967, pp. 212-226.

[39] Herup, E. J. *Low-Velocity Impact on Composite Sandwich Plates*, 1996, Ph.D. Dissertation, Air Force Institute of Technology, WPAFB, Ohio.

[40] Highsmith, A. L. and Reifsnider, K. L. "Stiffness Reduction Mechanisms in Composite Laminates". *Damage in Composite Materials, STP 775*, K. L. Reifsneider, Ed. , American Society for Testing Materials, Philadelphia, 1982, pp. 103-117.

[41] Hinrichsen, R. L. and Palazotto, A. N. "Nonlinear Finite Element Analysis of Thick Composite Plates Using Cubic Spline Functions". *AIAA Journal*, 1986, **24**, 1836-1842.

[42] Hinrichsen R. L. and Palazotto A. N. "Use of a Cubic Spline Function in Finite Elements". *Mathl. Comput. Modeling*, **10**(1), pp. 37-47, 1988.

[43] Hornbeck, R. W. *Numerical Methods*, 1975, New York: Quantum Publishers, Inc

[44] Kanematsu, H. H., Hirano, Y., Iyama, H. "Bending and Vibration of CFRP-faced Rectangular Sandwich Plates". *Composite Structures* **10**(2), pp. 145-163, 1988.

[45] Karpenko, L.I., Terletzki, V.A., Lyashchenko B.A., "About one Mechanism of Fracture of Oriented Plastics". *Strength of Materials*, 1972, vol. 1.

[46] Katona, M. G. and Zienkiewicz, O. C. "A Unified Set of Single-step Algorithms, Part 3: the Beta-m Method, a Generalization of the Newmark Scheme". *International Journal for Numerical Methods in Engineering*, 1985, **21**(7), pp. 1345-1359.

[47] Kerr, A. "Elastic and Viscoelastic Foundation Models Journal of Applied Mechanics", *ASME*, **31**, pp. 491-498, 1964.

[48] Kneifati, M., "Analysis of Plates on a Kerr Foundation Model", *Journal of Engineering Mechanics, ASCE*, **111**(11), pp. 1325-1342, 1985.

[49] Kwon, Y. W. and Bang , H. *The Finite Element Method Using MATLAB.* Boca Raton: CRC Press, 1997

[50] Laicok, M. R. "New Approach in the Determination of Interlaminar Shear Stresses from the Results of MSC/NASTRAN". *Computers and Structures*, 1986, **24**(4), pp. 651-656.

[51] Lee, J. D. "Three Dimensional Finite Element Analysis of Damage Accumulation in Composite Laminate". *Computers and Structures*, 1982, **15**(3), pp. 335-350.

[52] Lee, K. H, Xavier, P. B., Chew, C. H. "Static Response of Unsymmetrical Sandwich Beams Using an Improved Zig-Zag Model". *Composite Engineering*, **3**(3), pp. 235-248, 1993.

[53] Liu, F. L., Chen, H., "A General Bending Theory of Composite Sandwich Plates and its Application in Aircraft". *Composites: Design, Manufacture and Application (Proc. 8th Int. Conf. On Composite Mat.)*, 1991, Tsai, S. W. , Springer, G. S. (eds), SAMPE, Covina CA, 3-C-1 to 3-C-8.

[54] Lo, K. H., Christensen, R. M. and Wu, E. M. "A High-Order Theory of Plate Deformation. Part II: Laminated Pates". *ASME Journal of Applied Mechanics*, 1977, 44, pp. 669-676.

[55] Lo, K. H., Christensen R. M. and Wu, E. M. "Stress Solution Determination for High-Order Plate Theory". *International Journal of Solids and Structures*, 1978, **14**, pp. 655-662.

[56] Lu, X. and Liu, D. "An Interlaminar Shear Stress Continuity Theory for Both Thin and Thick Composite Laminate". *ASME Journal of Applied Mechanics*, 1992, **59**, pp. 502-509.

[57] Malvern , L. E. *Introduction to the Mechanics of a Continuous Medium*, 1969, Englewood Cliffs, New Jersey: Prentice-Hall, Inc.

[58] MacNeal, R. *Finite Elements: Their Design and Performance*, 1994, New York: M. Dekker

[59] Markus, S. and Nanasi, T. "Significance of In-Plane Inertia Forces in the Vibration Analysis of Three-Layered Circular Plates". *Journal of Sound and Vibration*, 1981, **763**, pp. 421-441.

[60] Mau, S.T. "A Refined Laminate Plate Theory". *Journal of Applied Mechanics*, 1973, **40**, pp. 606-607.

[61] Mead D. J. "The damping properties of elastically supported sandwich plates". *Journal of Sound and Vibrations*, 1972, **243**, pp. 275-295.

[62] Meirovitch, L. *Methods of Analytical Dynamics*, 1970, McGraw-Hill Book Co., New York.

[63] Mukarami, H. "Laminated Composite Plate Theory with Improved In-Plane Responses". *Journal of Applied Mechanics*, 1986, **53**, pp. 661-666.

[64] Mukhopadhyay, Sierakowsky. "On Sandwich Beams with Laminate Facings and Honeycomb Cores Subjected to Hydrothermal Loads: Part I, Analysis". *Journal of Composite Materials*, **24**, pp. 382-400, 1990.

[65] Monforton, G. R., Ibrahim, I. M. "Analysis of Sandwich Plates with Unbalanced Cross-Ply Faces". *International Journal of Mechanical Sciences*, **17**, pp. 227-238, 1975.

[66] Murthy, A. V. K. and Vellaichamy, S. "On Higher-Order Shear Deformation Theory of Laminated Composite Panels". *Composite Structures*, **8**, pp. 247-270, 1987.

[67] Noor, A. K. "Free Vibrations of Multilayered Composite Plates". *AIAA Journal*, **11**, pp. 1038-1039, 1973

[68] Novozhilov, V. V. *Theory of Elasticity.* Pergamon Press, 1961

[69] Ochoa, O. O. and Engblom J. J. "Analysis of Progressive Failure in Composites". *Composite Science and Technology*, **28**, pp. 87-102, 1987.

[70] Pai P.F. and Palazotto A. N. "Polar Decomposition Theory in Nonlinear Analysis of Solids and Structures". *Journal of Engineering Mechanics*, **121**(4), pp. 568-581, 1995(a).

[71] Pai P.F. and Palazotto A. N. "Nonlinear Displacement-Based Finite-Element Analyses of Composite Shells- A New Total Lagrangian Formulation". *International Journal of Solids and Structures*, 1995(b), **32**(20), pp. 3047-3073.

[72] Pagano, N. J. "Exact Solutions for Composite Laminates in Cylindrical Bending". *Journal of Composite Materials*, **3**, pp. 398-411, 1969.

[73] Pagano, N. J. "Exact Solutions for Rectangular Bidirectional Composite and Sandwich Plates". *Journal of Composite Materials*, **4**, pp. 20-34, 1970.

[74] Pagano, N. J. and Hatfield, S. J. "Elastic Behavior of Multilayered Bidirectional Composites". *AIAA Journal*, 1972, **10**, pp.931-933.

[75] Palazotto, A. N. and Dennis, S. T. *Nonlinear Analysis of Shell Structures.* 1992, AIAA Educational Series. Washington, D. C.: American Institute of Aeronautics and Astronautics, Inc.

[76] Pasternak, P. " On a New Method of Analysis of an Elastic Foundation by Means of Two Foundation Constants" (in Russian). *Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstvu I Arkhitekture*, Moscow, USSR, 1954.

[77] Petit, P. H. and Waddoups, M. E. "A Method of Predicting the Nonlinear Behavior of Laminated Composites". *Journal of Composite Materials*, 1969, 3, pp. 2-19.

[78] Pikul V. *General Technical Theory of Thin Elastic Plates and Shallow Shells* , M: Nauka, 1977 (in Russian).

[79] Pikul, V. "Physically Correct Models for Material of Elastic Shells", *Mechanics of Solids*, **30**(2), pp. 94-99, 1995.

[80] Plantema, F. J., *Sandwich Construction.* John Wiley, NY,1966

[81] Reddy, J. N. *Mechanics of Laminated Composite Plates: Theory and Analysis*, CRC Press, 1996.

[82] Reddy, J. N. *An Introduction to the Finite Element Method*, McGraw-Hill, 1993.

[83] Reddy, J. N. and Liu, C. F. "A Higher Order Shear Deformation Theory of Laminated Composite Shells". *International Journal of Engineering Science*, 1985, 23 (3), 319-330.

[84] Reddy, J. N. "A Simple Higher-Order Theory for Laminated Composite Plates". *Journal of Applied Mechanics*, 1984, 51, pp. 745-752.

[85] Reddy, J. N. and Pandley, A. K. "A First Ply Failure Analysis of Com posite Laminates". *Computers and Structures*, 1987, 25, pp. 371-393.

[86] Reddy, Y. S. N. and Reddy, J. N. "Linear and Non-Linear Failure Analysis of Composite Laminate with Transverse Shear". *Composite Science and Technology*, 44, 1992, pp. 227- 255.

[87] Reifsnider K. L. and Masters J. E. "An Investigation of Cumulative Damage Development in Quasi-Isotropic Graphite/Epoxy Laminates", *in Damage in Composite Materials, ASTM*

STP 775, K. L. Reifsnider, Ed., American Society for Testing and Materials, Philadelphia, 1982, pp. 40-62.

[88] Reissner, E., "Finite Deflections of Sandwich Plates". *Journal of Aeronautical Science* **15**(7), pp. 435-440, 1948.

[89] Ren, J. G. "A new Theory of Laminate Plate". *Composite Science and Technology,* **26**, pp. 225-239, 1986.

[90] "Research Interests and Broad Agency Announcement 97-1", *BAA announced by AFOSR*

[91] Rosen B.U., Dow N.F., "Fracture Mechanics of Fiber Composites", *Fracture, an Advanced Treatise, vol. 7, Fracture of Nonmetallic and Composite Materials,* edited by Liebowitz, H.

[92] Saada, A. S., *Elasticity Theory and Applications.* New York: Pergamon Press, 1993.

[93] Sadovski M.A., Pu S.L., Hussain M.A. "Buckling of Microfibers" Jbid., **34**(4), pp.295- 302, 1967.

[94] Savoia, M. and Reddy, J. N. "A Variational Approach to Three-dimensional Elasticity Solutions of Laminated Composite Plates". *Journal of Applied Mechanics,* **59**, S166- S175, 1992.

[95] Schuerch, H. "Prediction of Compressive Strength in Uniaxial Boron Fibermetal Matrix Composite Material", *AIAA Journal,* 4(1), pp.102-106, 1966.

[96] Seide, P. "An Improved Approximate Theory for Bending of Laminated Plates". Mech. Today, 1980, 5, pp. 451-466.

[97] Sierakowski R. L., Chaturvedi S. K. *Dynamic Loading and Characterization of Fiber- Reinforced Composites.* John Wiley and Sons, Inc., 1997.

[98] Shim V. P. W. and Yap K. Y. "Modeling Impact Deformation Of Foam-Plate Sandwich Systems". *International Journal of Impact Engineering,* Vol. 19, No 7, pp. 615-636, 1997.

[99] Srinivas, S., Joga Rao, C. V. and Rao, A. K. "An Exact Analysis for Vibration of Simply Supported Homogeneous and Laminated Thick Rectangular Plates". *Journal of Sound and Vibration,* **12**, pp. 187-199, 1970.

[100] Srinivas, S., and Rao, A. K. "Bending, Vibration and Buckling of Simply Supported Thick Orthotropic Rectangular Plates and Laminates". *International Journal of Solid and Structures*, **6**, pp. 1463-1481, 1970.

[101] Stoker, J. J. *Nonlinear Elasticity*, Gordon and Breach Science Publishers, 1968.

[102] Swift, G. W. and Heller, R. A. "Layered Beam Analysis". *Journal of the Engineering Mechanics Division*, ASCE, **100**, pp. 267-282, 1974.

[103] Sun, C. T. "Failure Analysis of Laminated Composites by Using Iterative Three Dimensional Finite Element Method". *Computers and Structures*, **33**, pp. 41-47, 1989.

[104] Sun, C. T. and Whitney, J. M. "Theories for the Dynamic Response of Laminated Plates". *AIAA Journal*, **11**(2), pp. 178-183, 1973.

[105] Talreja, R. "Transverse Cracking and Stiffness Reduction in Composite Laminates". *Journal of Composite Materials*, **19**, pp. 355-375, 1985.

[106] Tan, S. C. "A Progressive Failure Model of Composite Laminates Containing Openings". *Journal of Composite Materials*, **25**, pp. 556-577, 1991.

[107] Tay, T. E., Lam, K. Y. and Cen Z. "Analysis of Composite Structures with Distributed and Localized Damage by the Finite-Element Method". *Composite Structures*, **37**, pp. 135-143, 1997.

[108] Tay, T. E. and Lim, E. H. "Analysis of Stiffness Loss in Cross-Ply Composite Laminates". *Composite Structures*, **25**, pp. 419-425, 1993.

[109] Tolson, S. and Zabaras, N. "Finite Element Analysis of Progressive Failure in Laminated Composite Plates". *Computers and Structures*, **38**, pp. 361-376, 1991.

[110] Tsai, C. T. and Palazotto, A. N. "On the Finite Element Analysis of Non-Linear Vibration for Cylindrical Shells with High-Order Shear Deformation Theory". *International Journal of Non-Linear Mechanics*, **26**(3/4), pp. 379-388.

[111] Ugural A. C., Fenster, S. K. *Advanced Strength and Applied Elasticity.* 1995, Prentice Hall.

[112] Vlasov, B. F. "About One Case of Bending of Rectangular Thick Plate". *Bulletin of the Moscow State University, Mathematics and Mechanics.*1957, No. 2

[113] von Karman, T. "Festigkeitsprobleme in maschinenbau". *Ency. der. Math. Wissenschaften*, Vol. IV/4C, pp. 311-385. Teubner, Leipzig,1910.

[114] Washizu, K., *Variational Methods in Elasticity and Plasticity*, Oxford, Pergamon Press, 1982.

[115] Winkler, G. "Die Lehre von der Elasticitaet und Festigkeit". *Prog. Dominicus*, 1867.

[116] Whitney, J. M. *Structural Analysis of Laminated Anisotropic Plates.* Technomic Publishing Company,1987.

[117] Whitney, J. M. "The Effect of Transverse Shear Deformation in the Bending of Laminated Plates". *Journal of Composite Materials*, **3**, pp. 534-547, 1969.

[118] Whitney, J. M. "Shear Correction Factors for Orthotropic Laminates Under Static Loading". *Journal of Applied Mechanics*, **40**, pp. 302-304, 1971.

[119] Wu, E. M., "Phenomenological Anisotropic Failure Criterion". *Composite Materials*, **2**, pp. 353-431, 1974.

[120] Yamada, S. E. and Sun, C. T. "Analysis of Laminate Strength and its Distribution". *Journal of Composite Materials*, **12**, pp. 275-284, 1978.

[121] Yu, Y.Y. *Vibrations of Elastic Plates*, Springer, 1997.

[122] Yu, Y. Y. "A New Theory of Elastic Sandwich Plates. One-dimensional Case", *Journal of Applied Mechanics*, **26**, pp. 415-421, 1959.

Vita

Victor Y. Perel was born in [redacted] in Ukraine. He graduated from a high school in Rovno, Ukraine in 1979 and entered the Dnepropetrovsk State University in Dnepropetrovsk, Ukraine. He graduated from the university in 1984 with a degree of Master of Science in Physics. After graduation he worked in the Ukrainian Institute of Water Engineering in Rovno, Ukraine, where he was a physics instructor and was involved in research in analysis of composite structures. In January 1994 he entered the graduate program of the University of Dayton where he graduated with a Master of Science degree in Engineering Mechanics in December 1995. In June 1996 he entered the Ph.D. program of the Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, where he graduated with a Doctor of Philosophy degree in Structural Mechanics in June 2000.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 074-0188

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From – To)*
12-06-2000	Ph.D. Dissertation	May 1998 -- Jun 2000

4. TITLE AND SUBTITLE

Three-Dimensional Dynamic Stress Analysis of Sandwich Panels

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR

Perel, Victor

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME AND ADDRESS

Air Force Institute of Technology
Graduate School of Engineering and Management (AFIT/EN)
2950 P Street, Building 640
WPAFB OH 45433-7765

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME AND ADDRESS
AFOSR/NA
Dr. Steven Walker
801/ N. Randolph Street, Room 732
Arlington, VA 22203-1977
(703) 696-6962

10. SPONSOR/MONITOR'S ACRONYM
AFIT/DS/ENY/00-02

11. SPONSOR/MONITOR'S REPORT NUMBER
AFIT/DS/ENY/00-02

12. DISTRIBUTION/AVAILABILITY STATEMENT

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

13. SUPPLEMENTARY NOTES

14. ABSTRACT
A layerwise geometrically nonlinear theory for a thick sandwich plate was developed by introducing assumptions on a variation of transverse strains in the thickness direction of the faces and the core of the plate. An effect of transverse extensibility or compressibility of the core and the face sheets is taken into account, and the terms associated with transverse shear strain of the face sheets and the core are included into the expression for the strain energy. Displacements, obtained by integration of the strain-displacement relations, depend nonlinearly on a coordinate in the thickness direction, and are continuous at the boundaries between the face sheets and the core. The non-linear von Karman strain-displacement relations are used in order to provide a representation of the moderately large rotations. The in-plane stresses are computed from the constitutive relations in each ply of the face sheets, using each ply's material properties, and the transverse stresses are computed by substituting the in-plane stresses into equations of motion and by integrating the equations of motion. Such a method of computation of the transverse stress components allows one to obtain accurate results, because this method leads to satisfaction of conditions of continuity of the transverse stresses at the boundaries between the face sheets and the core, at the boundaries between the plies of the face sheets, and allows to satisfy stress boundary conditions at both the upper and lower external surfaces. A finite element formulation was developed for a sandwich cargo platform under its impact against the ground, modeled as an elastic Winkler foundation. This formulation was done for a plate in cylindrical bending, and a finite element program was written on the basis of this formulation, with the capability of taking account of damage progression in time. The damage prediction is performed with the use of the Hashin's and Tsai-Wu criteria by reducing at each step of time integration the appropriate material characteristics of those plies within a finite element in which failure occurs. The stresses and displacements, computed by this program, are shown to be in a good agreement with the known exact solutions of various static and dynamic problems. Example problems of stress and failure analysis of sandwich cargo-delivery platforms during their impact against the elastic foundations are considered. In these example problems, the stresses as functions of time are computed at certain locations in the platforms with account of degradation of material characteristics of the failing plies. The locations of the failures, the modes of failures and the times of their occurrence are defined by the program. The theory of the sandwich plates, presented in the dissertation, does not require many degrees of freedom in the finite element formulation and has a wide range of applicability. It can be used for analysis of both thick and thin sandwich plates, with thick and thin face sheets, with transversely flexible and transversely rigid faces and cores.

15. SUBJECT TERMS
Sandwich composite plates, finite element formulation, stress computation, failure criteria, progressive failure analysis

16. SECURITY CLASSIFICATION OF:	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
	UU	449	Professor Anthony N. Palazotto

a. REPORT	b. ABSTRACT	c. THIS PAGE			19b. TELEPHONE NUMBER *(Include area code)*
U	U	U			(937) 255-3636, ext 4599

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39-18

	Form Approved *OMB No. 074-0188*